Some Useful Vector Identities

¶ $A \cdot B \times C = B \cdot C \times A = C \cdot A \times B$

¶ $A \times (B \times C) = B(A \cdot C) - C(A \cdot B)$

¶ $\nabla(\psi V) = \psi \nabla V + V \nabla \psi$

¶ $\nabla \cdot (\psi A) = \psi \nabla \cdot A + A \cdot \nabla \psi$

¶ $\nabla \times (\psi A) = \psi \nabla \times A + \nabla \psi \times A$

¶ $\nabla \cdot (A \times B) = B \cdot (\nabla \times A) - A \cdot (\nabla \times B)$

¶ $\nabla \cdot \nabla V = \nabla^2 V$

¶ $\nabla \times \nabla \times A = \nabla(\nabla \cdot A) - \nabla^2 A$

¶ $\nabla \times \nabla V = 0$

¶ $\nabla \cdot (\nabla \times A) = 0$

¶ $\int_V (\nabla \cdot A)\, dv = \oint_S A \cdot ds$ (Divergence theorem)

¶ $\int_S (\nabla \times A) \cdot ds = \oint_C A \cdot dl$ (Stokes's theorem)

Gradient, Divergence, Curl, and Laplacian Operations

Cartesian Coordinates (x, y, z)

1) $$\nabla V = a_x \frac{\partial V}{\partial x} + a_y \frac{\partial V}{\partial y} + a_z \frac{\partial V}{\partial z}$$

2) $$\nabla \cdot A = \frac{\partial A_x}{\partial x} + \frac{\partial A_y}{\partial y} + \frac{\partial A_z}{\partial z}$$

3)

$$\nabla \times A = \begin{vmatrix} a_x & a_y & a_z \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ A_x & A_y & A_z \end{vmatrix}$$

$$= a_x\left(\frac{\partial A_z}{\partial y} - \frac{\partial A_y}{\partial z}\right) + a_y\left(\frac{\partial A_x}{\partial z} - \frac{\partial A_z}{\partial x}\right) + a_z\left(\frac{\partial A_y}{\partial x} - \frac{\partial A_x}{\partial y}\right)$$

4) $$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2}$$

Cylindrical Coordinates (r,ϕ,z)

1) $\nabla V = a_r \frac{\partial V}{\partial r} + a_\phi \frac{\partial V}{r\partial \phi} + a_z \frac{\partial V}{\partial z}$

2) $\nabla \cdot A = \frac{1}{r}\frac{\partial}{\partial r}(rA_r) + \frac{\partial A_\phi}{r\partial \phi} + \frac{\partial A_z}{\partial z}$

3)

$$\nabla \times A = \frac{1}{r}\begin{vmatrix} a_r & a_\phi r & a_z \\ \frac{\partial}{\partial r} & \frac{\partial}{\partial \phi} & \frac{\partial}{\partial z} \\ A_r & rA_\phi & A_z \end{vmatrix}$$

$$= a_r\left(\frac{\partial A_z}{r\partial \phi} - \frac{\partial A_\phi}{\partial z}\right) + a_\phi\left(\frac{\partial A_r}{\partial z} - \frac{\partial A_z}{\partial r}\right) + a_z\frac{1}{r}\left[\frac{\partial}{\partial r}(rA_\phi) - \frac{\partial A_r}{\partial \phi}\right]$$

4) $\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2}$

Spherical Coordnates (R,Θ,ϕ)

1) $\nabla V = a_R \frac{\partial V}{\partial R} + a_\Theta \frac{\partial V}{R\partial \Theta} + a_\phi \frac{1}{R\sin\Theta}\frac{\partial V}{\partial \phi}$

2) $\nabla \cdot A = \frac{1}{R^2}\frac{\partial}{\partial R}(R^2 A_R) + \frac{1}{R\sin\Theta}\frac{\partial}{\partial \Theta}(A_\Theta \sin\Theta) + \frac{1}{R\sin\Theta}\frac{\partial A_\phi}{\partial \phi}$

3)

$$\nabla \times A = \frac{1}{R^2\sin\Theta}\begin{vmatrix} a_R & a_\Theta R & a_\phi(R\sin\Theta) \\ \frac{\partial}{\partial R} & \frac{\partial}{\partial \Theta} & \frac{\partial}{\partial \phi} \\ A_R & RA_\Theta & (R\sin\Theta)A_\phi \end{vmatrix}$$

$$= a_R\frac{1}{R\sin\Theta}\left[(\frac{\partial}{\partial \Theta}(A_\phi \sin\Theta) - \frac{\partial A_\Theta}{\partial \phi}\right]$$

$$+ a_\Theta\frac{1}{R}\left[\frac{1}{\sin\Theta}\frac{\partial A_R}{\partial \phi} - \frac{\partial}{\partial R}(RA_\phi)\right]$$

$$+ a_\phi\frac{1}{R}\left[\frac{\partial}{\partial R}(RA_\Theta) - \frac{\partial A_R}{\partial \Theta}\right]$$

4)

$$\nabla^2 V = \frac{1}{R^2}\frac{\partial}{\partial R}\left(R^2\frac{\partial V}{\partial R}\right) + \frac{1}{R^2\sin\Theta}\frac{\partial}{\partial \Theta}\left(\sin\Theta\frac{\partial V}{\partial \Theta}\right) + \frac{1}{R^2\sin^2\Theta}\frac{\partial^2 V}{\partial \phi^2}$$

Three Basic General Orthogonal Coordinate System			
	Base Vectors	Metric Coefficients	Differential Volume
	$\mathbf{a}_{u1}$ $\mathbf{a}_{u2}$ $\mathbf{a}_{u3}$	h_1 h_2 h_3	dv
Cartesian Coord. (x,y,z)	$\mathbf{a}_x$ $\mathbf{a}_y$ $\mathbf{a}_z$	1 1 1	$dxdydz$
Cylindrical Coord. $(r.\Phi,z)$	$\mathbf{a}_r$ $\mathbf{a}_\Phi$ $\mathbf{a}_z$	1 r 1	$rdrd\Phi dz$
Spherical Coord. (R,Θ,Φ)	$\mathbf{a}_R$ $\mathbf{a}_\Theta$ $\mathbf{a}_\Phi$	1 R $R\sin\Theta$	$R^2\sin\Theta dRd\Theta d\Phi$
A	$A = A_1\mathbf{u}_1 + A_2\mathbf{u}_2 + A_3\mathbf{u}_3$		
dl	$dl = (h_1du_1)\mathbf{u}_1 + (h_2du_2)\mathbf{u}_2 + (h_3du_3)\mathbf{u}_3$		
ds	$ds = h_2h_3du_2du_3\mathbf{n}_1$		
dv	$dv = h_1h_2h_3du_1du_2du_3$		
∇V	$\nabla V \equiv \left(a_{u1}\frac{\partial V}{h_1\partial u_1} + a_{u2}\frac{\partial V}{h_2\partial u_2} + a_{u3}\frac{\partial V}{h_3\partial u_3} \right)$		
$\nabla \cdot A$	$\nabla \cdot A = \frac{1}{h_1h_2h_3}\left[\frac{\partial}{\partial u_1}(h_1h_3A_1) + \frac{\partial}{\partial u_2}(h_1h_3A_2) + \frac{\partial}{\partial u_3}(h_1h_2A_3) \right]$		
$\nabla \times A$	$\nabla \times A = \frac{1}{h_1h_2h_3}\begin{vmatrix} a_{u1}h_1 & a_{u2}h_2 & a_{u3}h_3 \\ \frac{\partial}{\partial u_1} & \frac{\partial}{\partial u_2} & \frac{\partial}{\partial u_3} \\ h_1A_1 & h_2A_2 & h_3A_3 \end{vmatrix}$		