Compactness is the topological property that generalizes to topological spaces a property of closed, bounded intervals \([a, b]\) developed in Chapter 2.

Recall that the Heine-Borel Theorem (Theorem 2.41) showed that if \(\mathcal{O}\) is a collection of open intervals whose union contains \([a, b]\), then there is a finite subcollection of \(\mathcal{O}\) whose union contains \([a, b]\).

A topological space which has this property for every covering by open sets is called *compact*. 
• Compactness is a subtle property whose ramifications are not immediately apparent; do not let the term “compact” suggest simply smallness of size. The intervals [0, 1] and (0, 1) have the same size, but [0, 1] is compact and (0, 1) is not.

• One of the main results of this chapter is that a subset of $\mathbb{R}^n$ is compact if and only if it is closed and bounded.

• Historically, compactness was intended to generalize to topological spaces the properties which characterize the closed and bounded subsets of $\mathbb{R}^n$.

• Several different properties that will be introduced in this chapter were put forward with varying degrees of success until it was recognized that compactness is the desired property.

• The reader may already know from calculus that a continuous, real-valued function $f : [a, b] \to \mathbb{R}$ whose domain is a closed interval $[a, b]$ attains a maximum and a minimum value.
This result is proved in the present chapter as a corollary to the more general theorem that any continuous, real-valued function whose domain is a compact space attains a maximum and a minimum value.
1 Compact Spaces and Subspaces

Definition 1. Let $A$ be a subset of a topological space $X$.

(1) An open cover of $A$ is a collection $\mathcal{O}$ of open subsets of $X$ whose union contains $A$.

(2) A subcover of an open cover $\mathcal{O}$ of $A$ is a subcollection $\mathcal{O}'$ of $\mathcal{O}$, whose union contains $A$.

(3) $X$ is said to be compact if every open cover of $X$ has a finite subcover.

In other words, for any collection $\mathcal{O}$ of open subsets of $X$ whose union is $X$, there exists a finite subcollection $\{O_i | 1 \leq i \leq n\}$ of $\mathcal{O}$ such that $X = \bigcup_{i=1}^{n} O_i$.

(4) $A$ is said to be compact if $A$ is a compact topological space as a subspace of $X$. 
Example 2. Consider the subspace $A = [0, 5]$ of $\mathbb{R}$ and the open cover $\mathcal{O} = \{(n - 1, n + 1) \mid n \in \mathbb{Z}\}$ of $A$.

- The subcollection $\mathcal{O}' = \{(-1, 1), (0, 2), (1, 3), (2, 4), (3, 5), (4, 6)\}$ is a subcover of $\mathcal{O}$ and happens to be the smallest subcover for $A$ that can be derived from $\mathcal{O}$.

- There are many other subcovers for $A$, however, since it is only required that the union of the members of the subcover contain $A$.

Since relatively open sets in the subspace topology for a subset $A$ of a space $X$ are the intersections of $A$ with open sets in $X$, the definition of compactness for subspaces can be restated as follows:

**Alternative Definition.** A subset $A$ of a space $X$ is compact if and only if every open cover of $A$ by open subsets of $X$ has a finite subcover.
Theorem 2.41. (Heine-Borel Theorem)

\([a, b]\): a closed and bounded interval in \(\mathbb{R}\).

\(\mathcal{O}\): a collection of open intervals whose union contains \([a, b]\).

\(\Rightarrow\) There exists a finite subset \(\{O_1, O_2, \cdots, O_n\}\) of \(\mathcal{O}\) such that

\([a, b] \subset \bigcup_{i=1}^{n} O_i\).

Example 3. (1) Any finite topological space is compact.

(2) Any indiscrete space is compact.

(3) Every closed and bounded interval in \(\mathbb{R}\) is compact.

(4) A subset of \(\mathbb{R}^n\) is compact if and only if it is closed and bounded, \(n \in \mathbb{N}\). (This will be proved later in Theorem 28.)

(5) \(\mathbb{R}\) with the cofinite topology is compact.

(6) An infinite discrete space is not compact.
(7) The open interval $(0, 1)$ is not compact.

(8) $\mathbb{R}^n$ is not compact for each $n \in \mathbb{N}$.

**Definition 4.** A family $\mathcal{A}$ of subsets of a space $X$ is said to have the finite intersection property (f.i.p.) if every finite subcollection of $\mathcal{A}$ has non-empty intersection.

**Example 5.** $\{[0, \frac{1}{n}] \mid n \in \mathbb{N}\}$ has the finite intersection property.

The duality between unions of open sets $O_\alpha$ and intersections of the corresponding closed sets $C_\alpha = X \setminus O_\alpha$ in a space $X$,

$$X \setminus \left( \bigcap C_\alpha \right) = \bigcup (X \setminus C_\alpha) = \bigcup O_\alpha$$

produces the following characterization of compactness in terms of closed sets.
Theorem 6. A space $X$ is compact if and only if every family of closed subsets in $X$ with the finite intersection property has a non-empty intersection.

Proof. ($\Rightarrow$) Assume that $X$ is compact and let $\mathcal{C} = \{C_\alpha \mid \alpha \in I\}$ be a family of closed subsets of $X$ satisfying the f.i.p.

Suppose that $\mathcal{C}$ has the empty intersection, i.e., $\bigcap_{\alpha \in I} C_\alpha = \emptyset$.

Let $\mathcal{O} = \{X \setminus C_\alpha \mid \alpha \in I\}$.

$\Rightarrow$ Since $\bigcup_{\alpha \in I} (X \setminus C_\alpha) = X \setminus \bigcap_{\alpha \in I} C_\alpha = X \setminus \emptyset = X$,

$\mathcal{O}$ is an open cover of $X$.

$\Rightarrow$ Since $X$ is compact, $\mathcal{O}$ has a finite subcover $\mathcal{O}' = \{X \setminus C_{\alpha_i} \mid i = 1, 2, \ldots, n, \alpha_i \in I\}$, i.e., $X = \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i})$. 
\[ X = \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}) = X \setminus \bigcap_{i=1}^{n} C_{\alpha_i}. \]

\[ \bigcap_{i=1}^{n} C_{\alpha_i} = \emptyset \text{ and } C_{\alpha_i} \in \mathcal{C}. \]

\[ \Rightarrow \mathcal{C} \text{ does not have the finite intersection property. A contradiction!} \]

\[ \Rightarrow \mathcal{C} \text{ must have the non-empty empty intersection, i.e., } \bigcap_{\alpha \in I} C_{\alpha} \neq \emptyset. \]

(\(\Leftarrow\)) Assume that every family of closed subsets in \(X\) with the finite intersection property has a non-empty intersection and let

\[ \mathcal{O} = \{O_{\alpha} \mid \alpha \in I\} \text{ be an open cover of } X. \]

Let \(\mathcal{C} = \{C_{\alpha} = X \setminus O_{\alpha} \mid \alpha \in I\}. \)

\[ \Rightarrow \text{Since } \bigcap_{\alpha \in I} C_{\alpha} = \bigcap_{\alpha \in I} (X \setminus O_{\alpha}) = X \setminus \bigcup_{\alpha \in I} O_{\alpha} = X \setminus X = \emptyset, \]

\(\mathcal{C}\) has the empty intersection.

\[ \Rightarrow \text{By assumption, } \mathcal{C} \text{ does not satisfy the f.i.p.} \]
⇒ There exist $C_{\alpha_1}, C_{\alpha_2}, \cdots, C_{\alpha_n} \in \mathcal{C}$ such that $\bigcap_{i=1}^{n} C_{\alpha_i} = \emptyset$.

$\Rightarrow X \setminus \bigcup_{i=1}^{n} O_{\alpha_i} = X \setminus \bigcup_{i=1}^{n} (X \setminus C_{\alpha_i}) = X \setminus (X \setminus \bigcap_{i=1}^{n} C_{\alpha_i}) = \bigcap_{i=1}^{n} C_{\alpha_i} = \emptyset$.

$\Rightarrow X = \bigcup_{i=1}^{n} O_{\alpha_i}$, i.e., $\emptyset$ has a finite subcover $\{O_{\alpha_i} \mid i = 1, 2, \cdots, n\}$.

$\Rightarrow X$ is compact. \qed

Remark 7. Let $X$ be a topological space. Then TFAE.

(1) $X$ is compact. i.e., every open cover of $X$ has a finite subcover.

(2) Let $U_\alpha$ be an open subset of $X$ for each $\alpha \in \mathcal{A}$. If $X \subset \bigcup_{\alpha \in \mathcal{A}} U_\alpha$,

then there exist $\alpha_1, \alpha_2, \cdots, \alpha_n \in \mathcal{A}$ such that $X \subset \bigcup_{i=1}^{n} U_{\alpha_i}$.

(3) Let $U_\alpha$ be an open subset of $X$ for each $\alpha \in \mathcal{A}$. If $X = \bigcup_{\alpha \in \mathcal{A}} U_\alpha$, 
then there exist $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{A}$ such that $X = \bigcup_{i=1}^{n} U_{\alpha_i}$.

(4) Let $U_{\alpha}$ be an open subset of $X$ for each $\alpha \in \mathcal{A}$. If $X \neq \bigcup_{i=1}^{n} U_{\alpha_i}$ for any finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{A}$, then $X \neq \bigcup_{\alpha \in \mathcal{A}} U_{\alpha}$.

(5) Let $C_{\alpha}$ be a closed subset of $X$ for each $\alpha \in \mathcal{A}$. If $\emptyset \neq \bigcap_{i=1}^{n} C_{\alpha_i}$ for any finitely many $\alpha_1, \alpha_2, \ldots, \alpha_n \in \mathcal{A}$, then $\emptyset \neq \bigcap_{\alpha \in \mathcal{A}} C_{\alpha}$.

(6) Let $\mathcal{C}$ be a family of closed subsets of $X$. If $\mathcal{C}$ has the finite intersection property, then $\bigcap_{C \in \mathcal{C}} C \neq \emptyset$.

(7) Every family of closed subsets of $X$ having the finite intersection property has non-empty intersection.
Theorem 8 (Cantor’s Theorem for Deduction). If \( \{E_n \mid n \in \mathbb{N}\} \) is a nested sequence of nonempty, closed and bounded subsets of \( \mathbb{R} \), then \( \bigcap_{n=1}^{\infty} E_n \) is not empty.

Proof. Let \( \{E_n \mid n \in \mathbb{N}\} \) be a nested sequence of nonempty, closed and bounded subsets of \( \mathbb{R} \). i.e., \( E_1 \supset E_2 \supset \cdots \supset E_n \supset E_{n+1} \cdots \),

\( E_n \) is a nonempty, closed and bounded subset of \( \mathbb{R} \), for all \( n \in \mathbb{N} \).

\( \Rightarrow \) Since \( E_1 \) is bounded, \( E_1 \subset [a, b] \) for some \( a < b \) in \( \mathbb{R} \).

\( \Rightarrow \) Since each \( E_n \) is closed in \( \mathbb{R} \) and \( E_n \subset E_1 \subset [a, b] \),

\( E_n \) is closed in \( [a, b] \) for the subspace topology.

\( \Rightarrow \) \( \{E_n \mid n \in \mathbb{N}\} \) is a collection of closed subsets of \( [a, b] \).

\( \Rightarrow \) Since \( \{E_n \mid n \in \mathbb{N}\} \) is nested and \( E_n \neq \emptyset \) for all \( n \in \mathbb{N} \),

\( \{E_n \mid n \in \mathbb{N}\} \) has the finite intersection property.
⇒ Since $[a, b]$ is compact, $\bigcap_{n=1}^{\infty} E_n \neq \emptyset$ by Theorem 6. □

**Corollary 9** (Cantor’s Nested Interval Theorem).

1. If \{[$a_n, b_n]$ | $n \in \mathbb{N}$\} is a nested sequence of closed and bounded intervals, then $\bigcap_{n=1}^{\infty} [a_n, b_n] \neq \emptyset$.

2. Furthermore if in addition $\lim_{n \to \infty} (b_n - a_n) = 0$, then $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is a singleton set.
Example 10. The requirement that $E_n$ of Cantor’s Theorem of Deduction be “bounded” and “closed” cannot be removed.

(1) $\{[n, \infty) \mid n \in \mathbb{N}\}$ is a nested sequence of nonempty closed subsets of $\mathbb{R}$ but $\bigcap_{n=1}^{\infty} [n, \infty) = \emptyset$. (boundedness is necessary!)

(2) $\{(0, \frac{1}{n}) \mid n \in \mathbb{N}\}$ is a nested sequence of nonempty bounded subsets of $\mathbb{R}$ but $\bigcap_{n=1}^{\infty} (0, \frac{1}{n}) = \emptyset$. (closedness is necessary!)
Theorem 11. Every closed subset of a compact space is compact.

Proof. Let \( X \) be a compact space and let \( A \) be a closed subset of \( X \).

Let \( \mathcal{O} \) be an open cover of \( A \) by open sets in \( X \).

\( \Rightarrow \) Since \( A \) is closed in \( X \), \( X \setminus A \) is open in \( X \). Let \( \mathcal{O}^* = \mathcal{O} \cup \{X \setminus A\} \).

\( \Rightarrow \) \( \mathcal{O}^* \) is an open cover of \( X \).

\( \Rightarrow \) Since \( X \) is compact, \( \mathcal{O}^* \) has a finite subcover \( \{O_1, O_2, \cdots, O_n\} \)

which may or may not contain \( X \setminus A \).

\( \Rightarrow \) Since \( X = (X \setminus A) \cup A = (X \setminus A) \cup (\bigcup_{i=1}^{n} O_i), A \subset \bigcup_{i=1}^{n} O_i, \)

(disregard \( X \setminus A \) if \( \{O_1, O_2, \cdots, O_n\} \) contains it)

\( \Rightarrow \) \( \{O_1, O_2, \cdots, O_n\} \) is a finite subcover of \( \mathcal{O} \) for \( A \).

\( \Rightarrow \) \( A \) is compact. \( \square \)
Theorem 12. Every compact subset of a Hausdorff space is closed.

Proof. Let $X$ be a Hausdorff space and let $A$ be a compact subset of $X$. To show that $X \setminus A$ is open in $X$, let $x \in X \setminus A$.

\[ \Rightarrow \] For each $y \in A$, since $X$ is Hausdorff, there exist disjoint open sets $U_y$ and $V_y$ in $X$ such that $y \in U_y$ and $x \in V_y$.

\[ \Rightarrow \{U_y \mid y \in A\} \text{ is an open cover of } A. \]

\[ \Rightarrow \] Since $A$ is compact, there exist $y_i \in A, (i = 1, 2, \ldots, n)$ such that $A \subset \bigcup_{i=1}^{n} U_{y_i}$. Let $U = \bigcup_{i=1}^{n} U_{y_i}$ and $V = \bigcap_{i=1}^{n} V_{y_i}$.

\[ \Rightarrow U \text{ and } V \text{ are disjoint open sets in } X, A \subset U \text{ and } x \in V. \]

i.e., $x \in V \subset X \setminus U \subset X \setminus A$.

\[ \Rightarrow X \setminus A \text{ is open in } X \text{ and } A \text{ is closed in } X. \]
Corollary 13. Let $X$ be a compact Hausdorff space and let $A \subset X$. Then $A$ is compact if and only if it is closed in $X$.

Proof. (only if) part follows from Theorem 12 and (if) part follows from Theorem 11. \qed


$A, B$: disjoint compact subsets of $X$.

$\Rightarrow$ There exist disjoint open sets $U$ and $V$ in $X$ such that $A \subset U$ and $B \subset V$.

Proof. Let $X$ be a Hausdorff space and let $A, B$ be disjoint compact subsets of $X$.

$\Rightarrow$ For each $x \in B$, there exist disjoint open sets $U_x$ and $V_x$ in $X$ such that $A \subset U_x$ and $x \in V_x$. 

(\therefore) \quad \text{Fix } x \in B \text{ and let } y \in A.

\Rightarrow \quad \text{Since } X \text{ is Hausdorff, } \exists \text{ disjoint open sets } U_y, V_y \text{ in } X
\quad \text{such that } y \in U_y \text{ and } x \in V_y.

\Rightarrow \quad \text{Since } \{U_y \mid y \in A\} \text{ is an open cover of } A \text{ and } A \text{ is compact,}
\quad \text{there exist } y_1, \ldots, y_n \in A \text{ such that } A \subset \bigcup_{i=1}^n U_{y_i}.

\text{Let } U_x = \bigcup_{i=1}^n U_{y_i} \text{ and } V_x = \bigcap_{i=1}^n V_{y_i}.

\Rightarrow \quad U_x \text{ and } V_x \text{ are disjoint open sets in } X \text{ such that }
\quad A \subset U_x \text{ and } x \in V_x.

\Rightarrow \quad \{V_x \mid x \in B\} \text{ is an open cover of } B.

\Rightarrow \quad \text{Since } B \text{ is compact, there exist } x_1, \ldots, x_m \in B \text{ such that }
\quad B \subset \bigcup_{j=1}^m V_{x_j}.
Let \( U = \bigcap_{j=1}^{m} U_{x_j} \) and \( V = \bigcup_{j=1}^{m} V_{x_j} \).

\[ \Rightarrow U \text{ and } V \text{ are disjoint open sets in } X \implies A \subset U \text{ and } B \subset V. \]

**Corollary 15.** \( X: \) a compact Hausdorff space.

\( A, B: \) disjoint closed subsets of \( X \).

\[ \Rightarrow \exists \text{ disjoint open sets } U, V \text{ in } X \text{ such that } A \subset U \text{ and } B \subset V. \]

**Proof.** Let \( X \) be a compact Hausdorff space and let \( A, B \) be disjoint closed subsets of \( X \).

\[ \Rightarrow A \text{ and } B \text{ are disjoint compact subsets of } X \text{ by Theorem 12.} \]

\[ \Rightarrow \text{ There exist disjoint open sets } U, V \text{ in } X \text{ such that } A \subset U \text{ and } B \subset V \text{ by Theorem 14.} \]

**Corollary 16.** Every compact Hausdorff space is normal.
Exercises 1

1. In Example 2, explain the sense in which the open subcover \( \mathcal{O}' = \{(-1, 1), (0, 2), (1, 3), (2, 4), (3, 5), (4, 6)\} \) is the smallest subcollection of \( \mathcal{O} = \{(n - 1, n + 1) \mid n \in \mathbb{Z}\} \) which contains \( A = [0, 5] \).

   \[ \text{Proof.} \] Let \( \mathcal{O}'' \) be a subcover of \( \mathcal{O} \) for \( A \).

   \[ \Rightarrow \] Since \((n - 1, n + 1)\) is the only member of \( \mathcal{O} \) containing \( n \) for each \( n \in \mathbb{Z} \),

   and since \( A \cap \mathbb{Z} = \{0, 1, 2, 3, 4, 5\} \),

   \( (-1, 1), (0, 2), (1, 3), (2, 4), (3, 5), (4, 6) \in \mathcal{O}'' \).

   \[ \Rightarrow \mathcal{O}' \subset \mathcal{O}'' . \]

   \[ \Rightarrow \mathcal{O}' \text{ is the smallest subcollection of } \mathcal{O} \text{ containing } A. \] \( \square \)
2. Give examples of each of the following.

(1) A closed subspace that is not compact.

(2) A compact subspace that is not closed.

(3) An open compact subspace.

(4) Two compact subsets whose intersection is not compact.

Proof. (1) \([0, \infty)\) is a closed subset of \(\mathbb{R}\) which is not compact because 
\[\{(-1, n) \mid n \in \mathbb{N}\}\] is an open cover of \([0, \infty)\) by open sets in \(\mathbb{R}\) which has 
no finite subcover for \([0, \infty)\).

(Another example) Let \(X = \mathbb{R} \cup \{(0, 1)\} \subset \mathbb{R}^2\) with the topology \(\mathcal{T}\) defined
\[\mathcal{T} = \{\emptyset\} \cup \{U \subset X \mid (0, 1) \in U\}.\]

\([i] X \setminus \{(0, 1)\}\) is closed in \(X\) because \(\{(0, 1)\}\) is open in \(X\).

\([ii] X \setminus \{(0, 1)\}\) is not compact because \(\{\{x, (0, 1)\} \mid x \in \mathbb{R}\}\) is an 
open cover of \(X \setminus \{(0, 1)\}\) which has no finite subcover.
(2) Consider $X = \mathbb{R} \cup \{(0, 1)\} \subset \mathbb{R}^2$ with the topology $\mathcal{T}$ defined by

$$\mathcal{T} = \{\emptyset\} \cup \{U \subset X \mid (0, 1) \in U\}.$$  

$\Rightarrow \{(0, 1)\}$ is a compact subset of $X$ because it is finite, but $\{(0, 1)\}$ is not closed because $X \setminus \{(0, 1)\}$ is not open in $X$.

(3) In a discrete space $X$, any finite subset of $X$ is an open compact subspace.

(4) Let $X = \{-\infty\} \cup \mathbb{Z} \cup \{\infty\}$ with $-\infty, \infty \notin \mathbb{Z}$ and give a topology on $X$ by taking a basis $\mathcal{B}$ as follows;

(i) all points of $\mathbb{Z}$ are isolated,

(ii) a basic neighbourhood of $-\infty$ is $\{-\infty\} \cup (\mathbb{Z} \setminus F)$ ($F \subset \mathbb{Z}$ finite) and

(iii) a basic neighbourhood of $\infty$ is $\{\infty\} \cup (\mathbb{Z} \setminus G)$ ($G \subset \mathbb{Z}$ finite).

i.e., $\mathcal{B} = \{\{z\} \mid z \in \mathbb{Z}\}$

$\cup\{U \subset X \mid -\infty \in U, \infty \notin U, \mathbb{Z} \setminus U \text{ is finite subset of } \mathbb{Z}\}$

$\cup\{V \subset X \mid \infty \in V, -\infty \notin V, \mathbb{Z} \setminus V \text{ is finite subset of } \mathbb{Z}\}$.

$\Rightarrow \mathcal{B}$ satisfies the conditions in Theorem 4.30;
(B1) \( X = \bigcup_{B \in \mathcal{B}} B. \)

(B2) For each \( B_1, B_2 \in \mathcal{B} \) and \( x \in B_1 \cap B_2 \), there exists a member \( B_x \in \mathcal{B} \) such that \( x \in B_x \subset B_1 \cap B_2 \).

\( \Rightarrow \) \( \mathcal{B} \) is a basis for a topology \( \mathcal{T} \) of \( X \).

Let \( A = \mathbb{N} \cup \{ -\infty \} \) and \( B = \mathbb{N} \cup \{ \infty \} \) be two subsets of \( X \).

\( \Rightarrow \) Under the topology \( \mathcal{T} \) of \( X \),

(a) \( X \) is compact,

(\( \because \)) Let \( \{ U_\alpha \mid \alpha \in \mathcal{A} \} \) be any cover of \( X \).

\( \Rightarrow \) There exist \( \alpha, \beta \in \mathcal{A} \) such that \( -\infty \in U_\alpha \) and \( \infty \in U_\beta \).

\( \Rightarrow \) There exist \( U, V \subset X \) and finite subsets \( F, G \subset \mathbb{Z} \) such that

\( -\infty \in U \subset U_\alpha, \infty \notin U, \mathbb{Z} \setminus U = F \) and

\( \infty \in V \subset U_\beta, -\infty \notin V, \mathbb{Z} \setminus V = G. \)

\( \Rightarrow \) \((X \setminus F) \cup (X \setminus G) \subset U \cup V \subset U_\alpha \cup U_\beta \) and since \( F \cup G \) is a finite subset of \( \mathbb{Z} \), \( \{ U_\alpha \mid \alpha \in \mathcal{A} \} \) has a finite subcover for \( X \).
⇒ \( X \) is compact.

(b) \( A \) and \( B \) are compact,

\((\because)\) Similar to that of (a).

(c) \( A \cap B = \mathbb{Z} \) is not compact.

\((\because)\) \( \mathbb{Z} \) is an infinite discrete subspace of \( X \).

(Note that \( X \) is not Hausdorff because the points \(-\infty \) and \( \infty \) have no disjoint open neighborhoods and that \( A \) and \( B \) are not closed subsets of \( X \).)

3. Prove the following.

(1) The union of a finite number of compact subsets of a space \( X \) is compact.

(2) If \( X \) is Hausdorff, then the intersection of any family of compact subspaces is compact.
Proof. (1) Let $X$ be a topological space and let $A_i, i = 1, 2, \cdots, n$ be compact subsets of $X$. Let $A = \bigcup_{i=1}^{n} A_i$ and let $\mathcal{O} = \{U_{\alpha} \mid \alpha \in \mathcal{A}\}$ be an open cover of $A$ by open sets $U_\alpha$ in $X$.

$\Rightarrow \mathcal{O}$ is an open cover of $A_i$ for each $i = 1, 2, \cdots, n$.

$\Rightarrow$ Since $A_i$ is compact for each $i = 1, 2, \cdots, n$, there exist $i_1, i_2, \cdots, i_n \in \mathcal{A}$ such that $A_i \subset \bigcup_{j=1}^{n_i} U_{i_j}$.

$\Rightarrow A = \bigcup_{i=1}^{n} A_i \subset \bigcup_{i=1}^{n} \bigcup_{j=1}^{n_i} U_{i_j}$ and $\{U_{i_j} \mid i = 1, 2, \cdots, n, j = 1, 2, \cdots, n_i\}$ is finite.

$\Rightarrow \{U_{i_j} \mid i = 1, 2, \cdots, n, j = 1, 2, \cdots, n_i\}$ is a finite subcover of $\mathcal{O}$ for $A$.

$\Rightarrow A$ is compact.
(2) Assume that $X$ is a Hausdorff space and let $A_\alpha, \alpha \in \mathcal{A}$ be a family of compact subsets $A_\alpha$ of $X$. Let $A = \bigcap_{\alpha \in \mathcal{A}} A_\alpha$.

$\Rightarrow$ Since $X$ is Hausdorff, $A_\alpha$ is closed in $X$ for each $\alpha \in \mathcal{A}$ by Theorem 12.

$\Rightarrow \bigcap_{\alpha \in \mathcal{A}} A_\alpha$ is closed in $X$.

$\Rightarrow$ Since \( \bigcap_{\alpha \in \mathcal{A}} A_\alpha \) is a closed subset of a compact set $A_\alpha$ for any $\alpha \in \mathcal{A}$, \( \bigcap_{\alpha \in \mathcal{A}} A_\alpha \) is compact by Theorem 11. \qed
4. Prove that a space $X$ is compact if and only if $X$ has a basis $\mathcal{B}$ for which every open cover of $X$ by members of $\mathcal{B}$ has a finite subcover.

Proof. (only if) It is obvious.

(if) Assume that $X$ has a basis $\mathcal{B}$ for which every open cover of $X$ by members of $\mathcal{B}$ has a finite subcover. Let $\mathcal{O}$ be an open cover of $X$.

⇒ For each $x \in X$, there exists $U_x \in \mathcal{O}$ such that $x \in U_x$.
⇒ Since $\mathcal{B}$ is a basis for $X$, there exists $B_x \in \mathcal{B}$ such that $x \in B_x \subset U_x$.
⇒ $\{B_x \mid x \in X\}$ is an open cover of $X$ by members of $\mathcal{B}$.
⇒ By assumption, there exist $x_1, x_2, \cdots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} B_{x_i}.$$ 

⇒ $X = \bigcup_{i=1}^{n} U_{x_i}$.
⇒ $\{U_{x_i} \mid i = 1, 2, \cdots, n\}$ is a finite subcover of $\mathcal{O}$ for $X$.
⇒ $X$ is compact. \qed
5. Show that the real line with the countable complement topology is not compact.

Proof. Let $X$ denote the real line with the countable complement topology. Let $A = \mathbb{R} \setminus \mathbb{N}$ and let $\mathcal{O} = \{A \cup \{n\} \mid n \in \mathbb{N}\}$.

$\Rightarrow$ (a) $\mathcal{O}$ is an open cover of $X$.

(∵) (i) Since $\mathbb{R} \setminus (A \cup \{n\}) = \mathbb{N} \setminus \{n\}$ is countable, $A \cup \{n\}$ is open in $X$.

(ii) $\bigcup_{n=1}^{\infty} (A \cup \{n\}) = X$.

(b) $\mathcal{O}$ has no finite subcover for $X$.

(∵) For each $n \in \mathbb{N}, A \cup \{n\}$ is the only member of $\mathcal{O}$ containing $n$. $\Rightarrow X$ is not compact. \hfill $\square$
2 Compactness and Continuity

Compactness is of importance in topology largely because of its relationships with continuity.

Some of these relationships are examined in this section.

**Theorem 17.** $X$: a compact space, $Y$: a topological space.

$f : X \rightarrow Y$ is a continuous surjective function.

$\Rightarrow Y$ is compact.

*Proof.* Let $\mathcal{O}$ be an open cover of $Y$.

$\Rightarrow$ Since $f : X \rightarrow Y$ is continuous, $f^{-1}(O)$ is open in $X$ for all $O \in \mathcal{O}$.

$\Rightarrow \mathcal{O}^* = \{f^{-1}(O) \mid O \in \mathcal{O}\}$ is an open cover of $X$.

$\Rightarrow$ Since $X$ is compact, $\mathcal{O}^*$ has a finite subcover

$\{f^{-1}(O_i) \mid O_i \in \mathcal{O}, i = 1, 2, \cdots, n\}$ for $X$. 


\[ \Rightarrow \text{Since } X = \bigcup_{i=1}^{n} f^{-1}(O_i) \text{ and } f : X \to Y \text{ is surjective,} \]

\[ Y = f(X) = f\left( \bigcup_{i=1}^{n} f^{-1}(O_i) \right) = \bigcup_{i=1}^{n} f(f^{-1}(O_i)) \subset \bigcup_{i=1}^{n} O_i. \]

\[ \Rightarrow \{ O_i \in \mathcal{O} \mid i = 1, 2, \ldots, n \} \text{ is a finite subcover of } \mathcal{O} \text{ for } Y. \]

\[ \Rightarrow Y \text{ is compact.} \]

In the language of invariants, Theorem 17 simply says that compactness is a continuous invariant.

**Corollary 18.** (1) Compactness is a continuous invariant. i.e., every continuous image of a compact space is compact.

(2) Compactness is a topological invariant. i.e., every homeomorphic image of a compact space is compact.
Theorem 19. \( X \): a compact space, \( Y \): Hausdorff.

\[ f : X \rightarrow Y \text{ a continuous function.} \]

\( \Rightarrow \) \( f : X \rightarrow Y \) is a closed function.

Proof. Let \( f : X \rightarrow Y \) be a continuous function, where \( X \) is compact and \( Y \) is Hausdorff and let \( C \) be a closed subset of \( X \).

\( \Rightarrow \) Since \( X \) is compact, \( C \) is compact by Theorem 11.

\( \Rightarrow \) Since \( f : X \rightarrow Y \) is continuous, \( f(C) \) is compact by Corollary 18.

\( \Rightarrow \) Since \( Y \) is Hausdorff, \( f(C) \) is closed in \( Y \) by Theorem 12.

\( \Rightarrow \) \( f : X \rightarrow Y \) is a closed function. \( \square \)

Theorem 20. \( X \): a compact space, \( Y \): Hausdorff.

\[ f : X \rightarrow Y \text{ a continuous bijective function.} \]

\( \Rightarrow \) \( f : X \rightarrow Y \) is a homeomorphism.
Proof. Let $X$ be a compact space and let $Y$ be a Hausdorff space.

Let $f : X \to Y$ be a continuous bijective function.

$\Rightarrow f : X \to Y$ is a closed function by Theorem 19.

Thus $f : X \to Y$ is a continuous, closed and bijective function.

$\Rightarrow f : X \to Y$ is a homeomorphism by Proposition 4.44. $\square$

Example 21 (Continuity of inverse functions). Theorem 20 would be of great value in elementary calculus if its proof were accessible at that level. Many laborious proofs of continuity could be avoided.
(1) The function $g : \mathbb{R}^+ \to \mathbb{R}^+$ defined by $g(x) = \sqrt{x}$ for all $x \in \mathbb{R}^+$ is continuous, where $\mathbb{R}^+ = \{x \in \mathbb{R} \mid x \geq 0\}$. 

\[ \therefore \text{Note that the function } f : \mathbb{R}^+ \to \mathbb{R}^+ \text{ via } f(x) = x^2, \]

for all $x \in \mathbb{R}^+$, is a continuous, bijective function and $g = f^{-1}$. Let $a \in \mathbb{R}^+$ and choose $b \in \mathbb{R}^+$ with $a < b$. 

\[ \Rightarrow \text{Since } f|_{[0, \sqrt{b}]} : [0, \sqrt{b}] \to [0, b] \text{ is a continuous bijective function from a compact space to a Hausforff space,} \]

\[ f|_{[0, \sqrt{b}]} : [0, \sqrt{b}] \to [0, b] \text{ is a homeomorphism.} \]

\[ \Rightarrow (f|_{[0, \sqrt{b}]})^{-1} = g|_{[0, b]} : [0, b] \to [0, \sqrt{b}] \text{ is continuous at } a. \]

\[ \Rightarrow g : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous at } a. \]

\[ \Rightarrow \text{Since } a \text{ was chosen arbitrarily, } g : \mathbb{R}^+ \to \mathbb{R}^+ \text{ is continuous.} \]
(2) $\sin^{-1} : [-1, 1] \to [-\frac{\pi}{2}, \frac{\pi}{2}]$ is continuous.
\[\left(\because\right) \sin : [-\frac{\pi}{2}, \frac{\pi}{2}] \to [-1, 1] \text{ is a continuous bijective function.}\]

(3) Define $f : [0, 1) \to S^1$ by $f(x) = (\cos 2\pi x, \sin 2\pi x)$ for $0 \leq x < 1$. Then $f : [0, 1) \to S^1$ is a continuous bijective function. But its inverse $f^{-1} : S^1 \to [0, 1)$ is not continuous.

Note that $f$ maps 0 to the point $(1, 0)$ and wraps the interval $[0, 1)$ around $S^1$ in the counterclockwise direction. In Figure 1, the sequence $\{y_n\}_{n=1}^{\infty}$ in $Y$ converges to $(1, 0)$, but the corresponding sequence $\{x_n\}_{n=1}^{\infty}, x_n = f^{-1}(y_n)$, does not converge to $f^{-1}((1, 0)) = 0$ in $X$. 
The reader should know from calculus the following maximum and minimum value theorem:

Figure 1: The map $f : [0, 1) \to S^1$
**Theorem 22** (Maximum and Minimum Value Theorem for $\mathbb{R}$).

$f : [a, b] \to \mathbb{R}$ is a continuous function.

$\Rightarrow \exists \ c, d \in [a, b]$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in [a, b]$.

*Proof.* Theorem 22 is a special case of the following Theorem 23. \qed

**Theorem 23** (Maximum and Minimum Value Theorem).

$X$ : a compact space, $f : X \to \mathbb{R}$ a continuous function.

$\Rightarrow \exists \ c, d \in X$ such that $f(c) \leq f(x) \leq f(d)$ for all $x \in X$.

*Proof.* Since $X$ is compact and $f : X \to \mathbb{R}$ is continuous, $f(X)$ is a compact subset of $\mathbb{R}$ by Corollary 18.

$\Rightarrow$ Since $\{(-n, n) \mid n \in \mathbb{N}\}$ is an open over of $f(X)$, $\exists \ n_1, \cdots, n_k \in \mathbb{N}$ such that $f(X) \subset \bigcup_{i=1}^{k} (-n_i, n_i)$. Let $m = \max\{n_1, \cdots, n_k\}$. 

36
\( \Rightarrow \) (1) Since \( f(X) \subset (-m, m) \), \( f(X) \) is bounded.

(2) Since \( \mathbb{R} \) is Hausdorff, \( f(X) \) is closed in \( \mathbb{R} \) by Theorem 12.

\( \Rightarrow \) \( f(X) \) is a closed and bounded subset of \( \mathbb{R} \).

\( \Rightarrow \) Since \( f(X) \) is bounded in \( \mathbb{R} \), there exist \( u = \) the least upper bound of \( f(X) \) by the Least Upper Bound Property of \( \mathbb{R} \) and \( l = \) the greatest lower bound of \( f(X) \) by Theorem 2.6.

\( \Rightarrow \) Since \( f(X) \) is closed, \( u, l \in f(X) \) by Theorem 4.7.

\( \Rightarrow \) There exists \( c, d \in X \) such that \( f(c) = l \) and \( f(d) = u \).

\( \Rightarrow \) \( f(c) \leq f(x) \leq f(d) \), for all \( x \in X \).

\( \square \)

Theorem 22 can be rewritten as in the following

**Corollary 24.** Every continuous function \( f : [a, b] \to \mathbb{R} \) has a maximum value and a minimum value.
Definition 25. Let \((X, d)\) and \((Y, d')\) be metric spaces. A function \(f : X \rightarrow Y\) is said to be uniformly continuous if \(\forall \epsilon > 0, \exists \delta > 0 \quad \Rightarrow \quad \) if \(x_1, x_2 \in X\) and \(d(x_1, x_2) < \delta\), then \(d'(f(x_1), f(x_2)) < \epsilon\).

Note that every uniformly continuous function is continuous.

Theorem 26. \((X, d)\): a compact metric space, \((Y, d')\): a metric sp.

\(f : (X, d) \rightarrow (Y, d')\) a continuous function.

\(\Rightarrow f : (X, d) \rightarrow (Y, d')\) is uniformly continuous.

Proof. Let \(f : (X, d) \rightarrow (Y, d')\) be continuous and let \(\epsilon > 0\) be given.

\(\Rightarrow\) Since \(f : (X, d) \rightarrow (Y, d')\) is continuous at each \(x \in X\),

\[\exists \delta_x > 0 \quad \Rightarrow \quad \forall y \in X \text{ with } d(x, y) < \delta_x, d'(f(x), f(y)) < \frac{1}{2} \epsilon.\]

\(\Rightarrow\) Since \(\{B_d(x, \frac{1}{2} \delta_x) \mid x \in X\}\) is an open cover of \(X\) and \(X\) is
compact, \( \exists x_1, x_2, \ldots, x_n \in X \implies X = \bigcup_{i=1}^{n} B_d(x_i, \frac{1}{2}\delta_{x_i}). \)

Let \( \delta = \min\{\frac{1}{2}\delta_{x_1}, \frac{1}{2}\delta_{x_2}, \ldots, \frac{1}{2}\delta_{x_n}\}. \)

\( \Rightarrow \) \( \delta > 0. \) Now let \( x, y \in X \) with \( d(x, y) < \delta. \)

\( \Rightarrow \) Since \( X = \bigcup_{i=1}^{n} B_d(x_i, \frac{1}{2}\delta_{x_i}), x \in B_d(x_j, \frac{1}{2}\delta_{x_j}) \) for some \( 1 \leq j \leq n. \)

\( \Rightarrow d(x_j, y) \leq d(x_j, x) + d(x, y) < \frac{1}{2}\delta_{x_j} + \delta \leq \delta_{x_j}, \) and

\[ d(x_j, x) \leq \frac{1}{2}\delta_{x_j} < \delta_{x_j}. \]

\( \Rightarrow d'(f(x_j), f(y)) < \frac{1}{2}\epsilon \) and \( d'(f(x_j), f(x)) < \frac{1}{2}\epsilon. \)

\( \Rightarrow d'(f(x), f(y)) < d'(f(x), f(x_j)) + d'(f(x_j), f(y)) < \frac{1}{2}\epsilon + \frac{1}{2}\epsilon = \epsilon. \)

\( \Rightarrow f : (X, d) \rightarrow (Y, d') \) is uniformly continuous.
Figure 2: $B(x_j, \delta_{x_j})$
Exercises 2

1. Prove Corollary 18.

**Corollary 18.**

(1) Compactness is a continuous invariant. i.e., every continuous image of a compact space is compact.

(2) Compactness is a topological invariant. i.e., every homeomorphic image of a compact space is compact.

**Proof.** (1) Let $f : X \to Y$ be a continuous function.

Suppose that $X$ is compact.

$\Rightarrow g : X \to f(X)$ defined by $g(x) = f(x)$ for all $x \in X$, is continuous and surjective.

$\Rightarrow f(X)$ is compact by Theorem 17.

(2) follows from (1) above directly. $\square$
2. Let \( \arctan x \) denote the inverse of the function \( \tan : \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R} \).
Assume that \( \tan x \) is continuous and prove that \( \arctan x \) is continuous.

\textit{Proof.} Let \( y \in \mathbb{R} \) and let \( -\frac{\pi}{2} < x < \frac{\pi}{2} \) such that \( \tan x = y \).

Choose \( a, b, c, d \) such that \( -\frac{\pi}{2} < a < x < b < \frac{\pi}{2}, \ c < y < d \) and
\( \tan a = c, \tan b = d \).

\( \Rightarrow \) Since \( \tan : \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \to \mathbb{R} \) is continuous, increasing and bijective,
\( \tan \mid_{[a,b]} : [a, b] \to [c, d] \) is continuous and bijective.

\( \Rightarrow \) \( \tan \mid_{[a,b]} : [a, b] \to [c, d] \) is a homeomorphism by Theorem 20.

\( \Rightarrow \) \( \arctan \mid_{[c,d]} : [c, d] \to [a, b] \) is a homeomorphism.

\( \Rightarrow \) \( \arctan : \mathbb{R} \to \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) is continuous at \( y \).

\( \Rightarrow \) Since \( y \) was chosen arbitrarily in \( \mathbb{R} \), \( \arctan : \mathbb{R} \to \left( -\frac{\pi}{2}, \frac{\pi}{2} \right) \) is
continuous on \( \mathbb{R} \). \( \square \)
3. **Definition.** A function \( f : \mathbb{R} \to \mathbb{R} \) is *strictly increasing function* provided that for all \( x, y \in \mathbb{R} \) with \( x < y \), \( f(x) < f(y) \).

Prove the following.

(1) Every strictly increasing function is injective.

(2) Let \( f : \mathbb{R} \to \mathbb{R} \) be a continuous, surjective, strictly increasing function from \( \mathbb{R} \) onto \( \mathbb{R} \). Then \( f^{-1} : \mathbb{R} \to \mathbb{R} \) is also continuous and strictly increasing.

**Proof.** (1) Let \( f : \mathbb{R} \to \mathbb{R} \) be a strictly increasing function.

Let \( x, y \in \mathbb{R} \) with \( x \neq y \).

\( \Rightarrow \) Either \( x < y \) or \( y < x \).

\( \Rightarrow \) Since \( f \) is strictly increasing, either \( f(x) < f(y) \) or \( f(y) < f(x) \).

\( \Rightarrow \) \( f(x) \neq f(y) \).

\( \Rightarrow \) \( f : \mathbb{R} \to \mathbb{R} \) is injective.

(2) Assume that \( f : \mathbb{R} \to \mathbb{R} \) is a continuous, surjective, and strictly increasing function. Let \( y \in \mathbb{R} \) and choose \( a, b, x \in \mathbb{R} \) with \( y = f(x) \) and \( a < x < b \).
\[ f|_{[a,b]} : [a, b] \to [f(a), f(b)] \] is a continuous bijective function.

\[ f|_{[a,b]} : [a, b] \to [f(a), f(b)] \] is a homeomorphism by Theorem 20.

\[ \Rightarrow \text{Since } f(a) < y < f(b), f^{-1} : \mathbb{R} \to \mathbb{R} \text{ is continuous at } y = f(x). \]

\[ \Rightarrow \text{Since } y \text{ was chosen arbitrarily, } f^{-1} : \mathbb{R} \to \mathbb{R} \text{ is continuous.} \]

Next, let \( y_1, y_2 \in \mathbb{R} \) with \( y_1 < y_2 \).

\[ \Rightarrow \text{Since } f : \mathbb{R} \to \mathbb{R} \text{ is bijective, there exists } x_1, x_2 \text{ such that } x_1 \neq x_2, \]

\[ f(x_1) = y_1 \text{ and } f(x_2) = y_2. \]

\[ \Rightarrow f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2). \]

\[
\begin{aligned}
\therefore & \quad \text{Suppose that } x_1 > x_2. \\
\Rightarrow & \quad \text{Since } f \text{ is strictly increasing, } y_1 = f(x_1) > f(x_2) = y_2. \\
& \quad \text{A contradiction!} \\
\Rightarrow & \quad \text{Since } x_1 \neq x_2, x_1 < x_2.
\end{aligned}
\]

\[ \Rightarrow f^{-1} : \mathbb{R} \to \mathbb{R} \text{ is strictly increasing.} \]

\[ \square \]
4. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function from a closed and bounded interval \([a, b]\) into \(\mathbb{R}\). Show that \( f([a, b]) \) is a closed and bounded interval.

Proof. Let \( f : [a, b] \to \mathbb{R} \) be a continuous function.
⇒ Since \([a, b]\) is compact and \(\mathbb{R}\) is Hausdorff, \( f([a, b]) \) is compact by Corollary 18.
⇒ (1) Since \(\mathbb{R}\) is Hausdorff, \( f([a, b]) \) is closed by Theorem 12.
(2) \( f([a, b]) \) is bounded.

\[
(\therefore) \quad \text{Since } \{(-n, n) \mid n \in \mathbb{N}\} \text{ is an open cover of } f([a, b]),
\]

there exist \(n_1, n_2, \ldots, n_k \in \mathbb{N}\) such that \( f([a, b]) \subset \bigcup_{i=1}^{n_k} (-n_i, n_i). \)

Let \( m = \max\{n_1, n_2, \ldots, n_k\}. \)
⇒ \( f([a, b]) \subset (-m, m). \)
⇒ \( f([a, b]) \) is bounded.
(3) Since $f([a, b])$ is connected by Corollary 5.9, $f([a, b])$ is an interval by Theorem 5.27.

$\Rightarrow f([a, b])$ is a closed and bounded interval.

5. (1) Prove that every compact subset of a metric space $X$ is closed and bounded.

(2) Give an example of a metric space having a closed and bounded subset that is not compact. (Hint: A bounded metric space which is not compact will suffice. Consider Hilbert space for more sophisticated examples.)

Proof. (1) Let $(X, d)$ be a metric space and let $A$ be a compact subset of $X$.

(i) Since $X$ is Hausdorff, $A$ is closed by Theorem 12.

(ii) Choose $x_0 \in X$ and consider $\mathcal{O} = \{B_d(x_0, n) \mid n \in \mathbb{N}\}$.

$\Rightarrow \mathcal{O}$ is an open cover of $A$ by open sets in $X$.

$\Rightarrow$ Since $A$ is compact, there exists $n_1, n_2, \cdots, n_k \in \mathbb{N}$ such that
\[ A \subset \bigcup_{i=1}^{n_k} B_d(x_0, n_i) \]. Let \( m = \max\{n_1, n_2, \ldots, n_k\} \).

\[ \Rightarrow A \subset B_d(x_0, m). \]
\[ \Rightarrow A \) is bounded.

(2) Consider the usual metric \( d \) on \( \mathbb{R}^n \) and the standard bounded metric \( d' \) defined by \( d'(x, y) = \min\{d(x, y), 1\} \) for each pair of points \( x, y \in \mathbb{R}^n \).

\[ \Rightarrow d \) and \( d' \) generate the same usual topology on \( \mathbb{R}^n \).

(See Exercise 3.5.6.)
\[ \Rightarrow \mathbb{R}^n \) is bounded and closed in the metric \( d' \) but \( \mathbb{R}^n \) is not compact in the usual topology because the open cover \( \{B_d(0, n) \mid n \in \mathbb{N}\} \) has no finite subcover for \( \mathbb{R}^n \). \]
6. Let $B([a, b], \mathbb{R})$ denote the collection of all bounded functions from a closed interval $[a, b]$ into $\mathbb{R}$, i.e., $f \in B([a, b], \mathbb{R})$ if and only if $f([a, b])$ is a bounded subset of $\mathbb{R}$. Assign $B([a, b], \mathbb{R})$ the supremum metric $\rho$: for $f, g \in B([a, b], \mathbb{R})$,

$$\rho(f, g) = \text{lub}\{|f(x) - g(x)| \mid x \in [a, b]\}.$$

Prove the following facts about $(B([a, b], \mathbb{R}), \rho)$.

(1) $(B([a, b], \mathbb{R}), \rho)$ is a metric space which contains the space $(C([a, b], \mathbb{R}), \rho')$ of Example 3.11 as a subspace.

(2) A sequence $\{f_n\}_{n=1}^{\infty}$ in $B([a, b], \mathbb{R})$ converges to a member $f \in B([a, b], \mathbb{R})$ with respect to the metric $\rho$ if and only if $\{f_n\}_{n=1}^{\infty}$ converges to $f$ uniformly.

(For this reason, $\rho$ is often called the uniform metric for $B([a, b], \mathbb{R})$.)

(3) $C([a, b], \mathbb{R})$ is a closed subspace of $B([a, b], \mathbb{R})$, but $C([a, b], \mathbb{R})$ is not compact.

(4) $C([a, b], \mathbb{R})$ is nowhere dense in $B([a, b], \mathbb{R})$. 
Proof. (1) Clearly $\rho(f, g) \geq 0$ for all $f, g \in B([a, b], \mathbb{R})$.

Let $f, g, h \in B([a, b], \mathbb{R})$.

(i) $\rho(f, g) = 0$.

$\iff \lub\{|f(x) - g(x)| \mid x \in [a, b]\} = 0.$

$\iff |f(x) - g(x)| = 0$ for all $x \in [a, b]$.

$\iff f(x) = g(x)$ for all $x \in [a, b]$.

$\iff f = g$.

(ii) $\rho(f, g) = \lub\{|f(x) - g(x)| \mid x \in [a, b]\}$

$= \lub\{|g(x) - f(x)| \mid x \in [a, b]\}$

$= \rho(g, f)$.

(iii) $\rho(f, h) = \lub\{|f(x) - h(x)| \mid x \in [a, b]\}$

$\leq \lub\{|f(x) - g(x)| + |g(x) - h(x)| \mid x \in [a, b]\}$

$\leq \lub\{|f(x) - g(x)| \mid x \in [a, b]\}$

$+ \lub\{|g(x) - h(x)| \mid x \in [a, b]\}$

$= \rho(f, g) + \rho(g, h)$.

Thus $\rho$ is a metric on $B([a, b], \mathbb{R})$. 
Moreover since $C([a, b], \mathbb{R}) \subset B([a, b], \mathbb{R})$ by applying Theorem 22 and $\rho'$ of Example 3.11 is a restriction of $\rho$, $C([a, b], \mathbb{R})$ is a subspace of $B([a, b], \mathbb{R})$. (See Definition 3.50.)

(2) (only if) Assume that a sequence $\{f_n\}_{n=1}^{\infty}$ in $B([a, b], \mathbb{R})$ converges to a member $f \in B([a, b], \mathbb{R})$ with respect to the metric $\rho$. Let $\epsilon$ be given.

\[ \Rightarrow \] There exists $n_0 \in \mathbb{N}$ such that $\rho(f_n, f) < \epsilon$ for all $n \geq n_0$.

\[ \Rightarrow \] For each $x \in [a, b]$, $|f_n(x) - f(x)| \leq \text{lub}\{|f_n(x) - f(x)| \mid x \in [a, b]\}$

\[ = \rho(f_n, f) < \epsilon \text{ for all } n \geq n_0. \]

\[ \Rightarrow \] $|f_n(x) - f(x)| < \epsilon$ for all $x \in [a, b]$ and for all $n \geq n_0$.

\[ \Rightarrow \] $\{f_n\}_{n=1}^{\infty}$ converges to $f$ uniformly.

(if) Assume that $\{f_n\}_{n=1}^{\infty}$ in $B([a, b], \mathbb{R})$ converges to $f$ in $B([a, b], \mathbb{R})$ uniformly and let $\epsilon > 0$ be given.

\[ \Rightarrow \] There exists $n_0 \in \mathbb{N}$ such that $|f_n(x) - f(x)| < \frac{\epsilon}{2}$ for all $x \in [a, b]$ and for all $n \geq n_0$. 

50
\[ \rho(f_n, f) = \text{lub}\{|f_n(x) - f(x)| \mid x \in [a, b]\} \leq \frac{\epsilon}{2} < \epsilon \text{ for all } n \geq n_0. \]

\[ \{f_n\}_{n=1}^{\infty} \text{ converges to } f \text{ with respect to the metric } \rho. \]

(3) (i) Let \( f \in B([a, b], \mathbb{R}) \) be a limit point of \( C([a, b], \mathbb{R}) \) in the metric space \( (B([a, b], \mathbb{R}), \rho) \).

\[ \Rightarrow \text{There exists a sequence } \{f_n\} \text{ of distinct members of } C([a, b], \mathbb{R}) \]

converging to \( f \) with respect to \( \rho \) by Theorem 3.27.

\[ \Rightarrow \{f_n\} \text{ converges to } f \text{ uniformly by (2) above.} \]

\[ \Rightarrow \text{Since } f_n \in C([a, b], \mathbb{R}) \text{ for each } n \in \mathbb{N}, f \in C([a, b], \mathbb{R}) \]

by Theorem 3.70.

\[ \Rightarrow C([a, b], \mathbb{R}) \text{ contains all its limit points.} \]

\[ \Rightarrow C([a, b], \mathbb{R}) \text{ is closed in } (B([a, b], \mathbb{R}), \rho) \text{ by Theorem 3.23.} \]

(ii) Let \( f : [a, b] \rightarrow \mathbb{R} \) denote the constant function defined by

\[ f(x) = 0 \text{ for all } x \in [a, b], \text{ and consider the open ball } B_{\rho}(f, n) \]

for each \( n \in \mathbb{N} \).

\[ \Rightarrow \{B_{\rho}(f, n) \mid n \in \mathbb{N}\} \text{ is an open cover of } C([a, b], \mathbb{R}). \]
But $\{B_\rho(f, n) \mid n \in \mathbb{N}\}$ has no finite subcover for $C([a, b], \mathbb{R})$.

(\therefore) Suppose that $C([a, b], \mathbb{R}) \subset \bigcup_{i=1}^{k} B_\rho(f, n_i)$

for some $n_1, n_2, \cdots, n_k \in \mathbb{N}$. Let $m = \max\{n_1, n_2, \cdots, n_k\}$

so that $C([a, b], \mathbb{R}) \subset B_\rho(f, m)$.

$\Rightarrow$ Choose $c \in \mathbb{R}$ with $-c < -m < m < c$ and define

g : [a, b] \to \mathbb{R} by $g(x) = \frac{2c}{b-a}(x-a) - c$ for all $x \in [a, b]$.

$\Rightarrow$ $g \in C([a, b], \mathbb{R})$ but $g \notin B_\rho(f, m)$. A contradiction!

$\Rightarrow$ $(C([a, b], \mathbb{R}), \rho)$ is not compact.
(4) Since $C([a, b], \mathbb{R})$ is closed in $B([a, b], \mathbb{R})$ by (3) above,
\[ \text{cl}_{B([a, b], \mathbb{R})} C([a, b], \mathbb{R}) = C([a, b], \mathbb{R}). \]

We claim that $\text{int}_{B([a, b], \mathbb{R})} C([a, b], \mathbb{R}) = \emptyset$.

(\therefore) Let $f \in C([a, b], \mathbb{R})$ and let $\epsilon > 0$ be given.
\[ \Rightarrow \quad \text{Define } g : [a, b] \to \mathbb{R} \text{ by } g(x) = f(x) \text{ if } a \leq x < b \text{ and } \\
\quad \quad \quad \quad \quad \quad \quad g(x) = f(b) + \frac{\epsilon}{2} \text{ if } x = b. \]
\[ \Rightarrow \quad g \in (B([a, b], \mathbb{R}) \setminus C([a, b], \mathbb{R})) \text{ and } g \in B_\rho(f, \epsilon). \]
\[ \Rightarrow \quad B_\rho(f, \epsilon) \text{ is not contained in } C([a, b], \mathbb{R}) \\
\quad \text{for any } \epsilon\text{-ball centered at } f. \]
\[ \Rightarrow \quad \text{int}_{B([a, b], \mathbb{R})} C([a, b], \mathbb{R})) = \emptyset. \]
\[ \Rightarrow \quad \text{int}_{B([a, b], \mathbb{R})} (\text{cl}_{B([a, b], \mathbb{R})} C([a, b], \mathbb{R})) = \text{int}_{B([a, b], \mathbb{R})} (C([a, b], \mathbb{R})) = \emptyset. \]
\[ \Rightarrow \quad C([a, b], \mathbb{R}) \text{ is nowhere dense in } (B([a, b], \mathbb{R}), \rho). \]
3 Properties Related to Compactness

It will be shown in this section that the compact subsets of $\mathbb{R}^n$ are precisely the closed and bounded sets.

We shall also examine some other properties related to compactness and equivalent to it in various situations.
Lemma 27. The unit $n$-cube $I^n$ is a compact subset of $\mathbb{R}^n$, where
$I^n = \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid 0 \leq x_i \leq 1, \text{ for all } i = 1, 2, \cdots, n\}$.

Proof. $I^1 = [0, 1]$ is a compact subset of $\mathbb{R}^1$ by the Heine-Borel Theorem (Theorem 2.39). We will prove the lemma for $n = 2$ and the analogue argument for $n > 2$ is left for the reader.

For any square $[a, b] \times [c, d]$ in $\mathbb{R}^2$, we shall refer to

$$[a, \frac{a+b}{2}] \times [c, \frac{c+d}{2}], \quad [a, \frac{a+b}{2}] \times [\frac{c+d}{2}, d],$$

$$[\frac{a+b}{2}, b] \times [c, \frac{c+d}{2}], \quad [\frac{a+b}{2}, b] \times [\frac{c+d}{2}, d]$$

as the four quarters of $[a, b] \times [c, d]$. 
Now suppose that \( \mathbb{I}^2 = [0, 1] \times [0, 1] \) is not compact.

Figure 3: The four quarters of a square
⇒ ∃ an open cover $\mathcal{O}$ of $\mathbb{I}^2$ such that $\mathcal{O}$ has no finite subcover for $\mathbb{I}^2$.

⇒ ∃ at least one quarter $Q_1 = [a_1, b_1] \times [c_1, d_1]$ of $\mathbb{I}^2$ such that

$\mathcal{O}$ has no finite subcover for $Q_1$. Note that $b_1 - a_1 = d_1 - c_1 = \frac{1}{2}$.

⇒ ∃ one quarter $Q_2 = [a_2, b_2] \times [c_2, d_2]$ of $Q_1$ such that $\mathcal{O}$ has no finite subcover for $Q_2$. Note that $b_2 - a_2 = d_2 - c_2 = \frac{1}{2^2}$.

Proceeding inductively, we define a nested sequence $\{Q_n \mid n \in \mathbb{N}\}$ of squares in $\mathbb{R}^2$

$$Q_n = [a_n, b_n] \times [c_n, d_n]$$

such that $\mathcal{O}$ has no finite subcover for $Q_n$, for all $n \in \mathbb{N}$,

$\{[a_n, b_n] \mid n \in \mathbb{N}\}$ and $\{[c_n, d_n] \mid n \in \mathbb{N}\}$ are nested and

$b_n - a_n = d_n - c_n = \frac{1}{2^n}$, for each $n \in \mathbb{N}$. 


⇒ By the Cantor’s Nested Interval Theorem (Theorem 2.39),

\[ \exists \text{ real numbers } p \in \bigcap_{n=1}^{\infty} [a_n, b_n] \text{ and } q \in \bigcap_{n=1}^{\infty} [c_n, d_n]. \]

⇒ \((p, q) \in \bigcap_{n=1}^{\infty} Q_n.\)

⇒ Since \(\mathcal{O}\) covers \(I^2\) and \((p, q) \in I^2, \exists O \in \mathcal{O}\) such that \((p, q) \in O.\)

⇒ Since \(\lim_{n \to \infty} \text{diam}(Q_n) = 0, \) there is \(n_0 \in \mathbb{N}\) such that \(Q_{n_0} \subset O.\)

⇒ \(Q_{n_0}\) is covered by a finite member of \(\mathcal{O},\) namely \(\{O\}.\)

This contradicts to the assumption that \(\mathcal{O}\) has no finite subcover for \(Q_{n_0}.\)

⇒ \(I^2\) must be compact. \(\square\)
Theorem 28. \( A \): a subset of \( \mathbb{R}^n \).

Then \( A \) is compact \( \iff \) \( A \) is closed and bounded.

Proof. \((\Rightarrow)\) Assume that \( A \) is compact.

\[ \Rightarrow (i) \] Since \( \mathbb{R}^n \) is Hausdorff, \( A \) is closed in \( \mathbb{R}^n \) by Theorem 12.

\[ (ii) \] To show that \( A \) is bounded, consider the point

\[ \textbf{0} = (0, 0, \cdots , 0) \in \mathbb{R}^n \] and the open cover \( \{ B(\textbf{0}, m) \mid m \in \mathbb{N} \} \) of \( \mathbb{R}^n \), where \( B(\textbf{0}, m) = \{ x \in \mathbb{R}^n \mid \| x \| < m \} \).

\[ \Rightarrow \{ B(\textbf{0}, m) \mid m \in \mathbb{N} \} \text{ covers } A. \]

\[ \Rightarrow \text{ Since } A \text{ is compact, } \exists n_0 \in \mathbb{N} \text{ such that } A \subset B(\textbf{0}, n_0). \]

\[ \Rightarrow A \text{ is bounded in } \mathbb{R}^n. \]

\((\Leftarrow)\) Assume that \( A \) is closed and bounded in \( \mathbb{R}^n \).

\[ \Rightarrow \text{ Since } A \text{ is bounded, there exists } b > 0 \text{ such that } \| x \| < b, \forall x \in A. \]
Let $J^n = \prod_{i=1}^{n} [-b, b]$

$= \{(x_1, x_2, \cdots, x_n) \in \mathbb{R}^n \mid -b \leq x_i \leq b, \forall i = 1, 2, \cdots, n\}$.

⇒ Since $J^n$ is homeomorphic with $I^n$ and $I^n$ is compact by Lemma 27, $J^n$ is compact by Corollary 18.

⇒ Since $A$ is a closed subset of $J^n$, $A$ is compact by Thm 11. □

**Definition 29.** A space $X$ is said to be *countably compact* if every countable open cover of $X$ has a finite subcover for $X$.

The reader is left the following exercises:

1. Every compact space is countably compact.
2. Countable compactness is a topological property.
3. There is a countably compact, non-compact space.
The next definition introduces a condition under which compactness and countable compactness are equivalent.

**Definition 30.** A space $X$ is said to have the *Lindelöf property* or to be a *Lindelöf space* if every open cover of $X$ has a countable subcover.

**Theorem 31.** Let $X$ be a Lindelöf space.

Then $X$ is compact $\iff$ it is countably compact.

**Proof.** ($\Rightarrow$) It is clear.

($\Leftarrow$) Assume that $X$ is a countably compact Lindelöf space.

Let $\mathcal{O}$ be an open cover of $X$.

$\Rightarrow$ Since $X$ is Lindelöf, $\mathcal{O}$ has a countable subcover $\mathcal{O}'$ for $X$.

$\Rightarrow$ Since $X$ is countably compact, $\mathcal{O}'$ has a finite subcover $\mathcal{O}''$ for $X$.

$\Rightarrow$ $\mathcal{O}''$ is a finite subcover of $\mathcal{O}$ for $X$.

$\Rightarrow$ $X$ is compact. \[\square\]
The next theorem shows that the Lindelöf property holds in an important class of spaces.

**Theorem 32** (Lindelöf Theorem). Every second countable space is a Lindelöf space.

*Proof.* Let $X$ be a second countable space with a countable basis $\mathcal{B}$ and let $\mathcal{O}$ be an open cover of $X$. For each $x \in X$, choose $O_x \in \mathcal{O} \ni x \in O_x$ and then $B_x \in \mathcal{B} \ni x \in B_x \subset O_x$.

$\Rightarrow$ Since $\mathcal{B}$ is countable, $\{B_x \mid x \in X\}$ is a countable open cover of $X$.

For each $B_x$ choose $O'_x \in \mathcal{O}$ such that $B_x \subset O'_x$.

$\Rightarrow \{O'_x \mid x \in X\}$ is a countable subcover of $\mathcal{O}$ for $X$.

$\Rightarrow X$ is Lindelöf. 

Since $\mathbb{R}^n$ is second countable by Corollary 4.28, Theorem 32 insures that $\mathbb{R}^n$ is Lindelöf, and Theorem 31 shows that the concepts of
compactness and countable compactness coincide for subsets of $\mathbb{R}^n$.

A subset of $\mathbb{R}^n$ is compact if and only if it is countably compact.

**Definition 33.** (1) A space $X$ has the *Bolzano-Weierstrass property* or is *limit point compact* if every infinite subset of $X$ has a limit point in $X$.

(2) A subset $A$ of $X$ has the *Bolzano-Weierstrass property* or is *limit point compact* if every infinite subset of $A$ has a limit point in $A$.

The Bolzano-Weierstrass property is evidently a topological invariant.
**Theorem 34.** Every compact space is limit point compact.

*Proof.* Let $X$ be a compact space.

Suppose that $X$ is not limit point compact.

$\Rightarrow \exists$ an infinite subset $A$ of $X$ such that $A$ has no limit point.

$\Rightarrow$ For each $x \in X$, since $x$ is not a limit point of $A$, there exists an open set $O_x$ in $X$ such that $x \in O_x$ and $O_x \cap (A \setminus \{x\}) = \emptyset$.

$\Rightarrow \{O_x \mid x \in X\}$ is an open cover of $X$.

$\Rightarrow$ Since $X$ is compact, there exist $x_1, x_2, \cdots, x_n \in X$ such that

$$X = \bigcup_{i=1}^{n} O_{x_i}.$$ 

$\Rightarrow$ Since $O_{x_i} \cap A$ is empty or a singleton set, 

$$A = A \cap X = A \cap \left( \bigcup_{i=1}^{n} O_{x_i} \right) = \bigcup_{i=1}^{n} (A \cap O_{x_i})$$ is finite.

It contradicts to the choice of an infinite set $A$.

$\square$
Example 35. (1) \([a, b]\) has the Bolzano-Weierstrass property.

(2) \((a, b)\) is not limit point compact.

(3) \(\mathbb{R}\) is not limit point compact.

(4) \(S = \{x = (x_1, x_2, \cdots, x_n, \cdots) \in \mathbb{H} \mid |x| = 1\}\) is bounded and closed in the Hilbert space \(\mathbb{H}\), but \(S\) is not limit point compact. (Thus \(S\) is not compact by Theorem 34. See Example 42(2).)

We recall that \(\mathbb{H} = \{(x_1, \cdots, x_n, \cdots) \in \mathbb{R}^\infty \mid x_i \in \mathbb{R}, \sum_{i=1}^{\infty} x_i^2 \}\) converges in \(\mathbb{R}\) with

\[
d(x, y) = \left(\sum_{i=1}^{\infty} (x_i - y_i)^2\right)^{\frac{1}{2}} \text{ and } |x| = \left(\sum_{i=1}^{\infty} x_i^2\right)^{\frac{1}{2}}
\]

for \(x = (x_1, \cdots, x_n, \cdots), y = (y_1, \cdots, y_n, \cdots) \in \mathbb{H}\).

Proof. (1) \([a, b]\) has the Bolzano-Weierstrass property by Theorem 34.
(2) Since \( \{ a + \frac{1}{n} \mid \frac{1}{n} < b - a \} \) is an infinite subset of \((a, b)\), which has no limit in \((a, b)\), \((a, b)\) is not limit point compact.

(3) Since \( \{ n \mid n \in \mathbb{N} \} \) has no limit point in \(\mathbb{R}\), \(\mathbb{R}\) is not limit point cpt.

(4) Let \( S = \{ \mathbf{x} = (x_1, x_2, \cdots, x_n, \cdots) \in \mathbb{H} \mid |\mathbf{x}| = 1 \}. \)

\[ \Rightarrow \] Clearly \( S \) is bounded and closed in the Hilbert space \( \mathbb{H} \).

We claim that \( S \) is not limit point compact.

\[
\begin{align*}
\therefore \quad & \text{Let } p_1 = (1, 0, 0, \cdots), p_2 = (0, 1, 0, \cdots), \cdots, \\
& p_n = (0, \cdots, 0, 1, 0, \cdots), \cdots \\
& \text{where } p_n \text{ has } n\text{th coordinate } 1 \text{ and all other coordinates } 0. \\
\Rightarrow & \text{Since } d(p_i, p_j) = \sqrt{2} \text{ for } i \neq j, \\
& \{ p_n \mid n \in \mathbb{N} \} \text{ has no limit point in } \mathbb{H}.
\end{align*}
\]
Lemma 36 (Lebesgue Number Lemma).

$(X,d)$: a limit point compact metric space.

$\mathcal{O}$: an open cover of $X$.

$\Rightarrow \exists \epsilon > 0 \implies \forall x \in X, B_d(x,\epsilon) \subset O$ for some $O \in \mathcal{O}$.

Proof. Suppose contrarily that for arbitrary $\epsilon > 0 \exists x \in X$ such that $B_d(x,\epsilon)$ is not contained in any member of $\mathcal{O}$.

$\Rightarrow$ For each $n \in \mathbb{N}$ there exists $x_n \in X$ such that $B_d(x_n,\frac{1}{n})$ is not contained in any member of $\mathcal{O}$.

$\Rightarrow \{x_n \mid n \in \mathbb{N}\}$ is an infinite subset of $X$. 
For each \( n \in \mathbb{N} \), since \( \mathcal{O} \) covers \( X \),
there exists \( O_n \in \mathcal{O} \) such that \( x_n \in O_n \).

⇒ Since \( O_n \) is open in \( X \), there exists \( \epsilon_n > 0 \) such that
\( B_d(x_n, \epsilon_n) \subset O_n \).

⇒ \( x_m \neq x_n \) for all \( m \in \mathbb{N} \) with \( \frac{1}{m} < \epsilon_n \).

( :: ) Suppose that \( x_m = x_n \) with \( \frac{1}{m} < \epsilon_n \).

⇒ \( B_d(x_m, \frac{1}{m}) = B_d(x_n, \frac{1}{m}) \).

⇒ \( B_d(x_m, \frac{1}{m}) \subset B_d(x_n, \epsilon_n) \subset O_n \). A contradiction!

⇒ For each \( x_n \), there exist infinitely many \( m \in \mathbb{N} \) with \( x_n \neq x_m \).

⇒ \( \{ x_n \mid n \in \mathbb{N} \} \) is infinite.

⇒ Since \( X \) is limit point compact, \( \{ x_n \mid n \in \mathbb{N} \} \) has a limit point \( a \in X \).
⇒ Since $\mathcal{O}$ covers $X$, there exists $O \in \mathcal{O}$ with $a \in O$
⇒ Since $O$ is open in $X$, there exists $\delta > 0$ with $B_d(a, \delta) \subset O$.
⇒ Since $a$ is a limit point of $\{x_n \mid n \in \mathbb{N}\}$, $B_d(a, \frac{\delta}{2})$ contains infinitely many points of $\{x_n \mid n \in \mathbb{N}\}$.
⇒ There exists $n_0 \in \mathbb{N}$ such that $\frac{1}{n_0} < \frac{\delta}{2}$ and $x_{n_0} \in B_d(a, \frac{\delta}{2})$.
⇒ $B_d(x_{n_0}, \frac{1}{n_0}) \subset B_d(a, \delta) \subset O \in \mathcal{O}$ which is a contradiction.

\[
\left(\because\right) \text{ Let } z \in B_d(x_{n_0}, \frac{1}{n_0}).
\]
⇒ $d(a, z) \leq d(a, x_{n_0}) + d(x_{n_0}, z) < \frac{\delta}{2} + \frac{1}{n_0} < \frac{\delta}{2} + \frac{\delta}{2} = \delta.$
⇒ $z \in B_d(a, \delta)$. 
\[\square\]
Figure 4: $B_d(x_{n_0}, \frac{1}{n_0}) \subset B_d(a, \delta)$
Lemma 37. \((X, d)\): a limit point compact metric space.

\[ \forall \epsilon > 0, \exists \text{ a finite subset } A_\epsilon \text{ of } X \implies \forall x \in X, B_d(x, \epsilon) \cap A_\epsilon \neq \emptyset. \]

Proof. Suppose that we have \(\epsilon > 0\) such that, for any finite subset \(A\) of \(X\), there exists \(x_A \in X\) such that \(B_d(x_A, \epsilon) \cap A = \emptyset\).

Now start with any point \(a_1 \in X\) and let \(A_1 = \{a_1\}\).

\[ \Rightarrow \text{ Since } A_1 \text{ is finite, there exists } a_2 \in X \text{ such that } B_d(a_2, \epsilon) \cap A_1 = \emptyset. \]

Let \(A_2 = \{a_1, a_2\}\). Then \(d(a, b) \geq \epsilon\), for all \(a, b \in A_2(a \neq b)\).

\[ \Rightarrow \text{ Since } A_2 \text{ is finite, there exists } a_3 \in X \text{ such that } B_d(a_3, \epsilon) \cap A_2 = \emptyset. \]

Let \(A_3 = \{a_1, a_2, a_3\}\). Then \(d(a, b) \geq \epsilon\), for all \(a, b \in A_3(a \neq b)\).

Now assume that for \(n \geq 2\), we have a set \(A_n = \{a_1, \cdots, a_n\}\) such that \(d(a, b) \geq \epsilon\), for all \(a, b \in A_n(a \neq b)\).

\[ \Rightarrow \text{ Since } A_n \text{ is finite, there exists } a_{n+1} \in X \text{ such that } B_d(a_{n+1}, \epsilon) \cap A_n = \emptyset. \]

Let \(A_{n+1} = \{a_1, a_2, \cdots, a_n, a_{n+1}\}\).
Then \( d(a, b) \geq \epsilon \), for all \( a, b \in A_{n+1}(a \neq b) \).

Inductively, we have an infinite set \( B = \{a_n \mid n \in \mathbb{N}\} \) such that \( d(a, b) \geq \epsilon \), for all \( a, b \in B(a \neq b) \).

\[ \Rightarrow \] Since \( X \) is a limit point compact space, \( B \) has a limit point \( a \in X \).

\[ \Rightarrow \] \( B_{\frac{\epsilon}{2}}(a, \frac{\epsilon}{2}) \) contains infinitely many points of \( B \).

\[ \Rightarrow \] There exist \( a_n, a_m \in B_{\frac{\epsilon}{2}}(a, \frac{\epsilon}{2}) \cap B \) with \( a_n \neq a_m \).

\[ \Rightarrow \] \( \epsilon \leq d(a_n, a_m) \leq d(a_n, a) + d(a, a_m) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \),

which is a contradiction. \( \square \)

**Theorem 38.** \((X, d)\): a metric space. Then

\( X \) is compact \( \Leftrightarrow \) \( X \) is limit point compact.
Proof. \((\Rightarrow)\) It is Theorem 34.

\((\Leftarrow)\) Assume that \(X\) is limit point compact and let \(\mathcal{O}\) be an open cover of \(X\).

\(\Rightarrow\) By Lemma 36 (Lebesgue Number Lemma), \(\exists \epsilon > 0\) such that for any \(x \in X\), \(B_d(x, \epsilon)\) is contained in some member of \(\mathcal{O}\).

\(\Rightarrow\) By Lemma 37, there exists a finite subset \(A_\epsilon = \{x_1, \cdots, x_n\}\) of \(X\) such that for any \(x \in X\), \(B_d(x, \epsilon) \cap A_\epsilon \neq \emptyset\).

Choose \(O_i \in \mathcal{O}\) such that \(B_d(x_i, \epsilon) \subset O_i\), for each \(i = 1, 2 \cdots, n\).

\(\Rightarrow\) For any \(x \in X\), since \(B_d(x, \epsilon) \cap A_\epsilon \neq \emptyset\), there exists \(x_i \in A_\epsilon\) such that \(x_i \in B_d(x, \epsilon)\) and hence \(x \in B_d(x_i, \epsilon) \subset O_i \subset \bigcup_{i=1}^{n} O_i\).

\(\Rightarrow\) \(X = \bigcup_{i=1}^{n} O_i\). i.e., \(\{O_1, O_2, \cdots, O_n\}\) is a finite subcover of \(\mathcal{O}\) for \(X\).

\(\Rightarrow\) \(X\) is compact. \(\blacksquare\)
**Definition 39.** Let \((X, d)\) be a metric space and let \(\epsilon > 0\).

(1) An \(\epsilon\)-net for \(X\) is a finite subset \(A_\epsilon\) of \(X\) such that for any \(x \in X\), \(B_d(x, \epsilon) \cap A_\epsilon \neq \emptyset\).

(2) \(X\) is said to be *totally bounded* if for any \(\epsilon > 0\) there exists an \(\epsilon\)-net for \(X\), that is, there exists a finite covering of \(X\) by \(\epsilon\)-balls.

(3) For an open cover \(\mathcal{O}\) of \(X\), a *Lebesgue number* for \(\mathcal{O}\) is a positive number \(\epsilon > 0\) with the property that every subset of \(X\) of diameter less than \(\epsilon\) is contained in some member of \(\mathcal{O}\).

Lemma 37 states that every metric space satisfying the Bolzano-Weierstrass property, and hence every compact metric space, is totally bounded.
Theorem 40 (Lebesgue Number Theorem).

\((X, d)\): a compact metric space.

⇒ Every open cover of \(X\) has a Lebesgue number.

Proof. Note that a metric space \(X\) is compact if and only if \(X\) is limit point compact by Theorem 38.

Let \(X\) be a compact metric space and let \(\mathcal{O}\) be an open cover of \(X\).

⇒ By Lebesgue Number Lemma (Lemma 36), there exists \(\epsilon > 0\) such that for any \(x \in X\), \(B_d(x, \epsilon)\) is contained in some member of \(\mathcal{O}\).

⇒ \(\epsilon\) is a Lebesgue number of \(\mathcal{O}\).

\[ \left(\therefore\right) \text{Let } A \text{ be any subset of } X \text{ having diameter } < \epsilon. \]

⇒ There exists \(a \in A\) such that \(A \subset B_d(a, \epsilon)\).

⇒ There exists \(O \in \mathcal{O}\) such that \(B_d(a, \epsilon) \subset O\) and hence \(A \subset O\).
The information about compact subsets of $\mathbb{R}^n$ so far may be summarized as follows;

**Theorem 41.** Let $A$ be a subset of $\mathbb{R}^n$. Then TFAE.

1. $A$ is compact.
2. $A$ is limit point compact.
   *i.e., $A$ has the Bolzano-Weierstrass property.*
3. $A$ is countably compact.
4. $A$ is closed and bounded.

**Proof.**

1. $\iff$ 2.: Since $\mathbb{R}^n$ is a metric space, it follows from Thm 38.

1. $\iff$ 3.: Since $\mathbb{R}^n$ is a Lindelöf space, it follows from Theorem 31.

1. $\iff$ 4.: It is Theorem 28. \qed
Example 42. (1) Every compact subset of a metric space is closed and bounded.

(2) A closed and bounded subset of a metric space may not be compact.

Proof. (1) Let \( A \) be a compact subset of a metric space \((X, d)\).

⇒ (i) Since \( X \) is Hausdorff, \( A \) is closed in \( X \) by Theorem 12.

(ii) Since \( \{B_d(x_0, n) \mid n \in \mathbb{N}\} \) \((x_0 \in X \) is any fixed point of \( X \))

has a finite subcover for \( A \), \( A \) is bounded.

(2) Consider \( S = \{x = (x_1, \cdots, x_n, \cdots) \in \mathbb{H} \mid \|x\| = 1\}\).

⇒ \( S \) is bounded and closed in \( \mathbb{H} \) but \( S \) is not limit point compact by Example 35 (4).

⇒ \( S \) is not compact by Theorem 38. \( \square \)
Note that $S$ in the proof of Example 42 (2) is bounded but not totally bounded.

Exercise 6.3.10 (3) at the end of this section establishes a criterion comparable to being closed and bounded which is equivalent to compactness in general metric spaces:

*A metric space is compact if and only if it is complete and totally bounded.*

There are many interesting relationships among the properties of compactness, connectedness, path connectedness, and their corresponding local properties.

One example (Theorem 44) that illustrates a famous characterization theorem is described here without proof.

**Definition 43.** A compact, connected and locally connected metric space is called a *Peano space* or *Peano continuum.*
• For example, closed and bounded intervals in $\mathbb{R}$, closed squares and closed disks in $\mathbb{R}^2$, closed cubes and closed balls in $\mathbb{R}^3$, and their higher dimensional analogues in $\mathbb{R}^n$ are all Peano spaces.

• The remarkable fact about Peano spaces is that for any Peano space $X$, there is a continuous function from the closed unit interval $I$ onto $X$. Such a function was first discovered for the case of the unit square in the late nineteenth century by the Italian mathematician Guiseppe Peano and was called a “space-filling curve.”

• In fact, the properties of being a Hausdorff space and the image of the closed unit interval under a continuous function characterize the Peano space.

• In order that a topological space $X$ be a Peano space it is necessary and sufficient that $X$ be Hausdorff and that there exist a continuous function from $I$ onto $X$. 
• The celebrated result is called the Hahn-Mazurkiewicz Theorem. One of its surprising consequences is the existence of space-filling curves from $I$ onto closed cubes and closed balls of any finite dimension.

**Theorem 44** (Hahn-Mazurkiewicz Theorem). *Let $X$ be a space. Then $X$ is a Peano space if and only if $X$ satisfies the following conditions.*

(1) *$X$ is Hausdorff space.*

(2) *There exists a continuous surjective function $f : [0, 1] \rightarrow X$.*

*Proof.* Omit. □
Theorem 45 (Peano space-filling curve). There exists a continuous surjective function \( f : I \to I \times I \) from the unit interval \( I \) onto the unit square \( I \times I \).

Proof. We shall construct a continuous, surjective function

\[
f : I \to I \times I
\]

as the limit of a sequence of continuous functions

\[
f_n : I \to I \times I.
\]

Choose \( f_1 : I \to I \times I \), one of the four choices described in Fig 5.

Define \( f_2 : I \to I \times I \) by applying the second pictures of Figure 6 in each of 4 smaller squares and continue this process inductively to get \( f_n : I \to I \times I \) such that \( d(f_n(x), f_{n+1}(x)) < \frac{1}{2^n} \) for all \( x \in I \), where \( d \) denotes the max metric on \( \mathbb{R}^2 \).
Figure 5: Choice of the function $f_1$
Figure 6: The function $f_n$
Define $f(x) = \lim_{n \to \infty} f_n(x)$ for each $x \in I$.

⇒ (1) $f : I \to I \times I$ is well-defined.

(∵) Since $\{f_n(x)\}$ is a Cauchy sequence in the complete metric space $I \times I$ for each $x \in I$, $f(x) = \lim_{n \to \infty} f_n(x)$ is uniquely defined for each $x \in I$.

(2) $f : I \to I \times I$ is continuous.

(∵) We first show that $\{f_n\}$ converges uniformly to $f$.

Let $\epsilon > 0$ be given.

⇒ Since $d(f_n(x), f_{n+1}(x)) < \frac{1}{2^n}$ for all $x \in I$ and for all $n \in \mathbb{N}$,

\[ d(f_n(x), f(x)) \leq \frac{1}{2^n} \text{ for each } n \in \mathbb{N}. \]

Take $n_0 \in \mathbb{N}$ such that $\frac{2}{2^{n_0}} < \epsilon$.

⇒ For any $n \geq n_0$ and for any $x \in I$,
\[
d(f_n(x), f(x)) \leq d(f_n(x), f_{n_0}(x)) + d(f_{n_0}(x), f(x))
\]
\[
< \frac{1}{2^n_0} + \frac{1}{2^n_0} = \frac{2}{2^n_0} < \epsilon.
\]
\[\Rightarrow \{f_n\} \text{ converges uniformly to } f.\]
\[\Rightarrow f : I \rightarrow I \times I \text{ is continuous by Theorem 3.70.}\]

(3) \(f : I \rightarrow I \times I\) is surjective.

(\because) Note first that since \(f(I)\) is a compact subset of \(I \times I\),
\[f(I) \text{ is closed in } I \times I. \text{ Let } y \in I \times I \text{ and let } \epsilon > 0 \text{ be given.}\]
\[\Rightarrow \text{ There exist a point } x \in I \text{ and an integer } m \in \mathbb{N} \text{ such that }\]
\[d(y, f_m(x)) < \frac{1}{2^m} \text{ by the construction of the functions } f_n.\]
\[\Rightarrow d(y, f_n(x)) \leq d(y, f_m(x)) + d(f_m(x), f_n(x)) < \frac{2}{2^m}\]
\[\quad \text{ for each } n \geq m. \text{ Take } n_0 \in \mathbb{N} \text{ such that } n_0 > m \text{ and } \frac{3}{2^n_0} < \epsilon.\]
$\Rightarrow$ Since $d(f_n(x), f_{n+1}(x)) < \frac{1}{2^n}$ for all $x \in I$ and for all $n \in \mathbb{N}$,

$$d(f_n(x), f(x)) \leq \frac{1}{2^n} \text{ for each } n \in \mathbb{N} \text{ and hence}$$

$$d(y, f(x)) \leq d(y, f_{n_0}(x)) + d(f_{n_0}(x), f(x))$$

$$< \frac{2}{2^{n_0}} + \frac{1}{2^{n_0}} = \frac{3}{2^{n_0}} < \epsilon.$$  

$\Rightarrow$ $B_d(y, \epsilon) \cap f(I) \neq \emptyset$.

$\Rightarrow$ Since $\epsilon$ was given arbitrarily, $y \in \overline{f(I)}$.

$\Rightarrow$ Since $f(I)$ is closed in $I \times I$, $y \in \overline{f(I)} = f(I)$.

$\Rightarrow f : I \to I \times I$ is surjective. \qed
Exercises 3

1. Prove the following.

   (1) Countable compactness, the Lindelöf property, and the Bolzano-Weierstrass property are topological invariants but are not hereditary.

   (2) Countable compactness, the Lindelöf property, and the Bolzano-Weierstrass property are inherited by closed subspaces.

Proof. (1) Countable compactness is a topological invariant.

Let \( X \) be a countably compact space and let \( f : X \to Y \) be a homeomorphism. Let \( \mathcal{O} = \{O_i \mid i \in \mathbb{N}\} \) be a countable open cover of \( Y \).

\( \Rightarrow \) Since \( f : X \to Y \) is continuous, \( f^{-1}(O_i) \) is open in \( X \) for each \( i \in \mathbb{N} \). And

\[ X = f^{-1}(Y) = f^{-1}\left(\bigcup_{i=1}^{\infty} O_i\right) = \bigcup_{i=1}^{\infty} f^{-1}(O_i). \]

\( \Rightarrow \) \( \{f^{-1}(O_i) \mid i \in \mathbb{N}\} \) is a countable open cover of \( X \).
⇒ Since $X$ is countably compact, there exists a finite subcover
\[
\{f^{-1}(O_{i,j}) \mid i_j \in \mathbb{N}, j = 1, 2, \cdots, n\}
\text{ of } \{f^{-1}(O_i) \mid i \in \mathbb{N}\}.
\]
⇒ $X = \bigcup_{j=1}^{n} f^{-1}(O_{i,j})$.
⇒ Since $f$ is surjective,
\[
Y = f(X) = f\left(\bigcup_{j=1}^{n} f^{-1}(O_{i,j})\right) = f\left(f^{-1}\left(\bigcup_{j=1}^{n} O_{i,j}\right)\right) = \bigcup_{j=1}^{n} O_{i,j}.
\]
⇒ $\{O_{i,j} \mid j = 1, 2, \cdots, n\}$ is a finite subcover of $\mathcal{O}$ for $Y$.
⇒ $Y$ is a countably compact space.  

(2) Countable compactness is not hereditary.

Proof. Consider the compact space $X = [3, 7]$ and $Y = (3, 5) \subset X$.
⇒ $X$ is countably compact but $Y$ is not countably compact because
the countable open cover $\{(3 + \frac{1}{n}, 5) \mid n \in \mathbb{N}\}$ has no finite
subcover.
(3) The Lindelöf property is a topological invariant.

**Proof.** Let $X$ be a Lindelöf space and let $f : X \rightarrow Y$ be a homeomorphism. Let $\mathcal{O} = \{ O_\alpha \mid \alpha \in \mathcal{A} \}$ be an open cover of $Y$.

$\Rightarrow$ Since $f : X \rightarrow Y$ is continuous, $f^{-1}(O_\alpha)$ is open in $X$ for each $\alpha \in \mathcal{A}$. And $X = f^{-1}(Y) = f^{-1}( \bigcup_{\alpha \in \mathcal{A}} O_\alpha) = \bigcup_{\alpha \in \mathcal{A}} f^{-1}(O_\alpha)$.

$\Rightarrow \{ f^{-1}(O_\alpha) \mid \alpha \in \mathcal{A} \}$ is an open cover of $X$.

$\Rightarrow$ Since $X$ is Lindelöf, there exists a countable subcover

$\{ f^{-1}(O_{\alpha_i}) \mid \alpha_i \in \mathcal{A}, i \in \mathbb{N} \}$ of $\{ f^{-1}(O_\alpha) \mid \alpha \in \mathcal{A} \}$.

$\Rightarrow X = \bigcup_{i \in \mathbb{N}} f^{-1}(O_{\alpha_i})$.

$\Rightarrow$ Since $f$ is surjective,

$Y = f(X) = f \left( \bigcup_{i \in \mathbb{N}} f^{-1}(O_{\alpha_i}) \right) = f \left( f^{-1}(\bigcup_{i \in \mathbb{N}} O_{\alpha_i}) \right) = \bigcup_{i \in \mathbb{N}} O_{\alpha_i}$.

$\Rightarrow \{ O_{\alpha_i} \mid i \in \mathbb{N} \}$ is a countable subcover of $\mathcal{O}$ for $Y$.

$\Rightarrow Y$ is a Lindelöf space. 

\[ \square \]
The Lindelöf property is not hereditary.

Proof. Note that there exists an uncountable well ordered set $X$. 

(See page 66 of [Mu])

⇒ Give an order topology on $X$, which is the topology generated by the basis $\{ (\alpha, \beta) \mid \alpha < \beta (\alpha, \beta \in X) \}$.

Consider the subspace $Y = \{ \alpha \in X \mid S_\alpha \text{ is uncountable} \}$, where $S_\alpha = \{ x \in X \mid x < \alpha \}$.

⇒ Since $X$ is well ordered, there exists the smallest element $\Omega$ of $Y$.

⇒ $S_\Omega = \{ \alpha \in X \mid \alpha < \Omega \}$ is a well ordered uncountable set and every section of $S_\Omega$ is countable. ($S_\Omega$ is called a minimal uncountable well ordered set.) Let $\overline{S_\Omega} = S_\Omega \cup \Omega \subset X$.

⇒ It is left for the readers to show the following.

(i) $S_\Omega$ is a subspace of $\overline{S_\Omega}$.

(ii) Since $\overline{S_\Omega}$ is compact, $\overline{S_\Omega}$ is a Lindelöf space.

(iii) $S_\Omega$ is not a Lindelöf space. \qed
(5) The Bolzano-Weierstrass property is a topological invariant.

**Proof.** Let $X$ have the Bolzano-Weierstrass property and let $f : X \to Y$ be a homeomorphism. Let $A$ be an infinite subset of $Y$.

⇒ Since $f : X \to Y$ is bijective, $f^{-1}(A)$ is an infinite subset of $X$.

⇒ Since $X$ is limit point compact, there exists a limit point $x \in X$ of $f^{-1}(A)$. Let $f(x) = y \in Y$.

⇒ $y$ is a limit point of $A$.

\[
\begin{align*}
(\because) & \quad \text{Let } U \text{ be an open subset of } Y \text{ containing } y. \\
\Rightarrow & \quad f^{-1}(U) \text{ is an open subset of } X \text{ containing } x. \\
\Rightarrow & \quad \text{Since } x \text{ is a limit point of } f^{-1}(A), (f^{-1}(U) \setminus \{x\}) \cap f^{-1}(A) \neq \emptyset. \\
\Rightarrow & \quad (U \setminus \{y\}) \cap A \neq \emptyset. \\
\Rightarrow & \quad Y \text{ is a limit point compact space.}
\end{align*}
\]
(6) The Bolzano-Weierstrass property is not hereditary.

Proof. Consider the compact space $X = [3, 7]$ and $Y = (3, 5) \subset X$.

$\Rightarrow X$ is limit point compact but $Y$ is not limit point compact

because the infinite subset $A = \{3 + \frac{1}{n} \mid n \in \mathbb{N}\}$ of $Y$ has no limit point in $Y$. \qed

(7) Countable compactness is inherited by closed subspaces.

Proof. Let $X$ be a countably compact space and let $A$ be a closed subset of $X$. Let $\mathcal{O}$ be a countable open cover of $A$ by open sets in $X$.

$\Rightarrow$ Since $A$ is closed in $X$, $X \setminus A$ is open in $X$. Let $\mathcal{O}^* = \mathcal{O} \cup \{X \setminus A\}$.

$\Rightarrow$ $\mathcal{O}^*$ is a countable open cover of $X$.

$\Rightarrow$ Since $X$ is countably compact, $\mathcal{O}^*$ has a finite subcover

$\{O_1, O_2, \ldots, O_n\}$ which may or may not contain $X \setminus A$. 

92
⇒ Since $X = (X \setminus A) \cup A = (X \setminus A) \cup (\bigcup_{i=1}^{n} O_i)$, $A \subset \bigcup_{i=1}^{n} O_i$.

(disregard $X \setminus A$ if $\{O_1, O_2, \cdots, O_n\}$ contains it)

⇒ $\{O_1, O_2, \cdots, O_n\}$ is a finite subcover of $\varnothing$ for $A$.

⇒ $A$ is countably compact. □

(8) The Lindelöf property is inherited by closed subspaces.

Proof. Let $X$ be a Lindelöf space and let $A$ be a closed subset of $X$.

Let $\varnothing$ be an open cover of $A$ by open sets in $X$.

⇒ Since $A$ is closed in $X$, $X \setminus A$ is open in $X$. Let $\varnothing^* = \varnothing \cup \{X \setminus A\}$.

⇒ $\varnothing^*$ is an open cover of $X$.

⇒ Since $X$ is Lindelöf, $\varnothing^*$ has a countable subcover

$\{O_n \in \varnothing^* \mid n \in \mathbb{N}\}$ which may or may not contain $X \setminus A$.

⇒ Since $X = (X \setminus A) \cup A = (X \setminus A) \cup (\bigcup_{n \in \mathbb{N}} O_n)$, $A \subset \bigcup_{n \in \mathbb{N}} O_n$.

(disregard $X \setminus A$ if $\{O_n \mid n \in \mathbb{N}\}$ contains it)
\( \Rightarrow \{ O_n \mid n \in \mathbb{N} \} \) is a countable subcover of \( \emptyset \) for \( A \).
\( \Rightarrow A \) is a Lindelöf space.

(9) The Bolzano-Weierstrass property is inherited by closed subspaces.

**Proof.** Let \( X \) be a limit point compact space and let \( A \) be a closed subset of \( X \). Let \( B \) be an infinite subset of \( A \).
\( \Rightarrow B \) is an infinite subset of \( X \).
\( \Rightarrow \) Since \( X \) is limit point compact, there exists a limit point \( x \in X \) of \( B \).
\( \Rightarrow \) Since \( B \subset A \), \( x \) is a limit point of \( A \).
\( \Rightarrow \) Since \( A \) is a closed subset of \( X \), \( x \in A \).
\( \Rightarrow B \) has a limit point \( x \) in \( A \).
\( \Rightarrow A \) is a limit point compact space.
2. Give an example of a Lindelöf space that is not second countable.

Proof. Let $X = \mathbb{R}$ with the cofinite topology $\mathcal{T}_f$.

$\Rightarrow$ $X$ is not a second countable space by Example 4.29.

To claim that $X$ is a Lindelöf space, let $\mathcal{O} = \{O_\alpha \in \mathcal{T}_f | \alpha \in \mathcal{A}\}$ be any open cover of $X$ for some index set $\mathcal{A}$.

Choose any $O_{\alpha_0} \in \mathcal{O}$ and write $O_{\alpha_0} = \mathbb{R} \setminus \{x_1, x_2, \cdots, x_n\}$.

$\Rightarrow$ Since $X = \bigcup_{\alpha \in \mathcal{A}} O_\alpha$, there exist $O_{\alpha_i} \in \mathcal{O}$ such that $x_i \in O_{\alpha_i}$.

$\Rightarrow$ $X = \mathbb{R} = O_{\alpha_0} \cup \{x_1, x_2, \cdots, x_n\} \subset \bigcup_{i=0}^{n} O_{\alpha_i}$.

$\Rightarrow$ $\{O_{\alpha_0}, O_{\alpha_1}, O_{\alpha_2}, \cdots, O_{\alpha_n}\}$ is a finite subcover of $\mathcal{O}$.

$\Rightarrow$ $X$ is compact.

$\Rightarrow$ $X$ is Lindelöf. □
3. Prove the following.

(1) Every uncountable subset of the real line $\mathbb{R}$ has a limit point. (Hint: The union of a countable family of finite sets is countable.)

(2) Every uncountable subset of $\mathbb{R}^n$ has a limit point.

Proof. (1) Suppose that there is an uncountable subset $A$ of $\mathbb{R}$ which has no limit points.

$\Rightarrow$ Since $A \cap [-m, m] \subset A$ for each $m \in \mathbb{N}$,

$A \cap [-m, m]$ has no limit point in $[-m, m]$ for each $m \in \mathbb{N}$.

$\Rightarrow$ Since the closed interval $[-m, m]$ is limit point compact,

$A \cap [-m, m]$ is finite for each $m \in \mathbb{N}$.

$\Rightarrow A = \bigcup_{m \in \mathbb{N}} (A \cap [-m, m])$ is at most countable. A contradiction!

$\Rightarrow$ Every uncountable subset of $\mathbb{R}$ has a limit point.

(2) Suppose that there is an uncountable subset $A$ of $\mathbb{R}^n$ which has no limit points. Let $J^m = \prod_{i=1}^{n} [-m, m]$ for each $m \in \mathbb{N}$. 
⇒ Since \((A \cap J^m) \subset A\) for each \(m \in \mathbb{N}\),
\[ A \cap J^m \text{ has no limit point in } J^m \text{ for each } m \in \mathbb{N}. \]
⇒ Since \(J^m\) is limit point compact by Theorem 41, \(A \cap J^m\) is finite for each \(m \in \mathbb{N}\).
⇒ \(A = \bigcup_{m \in \mathbb{N}} (A \cap J^m)\) is at most countable. A contradiction!
⇒ Every uncountable subset of \(\mathbb{R}^n\) has a limit point.

4. (1) Let \((X, d)\) be a metric space. Prove that every subset of \(X\) of diameter less than \(\epsilon\) is contained in an open ball of radius \(\epsilon\), \(\epsilon > 0\).

(2) Give an example to show that a set of diameter less than \(\epsilon\) may not be contained in an open ball of radius \(\frac{\epsilon}{2}\).

\textit{Proof.} (1) Let \(A\) be a subset of a metric space \((X, d)\) with \(\text{diam}(A) < \epsilon\) and choose any \(x \in A\).
⇒ \(A \subset B_d(x, \epsilon)\).
(2) Consider the discrete metric space \((X, d)\) with more than two elements, where

\[
d(x, y) = \begin{cases} 
0 & \text{if } x = y \text{ and } \\
\frac{\epsilon}{2} & \text{otherwise} 
\end{cases}
\]

and let \(x \in X\).

\[\Rightarrow \ \text{diam}(X) = \frac{\epsilon}{2} < \epsilon, \ B_d(x, \frac{\epsilon}{2}) = \{x\} \text{ and } X \text{ is not contained} \]

in \(B_d(x, \frac{\epsilon}{2})\). \hfill \square

5. Prove the following.

(1) A \(T_1\)-space \(X\) is countably compact if and only if it has the Bolzano-Weierstrass property.

(2) A metric space \(X\) is compact if and only if it is countably compact.

Proof. (1) (only if) Assume that \(X\) is a countably compact space and let \(A\) be an infinite subset of \(X\).

Suppose that \(A\) has no limit point in \(X\).

\[\Rightarrow A \text{ is closed in } X.\]
⇒ Since $X$ is countably compact, $A$ is countably compact by Exercise 6.3.1 (2).

But we claim that $A$ is not countably compact and arrives at a contradiction.

(∵) Since $A$ has no limit point, every subset of $A$ has no limit point in $X$.

⇒ $A$ is a closed and discrete subspace of $X$.

Now since $A$ is infinite, there exists a countable subset $B = \{x_n \mid n \in \mathbb{N}\}$ of $A$.

⇒ Since $A$ is discrete, $\mathcal{O} = \{(A \setminus B) \cup \{x_n\} \mid n \in \mathbb{N}\}$ is a countable open cover of $A$.

⇒ $\mathcal{O}$ has no finite subcover because $(A \setminus B) \cup \{x_n\}$ is an only subset of $A$ containing $x_n$ for each $n \in \mathbb{N}$.

⇒ $A$ is not countably compact.
(if) Suppose that $X$ is not countably compact.

$\Rightarrow$ There exists a countable open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ which has no finite subcover.

$\Rightarrow$ For each $n \in \mathbb{N}$, $X \setminus \bigcup_{i=1}^{n} O_i \neq \emptyset$.

$\Rightarrow$ Since $X \setminus O_1 \neq \emptyset$, there exists $x_1 \in X \setminus O_1$.

$\Rightarrow$ Since $x_1 \notin O_1$, there exist $x_2 \in X \setminus (O_1 \cup O_2)$ such that $x_1 \neq x_2$.

$\therefore$ If $X \setminus (O_1 \cup O_2) = \{x_1\}$, then choose $O_{x_1} \in \mathcal{O}$ such that $x_1 \in O_{x_1}$ and then $\{O_1, O_2, O_{x_1}\}$ is a finite subcover of $\mathcal{O}$.

A contradiction!

$\Rightarrow$ Inductively, for each $n \in \mathbb{N}$, there exists $x_n \in X \setminus \bigcup_{i=1}^{n} O_i$ such that $x_n \neq x_i (n > i)$ for all $i = 1, 2, \cdots, n - 1$.

$\Rightarrow$ Since $x_n \neq x_m$ for $n \neq m$, $A = \{x_n \mid n \in \mathbb{N}\}$ is an infinite subset of $X$.

$\Rightarrow$ $A$ has no limit point in $X$. 
(\therefore) Let \( x \in X \).
\[ \Rightarrow \text{Since } \mathcal{O} \text{ covers } X, \text{ there exists } n_0 \in \mathbb{N} \text{ such that } x \in O_{n_0}. \]
\[ \Rightarrow \text{Since } x \in O_n \text{ for all } n \geq n_0, O_{n_0} \cap A \text{ is finite.} \]
\[ \Rightarrow \text{Since } O_{n_0} \text{ is an open set containing } x \text{ and contains finitely many elements of } A \text{ and since } X \text{ is } T_1, \]
\[ x \text{ is not a limit point of } A. \]
\[ \Rightarrow A \text{ has no limit points in } X. \]
\[ \Rightarrow X \text{ is not limit point compact. A contradiction!} \]

(2) (only if) It is clear.

(if) Let \((X, d)\) be a countably compact metric space.
\[ \Rightarrow \text{Since every metric space is Hausdorff by Example 4.58(1),} \]
\[ X \text{ is limit point compact by Exercise 6.3.5(1) above.} \]
\[ \Rightarrow \text{Since } X \text{ is a metric space, } X \text{ is compact by Theorem 38.} \]
6. A topological space $X$ is said to be \textit{sequentially compact} if every sequence in $X$ has a convergent subsequence. For a second countable $T_1$ space $X$, show that the following are equivalent.

1. $X$ is compact.
2. $X$ is limit point compact.
3. $X$ is sequentially compact.
4. $X$ is countably compact.

\textit{Proof.} (1) $\Rightarrow$ (2): It is Theorem 34 for any space $X$.

(2) $\Rightarrow$ (3): Assume that $X$ is second countable and limit point compact. Let $\{x_n\}$ be a sequence in $X$.

$\Rightarrow$ If the set $A = \{x_n \mid n \in \mathbb{N}\}$ is finite, then there exists a constant subsequence of $\{x_n\}$. Assume that the set $A = \{x_n \mid n \in \mathbb{N}\}$ is infinite.

$\Rightarrow$ Since $X$ is limit point compact, $A$ has a limit point $x_0$ in $X$.

$\Rightarrow$ Since $X$ is second countable, it is first countable and hence there
exists a countable local basis \( B_{x_0} = \{B_n \mid n \in \mathbb{N}\} \) at \( x_0 \).

\[ \Rightarrow \] Since \( x_0 \) is a limit point of \( A \), \( \left( \bigcap_{i=1}^{j} B_i \right) \cap (A \setminus \{x_0\}) \neq \emptyset \).

Choose \( x_{n_j} \in \left( \bigcap_{i=1}^{j} B_i \right) \cap (A \setminus \{x_0\}) \) for each \( j \in \mathbb{N} \).

\[ \Rightarrow \] The sequence \( \{x_{n_j} \mid j \in \mathbb{N}\} \) is a subsequence of \( \{x_n\} \) converging to \( x_0 \).

\[ \Rightarrow \] \( X \) is sequentially compact.

(3) \( \Rightarrow \) (4): Assume that \( X \) is a second countable, sequentially compact, \( T_1 \)-space.

\[ \Rightarrow \] Since \( X \) is a \( T_1 \)-space, \( X \) is countably compact if and only if it is limit point compact by Exercise 6.3.5.

So to show that \( X \) is limit point compact, let \( A \) be an infinite subset of \( X \).

\[ \Rightarrow \] There exists a sequence \( \{x_n\} \) consisting all distinct elements of \( A \).

\[ \Rightarrow \] Since \( X \) is sequentially compact, there exists a subsequence \( \{x_{n_i}\} \)
of \{x_n\} converging to some \(x_0 \in X\).
\[\Rightarrow\] Since every neighborhood of \(x_0\) intersects \(\{x_{n_i} \mid i \in \mathbb{N}\}\),
\(x_0\) is a limit point of \(A\).
\[\Rightarrow\] \(X\) is limit point compact.
\[\Rightarrow\] \(X\) is countably compact by Exercise 6.3.5.

(4) \(\Rightarrow\) (1): Assume that \(X\) is second countable and countably compact.
\[\Rightarrow\] Since \(X\) is second countable, \(X\) is Lindelöf by Theorem 32.
\[\Rightarrow\] Since \(X\) is countably compact, \(X\) is compact by Theorem 31. \(\Box\)

7. Prove that a subset \(A\) of \(\mathbb{R}^n\) is compact if and only if every nested sequence \(\{A_n\}_n\) of relatively closed, non-empty subsets of \(A\) has non-empty intersection.
Proof. Let $A$ be a subset of $\mathbb{R}$.

(only if) Assume that $A$ is compact and let $\{A_n\}_{n=1}^{\infty}$ be a nested sequence of relatively closed, non-empty subsets of $A$.

Suppose that $\bigcap_{n=1}^{\infty} A_n = \emptyset$.

$\Rightarrow$ $A$ is closed in $\mathbb{R}$ by Theorem 28.

$\Rightarrow$ Since $A_n$ is closed in $A$, $A \setminus A_n$ is open in $A$ and

$$\bigcup_{n=1}^{\infty} (A \setminus A_n) = A \setminus \bigcap_{n=1}^{\infty} A_n = A \setminus \emptyset = A.$$

$\Rightarrow \{A \setminus A_n \mid n \in \mathbb{N}\}$ is an open cover of $A$.

$\Rightarrow$ Since $A$ is compact, there exist $n_1, n_2, \cdots, n_k \in \mathbb{N}$ such that

$$A = \bigcup_{i=1}^{k} (A \setminus A_{n_i}) = A \setminus \bigcap_{n=1}^{\infty} A_{n_i}.$$

Let $n_0 = \max\{n_1, n_2, \cdots, n_k\}$.

$\Rightarrow$ Since $\{A_n\}$ is nested, $A_{n_0} = \bigcap_{n=1}^{\infty} A_{n_i} = \emptyset$. A contradiction!

$\Rightarrow \bigcap_{n=1}^{\infty} A_n \neq \emptyset$. 
(if) Assume the condition. Since $A$ is compact if and only if $A$ is countably compact by Theorem 41, we will show that $A$ is countably compact. Suppose that $A$ is not countably compact.

$\Rightarrow$ There exists a countable open cover $\mathcal{O} = \{O_n \mid n \in \mathbb{N}\}$ which has no finite subcover. For each $n \in \mathbb{N}$, let $K_n = A \setminus (\bigcup_{i=1}^{n} O_i)$.

$\Rightarrow \{K_n\}_{n=1}^{\infty}$ is a nested sequence of relatively closed, non-empty subsets of $A$.

$\Rightarrow \bigcap_{n=1}^{\infty} K_n \neq \emptyset$ by assumption.

$\Rightarrow$ There exists $x \in \bigcap_{n=1}^{\infty} K_n$. i.e., $x \in K_n$ for all $n \in \mathbb{N}$.

$\Rightarrow$ Since $\mathcal{O}$ covers $X$, $x \in O_{n_0}$ for some $n_0 \in \mathbb{N}$.

$\Rightarrow x \notin K_n$ for all $n \geq n_0$ by construction of $K_n$. A contradiction!

$\Rightarrow A$ is countably compact.

$\Rightarrow A$ is compact by Theorem 41.

$\square$
8. Prove that if a metric space \((X, d)\) has an \(\epsilon\)-net for some positive number \(\epsilon\), then \((X, d)\) is bounded. Conclude that every totally bounded metric space is bounded.

Proof. (1) Let \(A_\epsilon\) be an \(\epsilon\)-net for a metric space \((X, d)\) and let

\[ \mathcal{O} = \{B_d(x, \epsilon) \mid x \in A_\epsilon\}. \]

\[ \Rightarrow X = \bigcup_{x \in A_\epsilon} B_d(x, \epsilon). \]

Put \(\delta = \text{diam}\left(\bigcup_{x \in A_\epsilon} B_d(x, \epsilon)\right)\) and let \(x_0 \in X\).

\[ \Rightarrow X \subset B_d(x_0, \delta). \]

(2) Assume that a metric space \((X, d)\) is totally bounded.

\[ \Rightarrow \text{There exists an } \epsilon\text{-net } A_\epsilon \text{ for any } \epsilon, \text{ say for example } \epsilon = 1. \]

\[ \Rightarrow (X, d) \text{ is bounded by (1) above.} \]

Thus every totally bounded metric space is bounded. \(\Box\)
9. Prove that every compact metric space is totally bounded, separable, and second countable.

Proof. Let \((X, d)\) be a compact metric space.

(1) Every compact metric space is totally bounded.

\((\because)\) Let \(\epsilon > 0\) be given.

\[ \Rightarrow \text{Since } (X, d) \text{ is limit point compact by Theorem 38, there exists an } \epsilon \text{-net } A_\epsilon \text{ by Lemma 37 } (X = \bigcup_{x \in A_\epsilon} B_d(x, \epsilon)). \]

Thus every compact metric space is totally bounded.

(2) Every compact metric space is separable.

\((\because)\) Since \((X, d)\) is totally bounded by (1) above, there exists an \(\frac{1}{n}\)-net \(A_{\frac{1}{n}}\) for each \(n \in \mathbb{N}\).

\[ \Rightarrow \text{Since } A_{\frac{1}{n}} \text{ is finite for each } n \in \mathbb{N}, A = \bigcup_{n \in \mathbb{N}} A_{\frac{1}{n}} \text{ is countable.} \]

To show that \(A\) is dense in \(X\), let \(x \in X\) and let \(U\) be an open subset of \(X\) containing \(x\).
There exists \( r > 0 \) such that \( B_d(x, r) \subset U \).

Choose \( m \in \mathbb{N} \) such that \( \frac{1}{m} < r \).

\[ \Rightarrow \] Since \( B_d(x, \frac{1}{m}) \cap A \neq \emptyset \), \( B_d(x, r) \cap A \neq \emptyset \) and hence \( U \cap A \neq \emptyset \).

Thus every compact metric space is separable.

(3) Every compact metric space is second countable.

\( \therefore \) Since \( (X, d) \) is a compact metric space, \( (X, d) \) is separable by (2) above.

\( \Rightarrow \) \( (X, d) \) is second countable by Theorem 4.27 (2).

Thus every compact metric space is second countable. \( \square \)
10. Prove the following.

(1) Completeness of metric spaces is not a topological invariant.

(2) Every compact metric space is complete.

(3) A metric space is compact if and only if it is complete and totally bounded. (Hint: The hard part is to prove that a complete and totally bounded metric space $X$ is compact. This can be done as follows by showing that $X$ has the Bolzano-Weierstrass property: Let $A$ be an infinite subset of $X$. Since $X$ is totally bounded, some infinite subset $A_1$ of $A$ must be contained in an open ball of radius 1. Show that there is a nested sequence $\{A_n\}_{n=1}^{\infty}$ of infinite subsets of $A$ such that $A_n$ is contained in a ball of radius $\frac{1}{n}$. For each $n$, let $x_n \in A_n$. Then $\{x_n\}_{n=1}^{\infty}$ is a Cauchy sequence.)

Proof. (1) Consider the real line $\mathbb{R}$ with the usual topology and its subspace $(-1, 1) \subset \mathbb{R}$.  

(i) $\mathbb{R}$ and $(0, 1)$ are homeomorphic under the homeomorphism $h : (-1, 1) \rightarrow \mathbb{R}$ defined by $h(x) = \frac{x}{1 - |x|}$ for each $x \in (-1, 1)$.

(ii) $\mathbb{R}$ is complete by Example 3.58(1).

(iii) $(-1, 1)$ is not complete because the sequence $\{1 - \frac{1}{n}\}$ is a Cauchy sequence in $(-1, 1)$ which does not converge in $(-1, 1)$.

(2) Let $(X, d)$ be a compact metric space.

Suppose that $(X, d)$ is not complete.

$\Rightarrow$ There exists a Cauchy sequence $\{x_n\}$ in $X$ such that $\{x_n\}$ does not converge.

$\Rightarrow \{x_n \mid n \in \mathbb{N}\}$ is an infinite subset of $X$ which has no limit point in $X$. (Prove this!)

$\Rightarrow (X, d)$ is not limit point compact.

$\Rightarrow (X, d)$ is not compact by Theorem 38. A contradiction!

$\Rightarrow (X, d)$ is complete.
(3) **(only if)** Let \((X, d)\) be a compact metric space.

Then \((X, d)\) is complete by (b) above. To show that \(X\) is totally bounded, let \(\epsilon > 0\) be given.

\[ \Rightarrow \text{Since } \{B_d(x, \epsilon) \mid x \in X\} \text{ is an open cover of } X, \text{ there exist finitely many points } x_1, x_2, \cdots, x_n \in X \text{ such that } X = \bigcup_{i=1}^{n} B_d(x_i, \epsilon). \]

\[ \Rightarrow A_\epsilon = \{x_1, x_2, \cdots, x_n\} \text{ is an } \epsilon\text{-net for } X. \]

\[ \Rightarrow (X, d) \text{ is totally bounded.} \]

**_(if)_** Assume that \((X, d)\) is a complete and totally bounded metric space.

\[ \Rightarrow \text{There exists an } \frac{1}{n} \text{-net } A_{\frac{1}{n}} \text{ for } X, \text{ for each } n \in \mathbb{N}. \]

\[ \Rightarrow X = \bigcup_{x \in A_{\frac{1}{n}}} B_d(x, \frac{1}{n}), \text{ and } A_{\frac{1}{n}} \text{ is finite for each } n \in \mathbb{N}. \]

To show that \((X, d)\) is limit point compact, let \(A\) be an infinite subset of \(X\).
Since $A_1$ is finite, there exists an infinite subset $A_1$ of $A$ such that

$$A_1 \subset B_d(a_1, \frac{1}{1}) \text{ for some } a_1 \in A_1.$$

Since $A_{\frac{1}{2}}$ is finite, there exists an infinite subset $A_2$ of $A_1$ such that $A_2 \subset B_d(a_2, \frac{1}{2})$ for some $a_2 \in A_{\frac{1}{2}}$.

Continuing this process inductively, if we have an infinite subset $A_{n-1}$ of $A_{n-2}$ such that $A_{n-1} \subset B_d(a_{n-1}, \frac{1}{n-1})$ for some $a_{n-1} \in A_{\frac{1}{n-1}}$, then since $A_{\frac{1}{n}}$ is finite, there exists an infinite subset $A_n$ of $A_{n-1}$ such that $A_n \subset B_d(a_n, \frac{1}{n})$ for some $a_n \in A_{\frac{1}{n}}$.

Thus we have an infinite subset $A_n$ of $A$ for each $n \in \mathbb{N}$ such that $A_1 \supset A_2 \supset \cdots \supset A_{n-1} \supset A_n \supset \cdots$ and $A_n \subset B_d(a_n, \frac{1}{n})$ for some $a_n \in A_{\frac{1}{n}}$ (Note that diam($A_n$) < $\frac{2}{n}$).

Choose $x_n \in A_n$ for each $n \in \mathbb{N}$.

$\Rightarrow \{x_n\}$ is a Cauchy sequence in $(X,d)$. 
\[
  \begin{align*}
  \text{(\therefore)} & \quad \text{Let } \epsilon > 0 \text{ be given. Take } n_0 \in \mathbb{N} \text{ such that } \frac{1}{n_0} < \frac{\epsilon}{2}.
  \\
  \quad \Rightarrow & \quad \text{For } m, n \geq n_0, \text{ since } x_n, x_m \in A_{n_0}, d(x_n, x_m) < \frac{\epsilon}{n_0} < \epsilon.
  \\
  \Rightarrow & \quad \text{Since } (X, d) \text{ is complete, } \{x_n\} \text{ converges to some point } x \in X.
  \\
  \Rightarrow & \quad \text{Since } x_n \in A \text{ for each } n \in \mathbb{N}, x \text{ is a limit point of } A.
  \\
  \Rightarrow & \quad (X, d) \text{ is limit point compact}.
  \\
  \Rightarrow & \quad \text{Since } (X, d) \text{ is a metric space, } X \text{ is compact by Theorem 38.} \quad \Box
\end{align*}
\]
11. Let \( \{C_{\alpha} \mid \alpha \in \mathcal{A}\} \) be a family of closed subsets of a compact metric space \( X \) such that \( \bigcap_{\alpha \in \mathcal{A}} C_{\alpha} = \emptyset \). Prove that there is a positive number \( \epsilon \) such that every subset of \( X \) of diameter less than \( \epsilon \) fails to intersect at least one member of \( \{C_{\alpha} \mid \alpha \in \mathcal{A}\} \).

**Proof.** Let \((X, d)\) be a compact metric space and let \( \{C_{\alpha} \mid \alpha \in \mathcal{A}\} \) be a family of closed subsets of \( X \) such that \( \bigcap_{\alpha \in \mathcal{A}} C_{\alpha} = \emptyset \).

Let \( U_{\alpha} = X \setminus C_{\alpha} \) for each \( \alpha \in \mathcal{A} \).

\( \Rightarrow \) Since \( \bigcap_{\alpha \in \mathcal{A}} C_{\alpha} = \emptyset \), \( \{U_{\alpha} \mid \alpha \in \mathcal{A}\} \) is an open cover of \( X \).

\( \Rightarrow \) Since \( X \) is a compact metric space, there exists a Lebesgue number \( \epsilon \) for \( \{U_{\alpha} \mid \alpha \in \mathcal{A}\} \) by Theorem 40.

\( \Rightarrow \) If \( A \subset X \) of diameter less than \( \epsilon \), then \( A \subset U_{\alpha} \) for some \( \alpha \in \mathcal{A} \) and hence \( A \cap C_{\alpha} = \emptyset \).

12. Use a Lebesgue number argument to make a new proof of Theorem 26.
Theorem 26. Let \((X, d)\) be a compact metric space and let \((Y, d')\) be a metric space. Let \(f : (X, d) \to (Y, d')\) be a continuous function. Then \(f : (X, d) \to (Y, d')\) is uniformly continuous.

Proof. Let \((X, d)\) be a metric space and let \((Y, d')\) be a metric space.

Let \(f : (X, d) \to (Y, d')\) be continuous and let \(\epsilon > 0\) be given.

\[\Rightarrow\] For each \(x \in X\), there exists \(\delta_x > 0\) such that \(f(B_d(x, \delta_x)) \subset B_{\delta'}(f(x), \frac{\epsilon}{2})\).

\[\Rightarrow\] Since \(X\) is compact, there exists a Lebesgue number \(\delta\) for the open cover \(\{B_d(x, \delta_x) \mid x \in X\}\) by Theorem 40.

\[\Rightarrow\] For each \(a \in X\), there exists \(x_a \in X\) such that \(B_d(a, \delta) \subset B_d(x_a, \delta_{x_a})\).

Now let \(a, b \in X\) with \(d(a, b) < \delta\).

\[\Rightarrow a, b \in B_d(a, \delta).\]

\[\Rightarrow f(a), f(b) \in f(B_d(a, \delta)) \subset f(B_d(x_a, \delta_{x_a})) \subset B_{\delta'}(f(x_a), \frac{\epsilon}{2}).\]

\[\Rightarrow d'(f(a), f(b)) \leq d'(f(a), f(x_a)) + d'(f(x_a), f(b)) < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.\]

\[\Rightarrow f : (X, d) \to (Y, d')\) is uniformly continuous. \(\square\)
13. Let $X$ denote the real line with the half-open interval topology defined in the proof of Example 4.31. Show that $X$ is a Lindelöf space.

Proof. Let $X$ denote the real line with the lower limit topology $\mathcal{T}_l$ generated by the basis $\mathcal{B} = \{[a, b) \subset \mathbb{R} \mid a < b\}$, defined in the proof of Example 4.31 and let $Y$ denote the real line with the usual topology $\mathcal{T}$ generated by the basis $\mathcal{B} = \{(a, b) \subset \mathbb{R} \mid a < b\}$. Note that every open set in $(Y, \mathcal{T})$ is open in the Sorgenfrey line $(X, \mathcal{T})$.

To show that $X$ is Lindelöf, it is sufficient to show that every open cover of $(X, \mathcal{T}_l)$ by basic open sets in the Sorgenfrey topology has a countable subcover. Let $\{[x_\alpha, y_\alpha) : \alpha \in \mathcal{A}\}$ be an open cover of $(X, \mathcal{T}_l)$ in the Sorgenfrey topology by basic open sets.

$\Rightarrow$ Since every subset of $(Y, \mathcal{T})$ is second countable, every subset of $(Y, \mathcal{T})$ is Lindelöf by Theorem 32. Consider $U = \bigcup_{\alpha \in \mathcal{A}} (x_\alpha, y_\alpha) \subset Y$.

$\Rightarrow$ Since $(Y, \mathcal{T})$ is Lindelöf, $U$ is Lindelöf as a subspace of $(Y, \mathcal{T})$. 

117
⇒ Since \( \{ (x_\alpha, y_\alpha) : \alpha \in \mathcal{A} \} \) is an open cover of \( U \) by open sets in \( (Y, \mathcal{T}) \), there exist countably many \( \alpha_1, \alpha_2, \cdots, \alpha_n, \cdots \in \mathcal{A} \) such that \( U = \bigcup_{i=1}^{\infty} (x_{\alpha_i}, y_{\alpha_i}) \).

⇒ \( U \subset \bigcup_{i=1}^{\infty} [x_{\alpha_i}, y_{\alpha_i}) \).

We claim that \( Y \setminus U \) is countable.

(∵) Let \( x \in Y \setminus U \).
⇒ \( x = x_\alpha \) for some \( \alpha \in \mathcal{A} \).
⇒ There exists \( q_x \in \mathbb{Q} \) such that \( x = x_\alpha < q_x < y_\alpha \) by Thm 2.8.
⇒ Since \( (x_\alpha, y_\alpha) \subset U \), \( (x, q_x) \subset U \).
⇒ The correspondence \( x \to q_x \) is 1-1.

(∵ Let \( x \neq x' \) in \( [0, 1) \setminus U \), say \( x = x_\alpha < x' = x_{\alpha'} \).
⇒ If \( q_{x'} \leq q_x \), then since \( x < x' = x_{\alpha'} < q_{x'} \leq q_x \),
\( x' \in (x, q_x) \subset U \).

But \( x' \notin U \). This contradiction implies that \( q_{x'} > q_x \).)
Since \( \mathbb{Q} \) is countable, \( Y \setminus U \) is countable.

Now for every \( a \in Y \setminus U \) pick an element 
\[
[x_{\alpha_a}, y_{\alpha_a}] \in \{(x_{\alpha}, y_{\alpha}) : \alpha \in \mathcal{A}\} \text{ such that } a \in [x_{\alpha_a}, y_{\alpha_a}].
\]

\( \Rightarrow \) \( \{[x_{\alpha_i}, y_{\alpha_i}] \mid i \in \mathbb{N}\} \cup \{[x_{\alpha_a}, y_{\alpha_a}] \mid a \in Y \setminus U\} \) forms a countable subcover of \( \{(x_{\alpha}, y_{\alpha}) : \alpha \in \mathcal{A}\} \).

\( \Rightarrow (X, \mathcal{T}_l) \) is a Lindelöf space. \(\square\)

14. Prove that second countability, separability, and the Lindelöf property are equivalent for a metric space.

**Proof.** (1) Every second countable space is separable by Theorem 4.26.
(2) Every separable space is second countable by Theorem 4.27 (2).
(3) Every second countable space is Lindelöf by Theorem 32.
(4) We will show that every Lindelöf space is second countable.

Let \( X \) be a Lindelöf space and consider an open cover 
\[
\mathcal{O}_n = \{B_d(x, \frac{1}{n}) \mid x \in X\} \text{ for each } n \in \mathbb{N}.
\]
⇒ There exists a countable subcover \( \mathcal{B}_n \) of \( \mathcal{O}_n \) for each \( n \in \mathbb{N} \).

Let \( \mathcal{B} = \bigcup_{n \in \mathbb{N}} \mathcal{B}_n \).

⇒ Since \( \mathcal{B}_n \) is countable for each \( n \in \mathbb{N} \), \( \mathcal{B} \) is countable.

To claim that \( \mathcal{B} \) is a basis for \( X \), let \( x \in X \) and let \( U \) be an open subset of \( X \) containing \( x \).

⇒ There exists \( r > 0 \) such that \( B_d(x, r) \subset U \).

⇒ Choose \( n_0 \in \mathbb{N} \) such that \( \frac{1}{n_0} < \frac{r}{2} \).

⇒ Since \( \mathcal{B}_{n_0} \) is an open cover of \( X \), there exists \( B_d(x_0, \frac{1}{n_0}) \in \mathcal{B}_{n_0} \)

such that \( x \in B_d(x_0, \frac{1}{n_0}) \).

⇒ \( B_d(x_0, \frac{1}{n_0}) \subset B_d(x, r) \subset U \) and hence \( \mathcal{B} \) is a basis for \( X \).
\[
\begin{align*}
\text{(\because)} & \quad \text{Let } y \in B_d(x_0, \frac{1}{n_0}). \\
\Rightarrow & \quad d(x, y) \leq d(x, x_0) + d(x_0, y) < \frac{1}{n_0} + \frac{1}{n_0} = \frac{2}{n_0} < r. \\
\Rightarrow & \quad y \in B_d(x, r).
\end{align*}
\]

\[\Rightarrow \text{Since } \mathcal{B} \text{ is countable, } \mathcal{B} \text{ is a countable basis for } X.\]
\[\Rightarrow X \text{ is second countable.} \quad \Box\]

15. Recall that every second countable space is separable by Theorem ?? and Lindelöf by Theorem 32. Give an example of a separable Hausdorff space that is not second countable.

\textit{Proof.} Let } X \text{ denote the real line with the lower limit topology generated by the basis } \mathcal{B} = \{[a, b) \subset \mathbb{R} \mid a < b\}, \text{ defined in the proof of Example 4.31.}
\[\Rightarrow X \text{ is a separable Hausdorff space which is not second countable by Exercise 4.3.6 (2).} \quad \Box\]
16. Prove the following.

(1) Every compact space is countably compact.

(2) Countable compactness is a topological property.

(3) (A challenging problem) There are countably compact spaces that are not compact.

Proof. (1) is clear from the definitions.

(2) Let $X$ be a countably compact space and let $f : X \to Y$ be a homeomorphism. Let $\mathcal{O}$ be a countable open cover of $Y$.

$\Rightarrow$ Since $f : X \to Y$ is a homeomorphism, $\{f^{-1}(O) \mid O \in \mathcal{O}\}$ is a countable open cover of $X$.

$\Rightarrow$ Since $X$ is countably compact, there exists a finite subcover $\{f^{-1}(O_1), f^{-1}(O_2), \cdots, f^{-1}(O_n)\}$ of $\{f^{-1}(O) \mid O \in \mathcal{O}\}$.

$\Rightarrow$ Since $f : X \to Y$ is a homeomorphism, $\{O_1, O_2, \cdots, O_n\}$ is a finite subcover of $\mathcal{O}$.

$\Rightarrow$ $Y$ is a countably compact space.
(3) (i) $\beta(\mathbb{N}) \setminus \{p\}$ is a countably compact non-compact space, where $p \in \beta(\mathbb{N}) \setminus \mathbb{N}$ and $\beta(\mathbb{N})$ is the Stone-Čech compactification of $\mathbb{N}$.

(ii) $\{0, 1\}^\mathbb{R} \setminus \{\text{a point}\}$ is a countably compact non-compact space, where $\{0, 1\}^\mathbb{R}$ is the product space.

(iii) The first uncountable ordinal space $\Omega$ in the order topology is a countably compact, non-compact Hausdorff space. $\Omega$ is not compact, we will show that $\Omega$ is countably compact.

$\therefore$ Let $\mathcal{U} = \{U_j \mid j \in \mathbb{N}\}$ be a countable open cover of $\Omega$.

$\Rightarrow$ There exists $U_k \in \mathcal{U}$ such that $U_k$ is unbounded.

(Otherwise the union or supremum of the upper bounds would be in $\Omega$ and then $\mathcal{U}$ can not cover $\Omega$.)

$\Rightarrow$ $U_k$ contains an unbounded interval $I$.

$\Rightarrow$ $I$ has a least element $\eta$, which is a successor ordinal.

(Prove this!)

$\Rightarrow$ $\Omega \setminus I = \{\xi \mid \xi < \eta\} = \eta$ which is a compact ordinal
because $\eta$ is a successor ordinal. \\
$\Rightarrow$ There exist some finite number $U_{j_1}, U_{j_2}, \cdots, U_{j_n} \in \mathcal{U}$ such that $\eta \subset \bigcup_{i=1}^{i=n} U_{j_i}$. \\
$\Rightarrow \{U_{j_1}, U_{j_2}, \cdots, U_{j_n}, U_k\}$ will cover $\Omega$. \\
$\Rightarrow \Omega$ is countably compact. \hfill $\square$

17. Does the topologist’s sine curve $T$ of Example 5.15(2) have the fixed point property? Justify your answer.

**Answer.** $T$ has the fixed point property. 

**Proof.** Let $T$ be the topologist’s sine curve in Example 5.15, where $T = A \cup B$ is a subspace of $\mathbb{R}^2$ with 

$A = \{(0, y) \in \mathbb{R}^2 \mid -1 \leq y \leq 1\}$ \n
and $B = \{(x, y) \in \mathbb{R}^2 \mid 0 < x \leq 1, y = \sin \frac{\pi}{x}\}$. 

124
\[ T = A \cup B \]

where

\[ A = \{(0, y) \in \mathbb{R} \mid -1 \leq y \leq 1\} \]

and

\[ B = \{(x, \sin \frac{\pi}{x}) \in \mathbb{R} \mid 0 < x \leq 1\} \]

Figure 5.1': The topologist’s sine curve \( T \)

Note that the interval \((0, 1]\) is homeomorphic with \( B \) under the homeomorphism \( g : (0, 1] \to B \) defined by \( g(x) = (x, \frac{\pi}{x}) \) \((x \in (0, 1])\) whose inverse is the restriction \( p_1|_B : B \to (0, 1] \) of the first coordinate projection \( p_1 : T \to [0, 1] \), defined by \( p_1(x, y) = x \)
for each \((x, y) \in T\). Also note that since \(T\) is closed and bounded in \(\mathbb{R}^2\), \(T\) is compact by Theorem 28.

Now let \(f : T \to T\) be a continuous function.

\(\Rightarrow\) Since \(A\) and \(B\) are path components of \(T\) by Example 5.39,
\(f(A) \subset A\) or \(f(A) \subset B\).

(Case 1) \(f(A) \subset A\).

\(\Rightarrow\) Since \(A\) is homeomorphic with a closed interval and an interval has the fixed point property by Example 5.33, \(f|_A : A \to A\) has a fixed point by Theorem 5.32.

\(\Rightarrow\) \(f : T \to T\) has a fixed point.

(Case 2) \(f(A) \subset B\).

\(\Rightarrow\) \(f(B) \cap A = \emptyset\) and hence \(f(B) \subset B\) and \(f(T) \subset B\).

(\(\because\)) Suppose that \(f(B) \cap A \neq \emptyset\).

\(\Rightarrow\) Since \(A\) and \(f(B)\) are path connected, \(A \cup f(B)\) is path connected by Theorem 5.44.

\(\Rightarrow\) Since \(A\) is a path component by Example 5.39, \(A \cup f(B) \subset A\).
⇒ $f(B) \subseteq A$.
⇒ Since $f : T \to T$ is continuous, $f(T) = f(\overline{B}) \subseteq \overline{f(B)} \subseteq \overline{A} = A$
    by applying Theorem 4.39.
⇒ $f(A) \subseteq A$. It contradicts to the assumption that $f(A) \subseteq B$.
⇒ Since $B$ is connected, $f(B)$ is a connected subset of $B$ and hence 
    $g^{-1}(f(B)) = p_1(f(B))$ is a connected subset of $(0, 1]$ 
    by Corollary 5.9.
⇒ $p_1(f(B))$ is an interval $I$ with the end points $a$ and $b$ such that 
    $0 \leq a \leq b \leq 1$ by Theorem 5.27.
On the other hand, since $f : T \to B$ and $p_1|_B : B \to (0, 1]$ are 
    continuous, $f(T)$ is a compact subset of $B$ and 
    $p_1 \circ f(T) \subseteq p_1(B) \subseteq (0, 1]$ is a compact subset of $(0, 1]$ 
    by Corollary 18.
⇒ $p_1 \circ f(T)$ is the closed interval $[a, b] \subseteq (0, 1]$ with $0 < a$. 

127
Consider the continuous function
\[ h = p_1|_{f(B)} \circ f|_{B} \circ g|_{[a,b]} : [a, b] \to [a, b], \]

where the following diagram commutes and the functions in the diagram are well defined;

\[
\begin{array}{ccc}
[a, b] & \xrightarrow{h} & [a, b] \\
g|_{[a,b]} & & \uparrow p_1|_{f(B)} \\
B & \xrightarrow{f|_{B}} & f(B)
\end{array}
\]

⇒ Since the closed interval \([a, b]\) has the fixed point property by Example 5.33, there exists \(x_0 \in [a, b]\) such that \(h(x_0) = x_0\).

⇒ \(g(x_0) = (x_0, \frac{\pi}{x_0}) \in B\) and

\[
f(x_0, \frac{\pi}{x_0}) = f(g(x_0)) = 1_B \circ f(g(x_0)) = (g \circ p_1) \circ f(g(x_0))
\]
\[ = g \circ (p_1 \circ f \circ g)(x_0) = g \circ h(x_0) = g(x_0) = (x_0, \frac{\pi}{x_0}). \]

\[ \Rightarrow (x_0, \frac{\pi}{x_0}) \in B \text{ is a fixed point of } f : T \to T. \]

\[ \Rightarrow f : T \to T \text{ has a fixed point.} \]

\[ \Rightarrow \text{The topologist’s sine curve } T \text{ has the fixed point property.} \]
4 One-point Compactification

- In this section, we consider one construction for answering the question
  “When can a topological space be considered to be a subspace of a compact topological space?”
- We shall see that the question can always be answered affirmatively by adding one additional point.
- It will be noted that this construction is of little value, however, unless the given space satisfies a local compactness condition.

Definition 46. Let $X$ be a topological space and $x \in X$.

1. $X$ is said to be *locally compact at $x$* if there exists an open set $U$ in $X$ such that $x \in U$ and $\overline{U}$ is compact.

2. $X$ is said to be *locally compact* if $X$ is locally compact at every point of $X$. 
Remark 47. (1) Local compactness is a topological property.
(2) Every compact space is locally compact.
(3) $X$ is locally compact at $x \in X$ if and only if there exists a local basis $\{B_\alpha \mid \alpha \in I\}$ at $x$ such that $\overline{B}_\alpha$ is compact for each $\alpha \in I$.

Example 48. (1) $\mathbb{R}^n$ is locally compact.
(2) The Hilbert space $\mathbb{H}$ is not locally compact.

Proof. (1) is clear. (2) Suppose that $\mathbb{H}$ is locally compact.
   $\Rightarrow \mathbb{H}$ is locally compact at the origin $0 = (0, 0, \cdots, 0, \cdots) \in \mathbb{H}$.
   $\Rightarrow \exists$ an open set $U$ in $\mathbb{H}$ such that $0 \in U$ and $\overline{U}$ is compact.
   $\Rightarrow$ There exists $r > 0$ such that $B(0, r) \subset U$.
   $\Rightarrow$ Since $\overline{B(0, r)} = B[0, r] \subset \overline{U}$ and $\overline{U}$ is compact, $B[0, r]$ is compact by Theorem 11.
⇒ Since $\mathbb{H}$ is metrizable, $B[0, r]$ is limit point compact by Thm 34.

But $B[0, r]$ is not limit point compact. A contradiction!

(∵) Let $p_1 = (r, 0, 0, \cdots, 0, \cdots)$,

\[ p_2 = (0, r, 0, \cdots, 0, \cdots), \]

\[ \vdots \]

\[ p_n = (0, \cdots, 0, r, 0, \cdots), \]

\[ \vdots \]

⇒ (i) $\{p_n \mid n \in \mathbb{N}\}$ is an infinite subset of $B[0, r]$ and

(ii) since $d(p_i, p_j) = \sqrt{2}r (i \neq j)$, $\{p_n \mid n \in \mathbb{N}\}$ has no limit point in $B[0, r]$.

⇒ $B[0, r]$ is not limit point compact.

⇒ $\mathbb{H}$ is not locally compact. \qed
Definition 49. Let $X$ be a topological space and choose $\infty \notin X$. ($\infty$ is called the point at infinity.)

Let $X_\infty = X \cup \{\infty\}$ and define a topology $\mathcal{T}_\infty$ on $X_\infty$ as follows: For $U \subset X_\infty$, $U \in \mathcal{T}_\infty$ will be determined by the following two conditions.

(a) If $\infty \notin U$, then $U \in \mathcal{T}_\infty$ if and only if $U$ is open in $X$.

(b) If $\infty \in U$, then $U \in \mathcal{T}_\infty$ if and only if $X_\infty \setminus U$ is a closed and compact subset of $X$.

Thus, $\mathcal{T}_\infty = \{U \subset X_\infty \mid U$ is open in $X$ and $\infty \notin U\}$

$$\cup \{U \subset X_\infty \mid X_\infty \setminus U$ is closed and compact in $X$, $\infty \in U\}.$$

Then $\mathcal{T}_\infty$ is a topology for $X_\infty$ (See Proposition 50).

The topological space $(X_\infty, \mathcal{T}_\infty)$ is called the one-point compactification of $X$. 
Proposition 50. $\mathcal{I}_\infty$ is a topology for the set $X_\infty$.

Proof. Note that if $\infty \in U \subset X_\infty$, then $X_\infty \setminus U = X \setminus U$.

(1) (i) Since $\emptyset$ is open in $X$, $\emptyset \in \mathcal{I}_\infty$.

(ii) Since $X_\infty \setminus X_\infty = \emptyset$ is closed and compact in $X$, $X_\infty \in \mathcal{I}_\infty$.

(2) Let $U_\alpha \in \mathcal{I}_\infty$, for $\alpha \in \mathcal{A}$.

$\Rightarrow$ (i) Assume that $\infty \in \bigcup_{\alpha \in \mathcal{A}} U_\alpha$.

$\Rightarrow$ There exists $\alpha_0 \in \mathcal{A}$ such that $\infty \in U_{\alpha_0}$.

$\Rightarrow$ $X_\infty \setminus U_{\alpha_0}$ is a closed and compact subset of $X$

and $X \setminus U_\alpha$ is closed in $X$ for each $\alpha \in \mathcal{A}$.

$\Rightarrow$ Since $X_\infty \setminus \bigcup_{\alpha \in \mathcal{A}} U_\alpha = \bigcap_{\alpha \in \mathcal{A}} (X_\infty \setminus U_\alpha) \subset X_\infty \setminus U_{\alpha_0}$,

$X_\infty \setminus \bigcup_{\alpha \in \mathcal{A}} U_\alpha$ is a closed and compact subset of $X$. 
⇒ \bigcup_{\alpha \in \mathcal{A}} U_{\alpha} \in \mathcal{T}_\infty. \text{ Thus } \bigcup_{\alpha \in \mathcal{A}} U_{\alpha} \in \mathcal{T}_\infty \text{ if } \infty \in \bigcup_{\alpha} U_{\alpha}.

(ii) Assume that \infty \notin \bigcup_{\alpha \in \mathcal{A}} U_{\alpha}.

⇒ \infty \notin U_{\alpha}, \text{ for all } \alpha \in \mathcal{A}.

⇒ U_{\alpha} \in \mathcal{T} \text{ for all } \alpha \in \mathcal{A}.

⇒ \bigcup_{\alpha \in \mathcal{A}} U_{\alpha} \in \mathcal{T}. \text{ Thus } \bigcup_{\alpha \in \mathcal{A}} U_{\alpha} \in \mathcal{T}_\infty \text{ if } \infty \notin \bigcup_{\alpha \in \mathcal{A}} U_{\alpha}.

⇒ \bigcup_{\alpha \in \mathcal{A}} U_{\alpha} \in \mathcal{T}_\infty \text{ by (i) and (ii) above.}

The union of any family of members of \mathcal{T}_\infty \text{ is a member of } \mathcal{T}_\infty.

(3) Let \(U_1, U_2 \in \mathcal{T}_\infty\).

(i) Assume that \infty \in U_1 \cap U_2. \text{ i.e., } \infty \in U_1 \text{ and } \infty \in U_2.

⇒ X_{\infty} \setminus U_1 \text{ and } X_{\infty} \setminus U_2 \text{ are closed and compact subsets of } X.

⇒ X_{\infty} \setminus (U_1 \cap U_2) = (X_{\infty} \setminus U_1) \cup (X_{\infty} \setminus U_2) \text{ is a closed, compact subset of } X.
\( \Rightarrow U_1 \cap U_2 \in \mathcal{T}_\infty. \)

(ii) Assume that \( \infty \notin U_1 \cap U_2 \). Say \( \infty \notin U_1 \) and \( \infty \in U_2 \).

\[ \Rightarrow X \setminus (U_1 \cap U_2) = (X \setminus U_1) \cup (X \setminus U_2) = (X \setminus U_1) \cup (X_\infty \setminus U_2) \] is closed in \( X \).

\[ \Rightarrow U_1 \cap U_2 \in \mathcal{T}. \]

\[ \Rightarrow U_1 \cap U_2 \in \mathcal{T}_\infty. \]

Similarly we can show that \( U_1 \cap U_2 \in \mathcal{T} \)

for \( \infty \in U_1 \) and \( \infty \notin U_2 \) or \( \infty \notin U_1 \) and \( \infty \notin U_2 \).

\[ \Rightarrow \text{Inductively, the intersection of any finite family of members of} \]

\( \mathcal{T}_\infty \) is a member of \( \mathcal{T}_\infty \). \qed
Theorem 51. Let $(X, \mathcal{T})$ be a topological space and let $(X_\infty, \mathcal{T}_\infty)$ be its one-pint compactification. Then

1. $(X, \mathcal{T})$ is a subspace of $(X_\infty, \mathcal{T}_\infty)$.
2. $(X_\infty, \mathcal{T}_\infty)$ is compact.
3. $X_\infty$ is Hausdorff if and only if $X$ is Hausdorff and locally cpt.
4. $X$ is a dense subset of $X_\infty$ if and only if $X$ is not compact.

Proof. (1) It is easy to see that $\mathcal{T} = \{ U \cap X \mid U \in \mathcal{T}_\infty \}$.

(2) Let $\mathcal{O}$ be an open cover of $X_\infty$.
   \[ \Rightarrow \] There exists an open set $U$ in $\mathcal{O}$ such that $\infty \in U$.
   \[ \Rightarrow X_\infty \setminus U = X \setminus U \text{ is compact}. \]
   \[ \Rightarrow \text{Since } X \setminus U \text{ is a subspace of } X_\infty \text{ and } \mathcal{O} \text{ is an open cover of} \]
   \[ X \setminus U \text{ by open sets in } X_\infty, \text{ there exist open sets } U_1, \cdots, U_n \in \mathcal{O} \]
such that \( X \setminus U \subset U_1 \cup \cdots \cup U_n \).

\[ \Rightarrow X_\infty \subset U \cup \left( \bigcup_{i=1}^{n} U_i \right) . \text{ Thus } X_\infty \text{ is compact.} \]

(3) \( (\Rightarrow) \) Assume that \( X_\infty \) is Hausdorff.

\[ \Rightarrow \] Since \( X \) is a subspace of \( X_\infty \), \( X \) is Hausdorff by Theorem 4.59.

To show that \( X \) is locally compact, let \( a \in X \).

\[ \Rightarrow \] Since \( a \neq \infty \), and since \( X_\infty \) is Hausdorff, there exist open sets \( U, V \) in \( X_\infty \) such that \( \infty \in U, a \in V \) and \( U \cap V = \emptyset \).

\[ \Rightarrow \] Since \( \infty \notin V \), \( V \) is open in \( X \). Moreover, \( V \subset X_\infty \setminus U \) and \( X_\infty \setminus U \) is a closed and compact subset of \( X \).

\[ \Rightarrow \] Since \( \text{cl}_{X}(V) \subset X_\infty \setminus U \), \( \text{cl}_{X}(V) \) is compact.

\[ \Rightarrow \] \( a \) has an open neighborhood \( V \) such that \( \text{cl}_{X}(V) \) is compact.

\[ \Rightarrow \] \( X \) is locally compact at \( a \) and hence \( X \) is locally compact.
(⇐) Assume that $X$ is Hausdorff and locally compact.

Let $a, b \in X_\infty$ with $a \neq b$. (To show that $X_\infty$ is Hausdorff, we only need to show that $a$ and $b$ have disjoint open nbd for $b = \infty$.) Assume that $b = \infty$.

$\Rightarrow$ Since $X$ is locally compact at $a \in X$, there exists an open set $V$ in $X$ such that $a \in V$ and $\text{cl}_X(V)$ is compact.

$\Rightarrow$ Since $\infty \in X_\infty \setminus \text{cl}_X(V)$ and $X \setminus (X_\infty \setminus \text{cl}_X(V)) = \text{cl}_X(V)$ is closed and compact in $X$, $X_\infty \setminus \text{cl}_X(V)$ is an open set in $X_\infty$.

$\Rightarrow$ Since $V$ is an open subset of $X_\infty$ containing $a$ and $V \cap (X_\infty \setminus \text{cl}_X(V)) = \emptyset$, $X_\infty$ is Hausdorff.
(4) $X$ is not compact.
\[
\Leftrightarrow \{\infty\} = X_\infty \setminus X \text{ is not open in } X_\infty.
\Leftrightarrow \text{Every open set in } X_\infty \text{ containing } \infty \text{ meets } X.
\Leftrightarrow \infty \in \text{cl}_{X_\infty}(X).
\Leftrightarrow X_\infty = \text{cl}_{X_\infty}(X).
\Leftrightarrow X \text{ is dense in } X_\infty.
\]

Remark 52. If $X$ is not locally compact Hausdorff, then $X_\infty$ is not Hausdorff. Since non-Hausdorff spaces are of limited interest, many texts define the one-point compactification only for locally compact Hausdorff spaces $X$.

Example 53. (1) The one-point compactification of the open interval $(0, 1)$ is homeomorphic with $S^1$.

(2) $\mathbb{R}_\infty$ is homeomorphic to a circle. Thus $\mathbb{R}_\infty \cong S^1$. 

(3) $\mathbb{R}^2_\infty$ is homeomorphic to the unit 2-sphere $S^2$, where $S^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$.

(4) In general, $\mathbb{R}^n_\infty$ is homeomorphic to the unit $n$-sphere $S^n$.

Proof. (1) Let $X = (0, 1)$ and define $f : X_\infty \to S^1$ by

$$f(t) = \begin{cases} (\cos 2\pi t, \sin 2\pi t) & \text{if } 0 < t < 1, \\ (1, 0) & \text{if } t = \infty. \end{cases}$$

\[\begin{array}{c}
\infty \\
\bullet \\
(0,1)
\end{array}\]

\[\begin{array}{c}
X_\infty \\
\longleftarrow f \\
S^1
\end{array}\]

Figure 7: The function $f : X_\infty \to S^1$
⇒ $f : X_\infty \to S^1$ is a bijective and continuous function.

⇒ Since $X_\infty$ is compact and $S^1$ is Hausdorff, $f : X_\infty \to S^1$ is a homeomorphism by Theorem 20.

⇒ The one-point compactification of the open interval $(0, 1)$ is homeomorphic with $S^1$.

(2) Let $C^1 = \{(x, y) \in \mathbb{R}^2 \mid x^2 + (y - 1)^2 = 1\}$ be the circle centered at $(0, 1)$ and of radius 1. Write $p = (0, 2) \in C^1$.

Define $g : \mathbb{R}_\infty \to C^1$ by

$$g(x) = \begin{cases} 
\overrightarrow{px} \cap (C^1 \setminus \{(0, 2)\}) & \text{if } x = (x_1, 0) \in \mathbb{R}^2, \\
(0, 2) & \text{if } x = \infty \in \mathbb{R}_\infty.
\end{cases}$$
i.e. \( g(x) = \begin{cases} \left( \frac{4x}{x^2 + 4}, 2 - \frac{8}{x^2 + 4} \right) & \text{if } x \in \mathbb{R}, \\ (0, 2) & \text{if } x = \infty. \end{cases} \)

Figure 8: The function \( g : \mathbb{R}_\infty \to C^1 \)

\( g : \mathbb{R}_\infty \to C^1 \) is a continuous and bijective function.

\( g : \mathbb{R}_\infty \to C^1 \) is a homeomorphism by Theorem 20.
⇒ $\mathbb{R}_\infty \cong C^1$ and hence $\mathbb{R}_\infty \cong S^1$.

(3) Let $C^2 = \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + (z - 1)^2 = 1\}$ be the sphere centered at $(0, 0, 1)$ and of radius 1. Write $P = (0, 0, 2) \in C^2$.

Define $g : \mathbb{R}^2_\infty \to C^2$ by

$$g(x) = \begin{cases} \overrightarrow{px} \cap (C^2 \setminus \{(0, 0, 2)\}) & \text{if } x = (x_1, x_2, 0) \in \mathbb{R}^3, \\ (0, 0, 2) & \text{if } x = \infty \in \mathbb{R}^2_\infty. \end{cases}$$

⇒ $g : \mathbb{R}^2_\infty \to C^2$ is a bijective and continuous function

⇒ $g : \mathbb{R}^2_\infty \to C^2$ is a homeomorphism by Theorem 20.

⇒ $\mathbb{R}^2_\infty \cong C^2$ and hence $\mathbb{R}^2_\infty \cong S^2$. 
Figure 9: The function \( g : \mathbb{R}_\infty^2 \rightarrow C^2 \)
(4) Define $g : \mathbb{R}^n_{\infty} \to S^n$ and $f : S^n \to \mathbb{R}^n_{\infty}$ by
for $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ or $x = \infty \in \mathbb{R}^n_{\infty}$, and
for $y = (y_1, \cdots, y_{n+1}) \in S^n$,

$$g(x) = \begin{cases} \left( \frac{2x_1}{\|x\|^2 + 1}, \cdots, \frac{2x_n}{\|x\|^2 + 1}, \frac{\|x\|^2 - 1}{\|x\|^2 + 1} \right) & \text{if } x \in \mathbb{R}^n, \\ (0, \cdots, 0, 1) & \text{if } x = \infty. \end{cases}$$

$$f(y) = \begin{cases} \left( \frac{y_1}{1 - y_{n+1}}, \cdots, \frac{y_n}{1 - y_{n+1}}, 0 \right) & \text{if } y_{n+1} \neq 1, \\ \infty & \text{if } y_{n+1} = 1. \end{cases}$$

$\Rightarrow f$ and $g$ are continuous and inverses to each other.
$\Rightarrow f$ and $g$ are homeomorphisms by Theorem 20.

(Compare this with those of (2) and (3) for $n = 1, 2.$) \hfill \square
Definition 54. • The method of defining the functions $g : \mathbb{R}^n \to S^n$ in the proof of (2) and (3) of Example 53 is called stereographic projection.

• The homeomorphism $f : S^n \to \mathbb{R}^n_\infty$ in the proof of (4) is called the stereographic projection.

• The point $x$ is said to be the stereographic projection of the corresponding point $g(x)$.

Forming the one-point compactification $X_\infty$ of a space $X$ is the simplest method of embedding a locally compact Hausdorff space in a compact Hausdorff space. We shall return to this problem in Chapter 8 with the Stone-Čech compactification which applies to a more general class of spaces, the class of completely regular spaces. The Stone-Čech compactification $\beta(X)$ of a space $X$ has the following properties, the second of which makes it more useful than the one-point compactification.
(1) $\beta(X)$ is a compact Hausdorff space in which $X$ is a dense subspace.

(2) Every bounded continuous function $f : X \to \mathbb{R}$ from $X$ to the real line $\mathbb{R}$ can be uniquely extended to a continuous function $f^* : \beta(X) \to \mathbb{R}$.

Several other aspects of the problem of extending continuous functions will also be addressed in Chapter 8.
1. Prove that local compactness is a topological property.

*Proof.* Let $X$ be a locally compact space and let $f : X \to Y$ be a homeomorphism. Let $y \in Y$.

$\Rightarrow$ Since $f^{-1}(y) \in X$, there exists an open subset $U$ of $X$ such that $f^{-1}(y) \in U$ and $\overline{U}$ is compact.

$\Rightarrow$ Since $f : X \to Y$ is a homeomorphism, $f(U)$ is an open subset of $Y$ containing $y = f(f^{-1}(y))$ and $\overline{f(U)} = f(\overline{U})$ is compact.

$\Rightarrow$ $Y$ is locally compact $y$.

$\Rightarrow$ Since $y$ was chosen arbitrarily, $Y$ is locally compact.

Thus local compactness is a topological property.
2. Let $X$ be a space and let $x \in X$ at which $X$ is locally compact. Prove that there is a local basis $\mathcal{B}$ at $x$ such that $\overline{B}$ is compact for each $B \in \mathcal{B}$.

Proof. Let $X$ be a space and let $x \in X$ at which $X$ is locally compact. 
⇒ There exists an open set $U$ in $X$ such that $x \in U$ and $\overline{U}$ is compact.
Let $\mathcal{B}_x = \{B \mid B$ is an open subset of $X$ and $x \in B \subset U\}.
⇒ \overline{B}$ is compact for each $B \in \mathcal{B}$. Moreover $\mathcal{B}_x$ is a local basis at $x$.
\[
\begin{align*}
\left( \because \right) & \text{ Let } V \text{ be an open subset of } X \text{ containing } x. \\
\Rightarrow & \quad x \in V \cap U \text{ and } V \cap U \in \mathcal{B}_x.
\end{align*}
\]

3. Show that Hilbert space is not locally compact at any point.
Proof. Note that $H$ is not locally compact at the origin $0 = (0, 0, \cdots, 0) \in H$ as shown in the proof of Example 48.
Let $a \in H$ and suppose that $H$ is locally compact at $a$.
$\Rightarrow \exists$ an open subset $U$ of $H$ such that $a \in U$ and $\overline{U}$ is compact.
Consider the function $f_a : H \to H$ defined by $f_a(x) = x - a$
for each $x \in H$.
$\Rightarrow f_a : H \to H$ is a homeomorphism.

\[
\begin{align*}
(\because) & \quad \text{Clearly } f_a : H \to H \text{ is a continuous function whose inverse is the continuous function } f_{-a} : H \to H. \\
& \quad \Rightarrow f_a : H \to H \text{ is a homeomorphism.}
\end{align*}
\]
$\Rightarrow f_a(U) = U - a = \{x - a \mid x \in U\}$ is an open subset of $H$,
$0 \in f_a(U)$ and $\overline{f_a(U)} = f_a(\overline{U})$ is compact.
$\Rightarrow H$ is locally compact at $0$. A contradiction!
$\Rightarrow H$ is not locally compact at $x$. \hfill \Box
4. Is the real line with the finite complement topology locally compact? Prove your answer.

Proof. The answer is “yes”. We will show that the real line with the finite complement topology is compact so that it is locally compact.

Let $X$ denote the real line with the cofinite topology and let $\mathcal{O}$ be an open cover of $X$. Choose any $O \in \mathcal{O}$.

$\Rightarrow$ Since $O$ is open in $X$, $X \setminus O = \{x_1, x_2, \cdots, x_n\}$.

$\Rightarrow$ Since $\mathcal{O}$ covers $X$, there exists $O_i \in \mathcal{O}$ such that $x_i \in O_i$ for each $i = 1, 2, \cdots, n$.

$\Rightarrow \{O, O_1, O_2, \cdots, O_n\}$ is a finite subcover of $\mathcal{O}$.

$\Rightarrow X$ is compact. \qed
5. If $X$ is a Hausdorff space, show that the requirement that $X_\infty \setminus U$ be closed in $X$ can be omitted in the definition of the topology for $X_\infty$.

Proof. Assume that $X$ is a Hausdorff space and let $X_\infty$ denote its one-point compactification. Let $U$ be a subset of $X_\infty$ containing $\infty$.

\[ \Rightarrow \text{Since every compact subset of } X \text{ is closed by Theorem 12, } X_\infty \setminus U \text{ is a closed and compact subset of } X \text{ if and only if } X_\infty \setminus U \text{ is a compact subset of } X \text{ for each } U \subset X_\infty. \]

\[ \Rightarrow \{U \subset X_\infty \mid X_\infty \setminus U \text{ is closed and compact in } X, \infty \in U\} = \{U \subset X_\infty \mid X_\infty \setminus U \text{ is a compact subset of } X, \infty \in U\}. \]

\[ \Rightarrow \text{The requirement that } X_\infty \setminus U \text{ be closed in } X \text{ can be omitted in the definition of the topology for } X_\infty. \] \[ \square \]
6. Prove that a space \( X \) is locally compact if and only if for each \( x \) in \( X \) there is a subspace \( A \) of \( X \) such that \( \overline{A} \) is compact and \( x \in \text{int}(A) \).

\textit{Proof. (only if)} Assume that \( X \) is locally compact and let \( x \in X \).
\( \Rightarrow \) Since \( X \) is locally compact at \( x \), there exists an open subset \( U \) of \( X \) such that \( x \in U \) and \( U \) is compact. Take \( A = \overline{U} \).
\( \Rightarrow \) \( A \) is compact and \( x \in \text{int}(A) = U \).

\textit{(if)} Assume that for each \( x \in X \), there is a compact subset \( A \) of \( X \) such that \( x \in \text{int}(A) \). Take \( U = \text{int}(A) \).
\( \Rightarrow \) \( U \) is an open subset of \( X \) containing \( x \) and \( \overline{U} = A \) is compact.
\( \Rightarrow \) \( X \) is locally compact at \( x \) for each \( x \in X \).
\( \Rightarrow \) \( X \) is locally compact. \( \square \)
7. Let $X$ be a locally compact Hausdorff space, $A$ a closed subset of $X$, and let $p$ be a point not in $A$. Prove that there are disjoint open sets $U$ and $V$ in $X$ such that $p \in U$ and $A \subset V$. (Hint: Consider the one-point compactification of $X$.)

Proof. Let $X$ be a locally compact Hausdorff space, $A$ a closed subset of $X$, and let $p \notin A$. Consider the one-point compactification $X_{\infty}$ of $X$. Note that $X_{\infty}$ is a compact Hausdorff space.
⇒ Since $X \setminus A$ is open in $X$, $X \setminus A$ is an open subset of $X_{\infty}$.
⇒ $X_{\infty} \setminus (X \setminus A) = A \cup \{\infty\}$ is a closed subset of the compact space $X_{\infty}$.
⇒ $A \cup \{\infty\}$ is a compact subset of $X_{\infty}$ by Theorem 11.
⇒ Since $\{p\}$ and $A \cup \{\infty\}$ are disjoint compact subsets of the Hausdorff space $X_{\infty}$, there exist disjoint open sets $U'$ and $V'$ in $X_{\infty}$ such that $p \in U'$ and $(A \cup \{\infty\}) \subset V'$ by Theorem 14.
Let $U = U' \cap X$ and $V = V' \cap X$.
⇒ $U$ and $V$ are disjoint open sets in $X$ with $p \in U$ and $A \subset V$. 

\[ \square \]
8. **Definition.** A point $p$ in a space $X$ is called an *isolated point* provided that $\{p\}$ is an open set.

Prove that a space $X$ is compact if and only if $\infty$ is an isolated point of $X_\infty$.

*Proof.* Let $X$ be a space and $X_\infty$ its one-point compactification.

Then we have the following equivalence:

$X$ is compact.

$\iff X_\infty \setminus \{\infty\} = X$ is a closed and compact subset of $X$.

$\iff \{\infty\}$ is an open subset of $X_\infty$.

$\iff \{\infty\}$ is an isolated point of $X_\infty$. \hfill \Box$

9. (1) Let $X$ be a Hausdorff space. Show that $X$ is locally compact if and only if each point of $X$ has a compact neighborhood.

In other words, $X$ is locally compact if and only if each point $x$ of $X$ belongs to the interior of a compact set $K_x$. 

(2) Give an example of a space $X$ for which each point has a compact neighborhood, but $X$ is not locally compact.

(Hint: By (1), $X$ cannot be a Hausdorff.)

Proof. (1) Let $X$ be a Hausdorff space.

(only if) Assume that $X$ is locally compact and let $x \in X$.
⇒ There exists an open subset $U$ of $X$ such that $x \in U$ and $\overline{U}$ is compact.
⇒ $\overline{U}$ is a compact neighborhood of $x$ in $X$.

(if) Assume that each point of $X$ has a compact neighborhood and let $x \in X$.
⇒ $x$ has a compact neighborhood $K$ in $X$.
⇒ There exists an open subset $U$ of $X$ such that $x \in U \subset K$.
⇒ Since $K$ is compact, $\overline{U}$ is compact by Theorem 11.
⇒ $X$ is locally compact at $x$.
⇒ Since $x$ was chosen arbitrarily, $X$ is locally compact.
(2) Let $X = \mathbb{R} \cup \{\ast\}$ be a subset of $\mathbb{R}^2$, where $\mathbb{R} = \{(x, 0) \in \mathbb{R}^2 \mid x \in \mathbb{R}^1\}$ and $\ast = (0, 1) \in \mathbb{R}^2$.

Topologize $X$ by setting a topology $\mathcal{T}$ on $X$ by $\mathcal{T} = \{\emptyset\} \cup \{A \subset X \mid \ast \in A\}$ as in Example 5.2(6).

$\Rightarrow$ For each $x \in \mathbb{R}^1$, $\{(x, 0), \ast\}$ is a compact neighborhood of $x$ in $X$ and $\{(0, 0), \ast\}$ is a compact neighborhood of $\ast$ in $X$.

We claim the following:

(i) For any nonempty open set $U \in \mathcal{T}$, $\overline{U} = X$.

$\therefore$ Let $U$ be a nonempty open subset of $X$ and let $K$ be a closed subset of $X$ containing $U$.

$\Rightarrow X \setminus K$ is an open subset of $X$.

$\Rightarrow$ Since $\ast \notin X \setminus K$, $X \setminus K = \emptyset$.

$\Rightarrow X = K$ and hence $\overline{U} = X$.

(ii) $X$ is not compact.

$\therefore$ $\{\{\ast, x\} \mid x \in \mathbb{R}\}$ is an open cover of $X$ which has no finite subcover.
$\Rightarrow X$ is not locally compact at any point of $X$. □

10. Generalize the method of stereographic projections in the proof of (2) and (3) of Example 53 to define $g : \mathbb{R}_\infty^n \rightarrow C^n$ for any natural number $n$.

Proof. Let
\[
C^n = \{(x_1, \cdots, x_{n+1}) \in \mathbb{R}^{n+1} \mid x_1^2 + \cdots x_n + (x_{n+1} - 1)^2 = 1\}
\]
be the $n$-sphere centered at $(0, 0, \cdots, 0, 1)$ and of radius 1.
Write $p = (0, 0, \cdots, 0, 2) \in C^2$. Define $g : \mathbb{R}_\infty^n \rightarrow C^n$ by $g(x) =$
\[
\begin{cases}
\overrightarrow{px} \cap (C^2 \setminus \{(0, 0, \cdots, 0, 2)\}) & \text{if } x = (x_1, x_2, \cdots, x_n, 0) \in \mathbb{R}^n, \\
(0, 0, \cdots, 0, 2) & \text{if } x = \infty \in \mathbb{R}_\infty^n.
\end{cases}
\]
$\Rightarrow g : \mathbb{R}_\infty^2 \rightarrow C^2$ is a bijective and continuous function.
$\Rightarrow g : \mathbb{R}_\infty^2 \rightarrow C^2$ is a homeomorphism by Theorem 20.
$\Rightarrow \mathbb{R}_\infty^n \cong C^n$ and hence $\mathbb{R}_\infty^n \cong S^n$. □
5 The Cantor Set and the Cantor Function

Of all the subsets of the real line, the Cantor set is probably the most fertile source of examples and counterexamples in topology.

In this section, we define this remarkable set and examine some of its properties.
Definition 55. The subset $C = \bigcap_{n=1}^{\infty} F_n$ of $[0, 1]$ is called the Cantor set, where $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1],$

\[
F_2 = [0, \frac{1}{3^2}] \cup [\frac{2}{3^2}, \frac{3}{3^2}] \cup [\frac{6}{3^2}, \frac{7}{3^2}] \cup [\frac{8}{3^2}, 1], \cdots,
\]

\[
F_n = [0, \frac{1}{3^n}] \cup [\frac{2}{3^n}, \frac{3}{3^n}] \cup \cdots \cup [\frac{3^n - 3}{3^n}, \frac{3^n - 2}{3^n}] \cup [\frac{3^n - 1}{3^n}, 1],
\]

\[
\vdots
\]

$I = [0, 1]$

\[
F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]
\]

\[
F_2 = [0, \frac{1}{9}] \cup [\frac{2}{9}, \frac{3}{9}] \cup [\frac{6}{9}, \frac{7}{9}] \cup [\frac{8}{9}, 1]
\]

\[
\vdots
\]

(Note that $C$ is closed.)
Definition 56 (Alternative definition of the Cantor set). For \( x \in \mathbb{R} \) with \( 0 \leq x \leq 1 \), \( x \) can be expressed as a sum of powers of 3;

\[
x = \sum_{n=1}^{\infty} x_n 3^{-n} (x_n = 0, 1, 2).
\]

In this case, the expression \( x = 0.x_1x_2x_3 \cdots \) is called a ternary expansion of \( x \).

\[
\frac{1}{3} = \frac{1}{3} + 0 \cdot \frac{1}{3^2} + 0 \cdot \frac{1}{3^3} + 0 \cdot \frac{1}{3^4} + \cdots
\]

\[
= 0 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3^2} + 2 \cdot \frac{1}{3^3} + 2 \cdot \frac{1}{3^4} + \cdots, \text{ and}
\]

\[
\frac{1}{9} = 0 \cdot \frac{1}{3} + 1 \cdot \frac{1}{3^2} + 0 \cdot \frac{1}{3^3} + 0 \cdot \frac{1}{3^4} + \cdots
\]

\[
= 0 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3^2} + 2 \cdot \frac{1}{3^3} + 2 \cdot \frac{1}{3^4} + \cdots.
\]
Note that $\frac{1}{3}$ and $\frac{1}{9}$ have the two ternary expansions as shown above.

Observe the following.

(1) If $x = 0.1**\cdots$ as a ternary expansion, then $\frac{1}{3} \leq x \leq \frac{2}{3}$
but $\frac{1}{3} (= 0.1000\cdots) = 0.0222\cdots$ and $\frac{2}{3} = 0.2000\cdots$ as a ternary expansion without 1’s.
Thus $F_1$ excludes all numbers $x$ which absolutely require 1 in the first place of their ternary expansions $x = 0.1**\cdots$
($\frac{1}{3} < x < \frac{2}{3}$).

(2) If $x = 0.01**\cdots$ as a ternary expansion, then $\frac{1}{9} \leq x \leq \frac{2}{9}$
but $\frac{1}{9} (= 0.0100\cdots) = 0.0022\cdots$ and $\frac{2}{9} = 0.0200\cdots$ as ternary expansions without 1’s.
If $x = 0.21 \ast \ast \ast \cdots$ as a ternary expansion, then $\frac{7}{9} \leq x \leq \frac{8}{9}$ but $\frac{7}{9} (= 0.2100 \cdots) = 0.2022 \cdots$ and $\frac{8}{9} = 0.2200 \cdots$ as ternary expansions without 1’s.

Thus $F_2$ excludes all numbers $x$ which absolutely require 1 in the second place of their ternary expansions $x = 0.01 \ast \ast \ast \cdots$ $(\frac{1}{9} < x < \frac{2}{9})$ or $x = 0.21 \ast \ast \ast \cdots$ $(\frac{7}{9} < x < \frac{8}{9})$.

(3) Similarly, $F_n$ excludes all numbers $x$ which absolutely require 1 in the $n$th place of their ternary expansions $x = 0.0 \cdots 01 \ast \ast \ast \cdots$.

Thus the Cantor set $C$ can be defined by

$$C = \{0 \leq x \leq 1 \mid x = 0.x_1x_2 \cdots \text{ as a ternary expansion with } x_n = 0, 2\}.$$
Example 57. (1) The Cantor set $C$ contains the end points $0$, $1$, $\frac{1}{3}$,
\[
\frac{2}{3}, \ \frac{1}{9}, \ \frac{2}{9}, \ \frac{7}{9}, \ \frac{8}{9}, \ \cdots
\]
of the open intervals deleted to define the sets \(\{F_n\}_{n=1}^{\infty}\). However these are not the only members of the Cantor set. In particular, the number $\frac{1}{4}$ belongs to $C$.

(2) The Cantor set $C$ is a closed uncountable subset of $[0,1]$ which contains no proper interval. i.e., $C$ is a closed, uncountable, nowhere dense subset of $[0,1]$.

(3) The total length $L$ of the removed intervals in the definition of $C$ is equal to $1$. 
Proof. (1) Consider the convergent series
\[ x = \sum_{n=1}^{\infty} 2 \cdot 3^{-2n} = \frac{2}{3^2} + \frac{2}{3^4} + \frac{2}{3^6} + \cdots \in C. \]
\[ \Rightarrow x = \frac{1}{3^2} (2 + x). \]
\[ \Rightarrow x = \frac{1}{4} \in C. \]

(2) Clearly, \( C \) is closed and contains no proper interval.
\[ \Rightarrow \text{int}(\overline{C}) = \text{int}(C) = \emptyset. \]
\[ \Rightarrow C \text{ is a nowhere dense, closed subset of } \mathbb{R}. \]

We claim that \( C \) is uncountable. Note that \( C \) is not finite.

Suppose that \( C \) is countably infinite.
\[ \Rightarrow \text{There exists a bijective function } f : \mathbb{N} \to C. \]
\[ \Rightarrow \text{For each } n \in \mathbb{N}, f(n) \in C. \text{ Write } f(n) = 0.n_1n_2n_3 \cdots \text{ as a ternary} \]
expansion with \( n_i = 0, 2(i \in \mathbb{N}) \).

Consider \( x = 0.x_1x_2x_3 \cdots \) as a ternary expansion defined as follows: For each \( n \in \mathbb{N} \), \( x_n = \begin{cases} 2 & \text{if } n_n = 0, \\ 0 & \text{if } n_n = 2. \end{cases} \)

\( \Rightarrow x \in C \) and \( x \neq f(n) \), for all \( n \in \mathbb{N} \).

\( \Rightarrow x \in C \setminus f(\mathbb{N}) \).

\( \Rightarrow f : \mathbb{N} \to C \) is not onto. A contradiction!

\( \Rightarrow C \) is not countably infinite.

\( \Rightarrow \) Since \( C \) is not finite, \( C \) is uncountable.

(3) In the construction of the Cantor set \( C \) from the intersection of the sets \( F_1, F_2, F_3, \cdots \), the open intervals removed were of length
\[
L = \frac{1}{3} + \frac{2}{9} + \frac{4}{27} + \frac{8}{81} + \cdots + \frac{2^{n-1}}{3^n} + \cdots
\]

\[
= \frac{1}{3} \left( 1 + \frac{2}{3} + \left( \frac{2}{3} \right)^2 + \left( \frac{2}{3} \right)^3 + \cdots + \left( \frac{2}{3} \right)^{n-1} + \cdots \right)
\]

\[
= \frac{1}{3} \left( \frac{1}{1 - \frac{2}{3}} \right) = 1.
\]
Students of analysis may observe that the Cantor set $C$, as defined here, has measure 0.

The measure of a set is not a topological property, however.

There are other representations of the Cantor set in which the measure is positive.

**Definition 58.** A nonempty closed subset $A$ of a space $X$ is said to be *perfect* if every point of $A$ is a limit point of $A$.

**Theorem 59.** The Cantor set $C$ is a compact, perfect, totally disconnected metric space.

*Proof.* (1) Since $C$ is closed and bounded in $\mathbb{R}$, $C$ is compact.

(2) Since $C$ contains no proper interval, $C$ is totally disconnected. (i.e., every component of $C$ is a single point.)
(3) To show that $C$ is perfect, let $x \in C$ and let $\epsilon > 0$.

⇒ Take $n_0 \in \mathbb{N}$ such that $\frac{2}{3^{n_0}} < \epsilon$.

Express $x = 0.x_1x_2x_3 \cdots$ as a ternary expression
where $x_n \in \{0, 2\}$ for each $n \in \mathbb{N}$.

⇒ Define $y = 0.y_1y_2y_3 \cdots$ by setting

\[
y_n = \begin{cases} 
  x_n & n \neq n_0, \\
  2 & n = n_0 \text{ and } x_{n_0} = 0, \\
  0 & n = n_0 \text{ and } x_{n_0} = 2.
\end{cases}
\]

⇒ $y \in C$ and $|x - y| = \frac{2}{3^{n_0}} < \epsilon$.

⇒ $(B(x, \epsilon) \setminus \{x\}) \cap C \neq \emptyset$.

⇒ $x$ is a limit point of $C$.

⇒ $C$ is perfect.

□
There is an important extension of the preceding theorem that illustrates a case in which the classification problem has been solved; Not only is the Cantor set a compact, perfect, totally disconnected metric space, but any topological space with these four properties is homeomorphic to the Cantor set.

Thus any two compact, perfect, totally disconnected metric spaces are homeomorphic, and the Cantor set may be considered the prototype of this genre.

In order to move on to other subjects, the proof of this classification theorem is omitted.

Further details can be found in the suggested reading list for this chapter.

Next we introduce the Cantor function.
Definition 60. We define the Cantor function \( f : [0, 1] \to [0, 1] \) as a limit of the following sequence of functions \( f_n : [0, 1] \to [0, 1] \).

Let \( f_0(x) = x \). Then \( f_{n+1}(x) \) will be defined in terms of \( f_n(x) \) as follows;

\[
    f_{n+1}(x) = \begin{cases} 
        0.5f_n(3x) & \text{if } 0 \leq x \leq \frac{1}{3}, \\
        0.5 & \text{if } \frac{1}{3} \leq x \leq \frac{2}{3} \text{ and} \\
        0.5 + 0.5f_n\left(3\left(x - \frac{2}{3}\right)\right) & \text{if } \frac{2}{3} \leq x \leq 1.
    \end{cases}
\]
Figure 10: The graph of the Cantor function
Remark 61. Observe that $f_n$ converges uniformly to $f$. Also notice that the choice of starting function does not really matter, provided $f_0(0) = 0$ and $f_0(1) = 1$ and $f_0$ is bounded. For the alternative definition of the Cantor function $f$, see Exercise 6.5.9.

**Theorem 62.** The Cantor function $f : [0, 1] \to [0, 1]$ has the following properties.

1. $\lim_{x \to 0^+} f(x) = 0$ and $\lim_{x \to 1^-} f(x) = 1$.

2. $f|_C : C \to [0, 1]$ is a surjective function.

3. $f : [0, 1] \to [0, 1]$ is continuous.

4. $f$ is not differentiable at any member of the Cantor set $C$ but has derivative 0 on $[0, 1] \setminus C$.

5. $f$ is uniformly continuous but not absolutely continuous.

*Proof.* Omit. \qed
1. Show that \( \frac{1}{36} \) is a member of the Cantor set.

**Proof.** Let \( \frac{1}{36} = \frac{a_1}{3^1} + \frac{a_2}{3^2} + \frac{a_3}{3^3} + \frac{a_4}{3^4} + \frac{a_5}{3^5} + \cdots \).

⇒ \( \frac{3}{36} = \frac{1}{12} = a_1 + \frac{a_2}{3^1} + \frac{a_3}{3^2} + \frac{a_4}{3^3} + \frac{a_5}{3^4} + \cdots \) and \( a_1 = 0 \).

⇒ \( \frac{3}{12} = \frac{1}{4} = a_2 + \frac{a_3}{3^1} + \frac{a_4}{3^2} + \frac{a_5}{3^3} + \cdots \) and \( a_2 = 0 \).

⇒ \( \frac{3}{4} = a_3 + \frac{a_4}{3^1} + \frac{a_5}{3^2} + \cdots \) and \( a_3 = 0 \).

⇒ \( \frac{3^2}{4} = \frac{9}{4} = a_4 + \frac{a_5}{3^1} + \cdots \) and \( a_4 = 2 \).

⇒ \( \frac{9}{4} - 2 = \frac{1}{4} = \frac{a_5}{3^1} + \cdots \).

⇒ \( \frac{3}{4} = a_5 + \frac{a_6}{3^1} + \cdots \) and \( a_5 = 0 \).

⇒ \( \frac{3^2}{4} = a_6 + \frac{a_7}{3^1} + \cdots \) and \( a_6 = 2 \).
\[\therefore\]
\[\Rightarrow a_1 = a_2 = a_3 = 0, a_4 = 2, a_5 = 0, a_6 = 2, a_7 = 0, a_8 = 2, a_9 = 0, \cdots.\]
\[\Rightarrow \frac{1}{36} = \frac{2}{3^4} + \frac{2}{3^6} + \frac{2}{3^8} + \frac{2}{3^{10}} + \cdots = \sum_{n=1}^{\infty} \frac{2}{3^{2n+2}}\text{ which is a ternary expression of } \frac{1}{36} \text{ without 1's.}\]
\[\Rightarrow \frac{1}{36} \in C.\]
2. Show that each set $F_n$ defined in the construction of the Cantor set has $2^n$ components.

Proof. Let $c_n$ denote the number of components of $F_n$ defined in the construction of the Cantor set.

⇒ Since $F_1 = [0, \frac{1}{3}] \cup [\frac{2}{3}, 1]$ has two components, $c_1 = 2 = 2^1$.

Now Assume that $c_{n-1} = 2^{n-1}$. In other words, $F_{n-1}$ consists of $2^{n-1}$ components which are closed intervals.

⇒ Since the middle thirds of each closed interval of $F_{n-1}$ is removed to get $F_n$ by the construction of the Cantor set, $F_n$ has $2c_{n-1} = 2 \cdot 2^{n-1}$ components and hence $c_n = 2^n$.

⇒ $F_n$ has $2^n$ components for each $n \in \mathbb{N}$ by induction. □
3. (1) If $x$ is a member of $[0, 1)$ with a binary expansion

$$x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, a_n \in \{0, 1\},$$

show that the ternary expansion

$$\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$$

represents a point of the Cantor set $C$.

(2) Use (1) to define a one-to-one function from $[0, 1)$ into $C$. Take into account the fact that some members of $[0, 1)$ have more than one binary expansion.

(3) Conclude that $C$ is uncountable.

**Proof.** (1) Let $x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, a_n \in \{0, 1\}$.

$\Rightarrow$ Since $2a_n \in \{0, 2\}$, $\sum_{n=1}^{\infty} \frac{2a_n}{3^n}$ is a ternary expression without 1’s.

$\Rightarrow \sum_{n=1}^{\infty} \frac{2a_n}{3^n} \in C$. 
(2) Define \( f : [0, 1) \to C \) by
\[
f(x) = \sum_{n=1}^{\infty} \frac{2a_n}{3^n},
\]
for each \( x = \sum_{n=1}^{\infty} \frac{a_n}{2^n}, a_n \in \{0, 1\}. \)

\( \Rightarrow \) \( f : [0, 1) \to C \) is a well-defined function.

\[ (\because) \]
Let \( x = \sum_{n=1}^{\infty} \frac{a_n}{2^n} \) and \( y = \sum_{n=1}^{\infty} \frac{b_n}{2^n}, (a_n, b_n \in \{0, 1\}) \). Suppose that \( x = y \). Let \( n = \min\{i \mid a_i \neq b_i\} \) and say \( a_n = 0, b_n = 1 \).

\( \Rightarrow \) \( a_i = b_i \) for all \( i \leq n, a_n = 0, b_n = 1 \)
\[ a_{n+1} = a_{n+2} = \cdots = 1 \quad \text{and} \quad b_{n+1} = b_{n+2} = \cdots = 0. \]

\( \Rightarrow \) \( 2a_i = 2b_i \) for all \( i \leq n, 2a_n = 0, 2b_n = 2 \)
\[ 2a_{n+1} = 2a_{n+2} = \cdots = 2 \quad \text{and} \quad 2b_{n+1} = 2b_{n+2} = \cdots = 0. \]

\( \Rightarrow \) \( \sum_{n=1}^{\infty} \frac{2a_n}{3^n} = \sum_{n=1}^{\infty} \frac{2b_n}{3^n} \).

\( \Rightarrow \) \( f(x) = f(y) \).
(3) Since clearly \( f : [0, 1) \rightarrow C \) is injective and \([0, 1)\) is uncountable, 
\( C \) is uncountable. \(\square\)

4. Example 57 showed that the Cantor set is nowhere dense in \( \mathbb{R} \). Conclude, however, that as a metric space in its own right, with 
\[ d(a, b) = |a - b| \] for \( a, b \in C \), the Cantor set is of the second category.

Proof. Since \( C \) is a closed subset of the metric space \([0, 1]\) and 
since \(([0, 1], d)\) is complete by Example 4.58, \( C \) is a complete metric space by Theorem 3.59.
\[ \Rightarrow C \text{ is of the second category by the Baire category theorem} \]
(Theorem 3.63). \(\square\)

5. Prove that a subspace \( A \) of a space \( X \) is perfect if and only if \( A \) is closed and has no isolated points.

Proof. Recall that a closed subset of \( X \) is said to be perfect if every point of \( A \) is a limit point of \( A \). Let \( A \) be a closed subset of \( X \).
We will show that $A$ is perfect if and only if $A$ has no isolated points.

**(only if)** Suppose that there exists an isolated point $p$ of $A$.

$\Rightarrow p \in A$, $\{p\}$ is an open subset of $A$.
$\Rightarrow$ There exists an open subset $U$ of $X$ such that $U \cap A = \{p\}$.
$\Rightarrow U$ is an open subset of $X$ containing $p$ such that $(U \setminus \{p\}) \cap A = \emptyset$.
$\Rightarrow p$ is not a limit point of $A$.
$\Rightarrow A$ is not perfect.

**(if)** Suppose that $A$ is not perfect.

$\Rightarrow$ There exists an element $p \in A$ such that $p$ is not a limit point of $A$.
$\Rightarrow$ There is an open subset $U$ of $X$ containing $p$

$\quad$ such that $(U \setminus \{p\}) \cap A = \emptyset$.
$\Rightarrow U \cap A = \{p\}$.
$\Rightarrow \{p\}$ is open in $A$.
$\Rightarrow \{p\}$ is an isolated point of $A$. □
6. (1) Prove that every perfect subset of \([0, 1]\) is uncountable.

(Hint: Assume to the contrary that \(\{a_n\}_{n=1}^{\infty}\) is a countable, closed, perfect subset of \([0, 1]\). Show that there is a nested sequence 
\(\{V_n\}_{n=1}^{\infty}\) of non-empty open subsets of \([0, 1]\) such that \(\overline{V}_n\) does not contain \(a_n\). Consider \(\bigcap_{n=1}^{\infty} V_n\).)

(2) Use part (1) to give a new proof that every non-degenerate interval in \(\mathbb{R}\) is uncountable.

**Proof.** (1) Let \(A\) be a perfect subset of \([0, 1]\). (\(A\) is a closed subset of 
\([0, 1]\) and every point of \(A\) is a limit point of \(A\).)

If \(A\) is finite, since every point of \(A\) is isolated, \(A\) is not perfect.

Suppose that \(A\) is countable and write \(A = \{a_n \mid n \in \mathbb{N}\}\).

\(\Rightarrow\) Since \(A \setminus \{a_1\} \neq \emptyset\), we may choose \(x_1 \in A \setminus \{a_1\}\) and 
\(0 < \delta_1 < |x_1 - a_1|\). Let \(A_1 = A \cap B_d[x_1, \delta_1]\),

where \(B_d[x_1, \delta_1] = \{x \in \mathbb{R} \mid d(x_1, x) \leq \delta_1\}\).

\(\Rightarrow\) (i) Since \(B_d[x_1, \delta_1]\) is compact and \(A\) is closed, \(A_1\) is a compact
subset of $[0, 1]$, $(a_1 \notin A_1$ and $A_1 \subset A$).

(ii) Since $x_1$ is a limit point of $A$, $B_d(x_1, \delta_1) \cap A$ is an infinite subset of $A$.

\[ \Rightarrow \text{We may choose } x_2 \in B_d(x_1, \delta_1) \cap (A \setminus \{a_2\}) \text{ and } 0 < \delta_2 \text{ such that } B_d(x_2, \delta_2) \subset B_d(x_1, \delta_1). \text{ Let } A_2 = A \cap B_d[x_2, \delta_2]. \]

\[ \Rightarrow (i) \text{ } A_2 \text{ is a compact subset of } [0, 1], a_2 \notin A_2 \text{ and } A_2 \subset A_1 \subset A. \]

(ii) Since $x_2$ is a limit point of $A$, $B_d(x_2, \delta_2) \cap A$ is an infinite subset of $A$.

\[ \Rightarrow \text{We may choose } x_3 \in B_d(x_2, \delta_2) \cap (A \setminus \{a_3\}) \text{ and } 0 < \delta_3 \text{ such that } B_d(x_3, \delta_3) \subset B_d(x_2, \delta_2). \text{ Let } A_3 = A \cap B_d[x_3, \delta_3]. \]

\[ \vdots \]

Continuing this process, we obtain a nested sequence $\{A_n\}$ of nonempty compact subsets of $A$ such that $a_n \notin A_n$ for each $n \in \mathbb{N}$.

\[ \Rightarrow \bigcap_{n \in \mathbb{N}} A_n \neq \emptyset \text{ by Theorem 8.} \]
But since $a_n \notin A_n$ for each $n \in \mathbb{N}$, $\bigcap_{n \in \mathbb{N}} A_n = \emptyset$. A contradiction!

$\Rightarrow$ $A$ is not countable.

$\Rightarrow$ Since $A$ is not finite, $A$ is uncountable.

(2) Note the following.

(i) Since $C$ is a perfect subset of $[0, 1]$, $C$ is uncountable by (1) above.

(ii) Since $(0, 1) = [0, 1] \setminus \{0, 1\}$, $(0, 1] = [0, 1] \setminus \{0\}$ and

$[0, 1) = [0, 1] \setminus \{1\}$, $(0, 1)$, $(0, 1]$, $[0, 1)$ are uncountable.

Now in general, let $X$ be a non-degenerate interval in $\mathbb{R}$.

$\Rightarrow$ Since $X$ is homeomorphic with one of $[0, 1], (0, 1], [0, 1), (0, 1)$,

$X$ is uncountable. \qed
7. Prove that every perfect, compact Hausdorff space is uncountable.

Proof. Let $X$ be a perfect, compact Hausdorff space.

⇒ Since $X$ is perfect and Hausdorff, every nonempty open subset meets $X$ in infinitely many points of $X$ and hence $X$ is infinite.

⇒ There exist distinct $x, y \in X (x \neq y)$.

⇒ Since $X$ is Hausdorff, there exist disjoint open sets $U, V$ in $X$ such that $x \in U$ and $y \in V$.

⇒ For $x \in U$ and for each $a \in X \setminus U$, since $X$ is Hausdorff, there exist disjoint open sets $U_a$ and $O_a$ in $X$ such that $x \in U_a$ and $a \in O_a$.

⇒ Since $\{O_a \mid a \in X \setminus U\}$ is an open cover of $X \setminus U$ and $X \setminus U$ is compact, there exist $a_1, a_2, \cdots, a_n \in X \setminus U$

such that $X \setminus U \subset \bigcup_{i=1}^{n} O_{a_i}$.

Let $U_0 = \bigcap_{i=1}^{n} U_{a_i}$. Then $x \in U_0 \subset \overline{U_0} \subset X \setminus \bigcup_{i=1}^{n} O_{a_i} \subset U$.

⇒ Thus we have a nonempty open set $U_0$ in $X$. ⇒ $U_0 \subset \overline{U_0} \subset U$. 
Similarly from $y \in V$, we have a nonempty open set $U_1$ in $X$ such that $y \in U_1 \subset \overline{U}_1 \subset V$.

$\Rightarrow$ Since $U \cap V = \emptyset$, $\overline{U}_0 \cap \overline{U}_1 = \emptyset$.

$\Rightarrow$ Thus we have nonempty open sets $U_0$ and $U_1$ in $X$ such that $\overline{U}_0 \cap \overline{U}_1 = \emptyset$.

$\Rightarrow$ Since $\overline{U}_0$ and $\overline{U}_1$ are compact Hausdorff again and since $U_0$ and $U_1$ are infinite, by applying the above method to $U_0$, we have non-empty open sets $U_{00}$ and $U_{01}$ in $\overline{U}_0$ such that $\overline{U}_{00} \cap \overline{U}_{01} = \emptyset$ and by applying the above method to $U_1$, we have non-empty open sets $U_{10}$ and $U_{11}$ in $\overline{U}_1$ such that $\overline{U}_{10} \cap \overline{U}_{11} = \emptyset$.

$\Rightarrow$ $\overline{U}_{00}$ and $\overline{U}_{01}$ and $\overline{U}_{10}$ and $\overline{U}_{11}$ are compact Hausdorff again, and hence for each pair $(i, j)$ with $i, j \in \{0, 1\}$, there exist non-empty open sets $U_{i0j}$ and $U_{i1j}$ in $\overline{U}_{ij}$ such that $\overline{U}_{i0j} \cap \overline{U}_{i1j} = \emptyset$.

$\Rightarrow$ Inductively for each sequence $i = (i_1, i_2, \cdots, i_n, \cdots)$ with $i_n \in \{0, 1\}$, (i.e., $i \in \{0, 1\}^\mathbb{N}$) we have a sequence of nonempty open sets $U_{i_1}, U_{i_1i_2}, \cdots, U_{i_1i_2\cdots i_n}, \cdots$ such that
\[ U_{i_1} \supset U_{i_1 i_2} \supset \cdots \supset U_{i_1 i_2 \cdots i_n} \supset \cdots \quad \text{and} \quad U_{i_1} \cap U_{i'_1} = \emptyset, \]
\[ U_{i_1 i_2} \cap U_{i_1 i'_2} = \emptyset \quad \text{and so on,} \]
where \( i'_j = 0 \) if \( i_j = 1 \) and \( i'_j = 1 \) if \( i_j = 0 \).
\[ \Rightarrow U_{i_1} \cap U_{i_1 i_2} \cap \cdots \cap U_{i_1 i_2 \cdots i_n} \cap \cdots \neq \emptyset \] by Theorem 6.

Choose \( x_i \in U_{i_1} \cap U_{i_1 i_2} \cap \cdots \cap U_{i_1 i_2 \cdots i_n} \cap \cdots \) and set \( f(i) = x_i \).
\[ \Rightarrow f : \{0,1\}^N \rightarrow X \] is a well-defined injective function.
\[ \Rightarrow \text{Since } \{0,1\}^N \text{ is uncountable, } X \text{ is uncountable.} \]

8. Let \( a = .a_1 a_2 a_3 \cdots \) and \( b = .b_1 b_2 b_3 \cdots \) be points of the Cantor set whose indicated ternary expansions into 0’s and 2’s agree for the first \( n_0 \) terms and disagree at the next term:
\[ a_n = b_n, 1 \leq n \leq n_0, \quad \text{and} \quad a_{n_0 + 1} \neq b_{n_0 + 1}. \] How close can \( a \) and \( b \)?

**Answer.** \( d(a, b) \leq \frac{2}{3^{n_0} + 1} + \cdots \leq \frac{1}{3^{n_0}}. \)
9. Define a function \( f : C \to [0, 1] \) from the Cantor set to \([0, 1]\) as follows:

For \( x \in C \), consider the ternary expansion \( x = \sum_{n=1}^{\infty} \frac{x_n}{3^n}, x_n \in \{0, 2\} \) and define \( f(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^{n+1}} \).

(1) Show that \( f \) is a non-decreasing function (i.e., if \( x \leq y \), then \( f(x) \leq f(y) \)).

(2) Show that \( f \) maps \( C \) onto \([0, 1]\). Conclude from this that \( C \) is uncountable.

(3) Show that

\[
\begin{align*}
  f\left(\frac{1}{3}\right) &= f\left(\frac{2}{3}\right), \\
  f\left(\frac{1}{9}\right) &= f\left(\frac{2}{9}\right), \\
  f\left(\frac{7}{9}\right) &= f\left(\frac{8}{9}\right).
\end{align*}
\]

(4) Show, in general, that \( f(a_n) = f(b_n) \),

where \( a_n, b_n \) are end points of one of the middle third intervals deleted from \( F_n \) in the construction of \( C \).
(5) Define an extension $F : [0, 1] \to [0, 1]$ of $f$ as follows; 
If $x$ belongs to the interval $(a_n, b_n)$ of part (4), then $f(a_n) = f(b_n)$. 
Define $F(x)$ to have this common value. 
Thus, we extend $f$ to $[0, 1]$ by defining the extension to be constant 
on the intervals deleted to form $C$. 
The function $F$ is called the Cantor function. Show that the Cantor function is continuous and sketch its graph.

Proof. (1) $f$ is a non-decreasing function:
Assume that $x = \sum_{n=1}^{\infty} \frac{x_n}{3^n} < y = \sum_{n=1}^{\infty} \frac{y_n}{3^n}$, where $x_n, y_n \in \{0, 2\}$.
$\Rightarrow$ There exists $n_0 \in \mathbb{N}$ such that
$x_n = y_n, 1 \leq n \leq n_0$, and $x_{n_0+1} < y_{n_0+1}$. 
$\Rightarrow f(x) = \sum_{n=1}^{\infty} \frac{x_n}{2^{n+1}} \leq f(y) = \sum_{n=1}^{\infty} \frac{y_n}{2^{n+1}}$. 
$\Rightarrow f : C \to [0, 1]$ is a non-decreasing function.
(2) \textbf{f maps } C \textbf{ onto } [0, 1]:

Let \( x \in [0, 1] \).

\[ \Rightarrow x = \sum_{n=1}^{\infty} \frac{x_n}{2^n}, \text{ where } x_n \in \{0, 1\}. \]

\[ \Rightarrow x = \sum_{n=1}^{\infty} \frac{2x_n}{2^{n+1}}, \text{ where } 2x_n \in \{0, 2\}. \]

\[ \Rightarrow \sum_{n=1}^{\infty} \frac{2x_n}{3^n} \in C \text{ and } f(\sum_{n=1}^{\infty} \frac{2x_n}{3^n}) = \sum_{n=1}^{\infty} \frac{2x_n}{2^{n+1}} = x. \]

\[ \Rightarrow f : C \rightarrow [0, 1] \text{ is surjective}. \]

\[ \Rightarrow \text{Since } [0, 1] \text{ is uncountable, } C \text{ is uncountable.} \]
(3) $f\left(\frac{1}{3}\right) = f\left(\frac{2}{3}\right)$, 
$f\left(\frac{1}{9}\right) = f\left(\frac{2}{9}\right)$, 
$f\left(\frac{7}{9}\right) = f\left(\frac{8}{9}\right)$:

$$\frac{1}{3} = \frac{0}{3} + \sum_{n=2}^{\infty} \frac{2}{3^n}, \quad \frac{2}{3} = \frac{2}{3} + \sum_{n=2}^{\infty} 0,$$

$$\frac{1}{9} = \frac{0}{3} + \frac{0}{3^2} + \sum_{n=3}^{\infty} \frac{2}{3^n}, \quad \frac{2}{9} = \frac{0}{3} + \frac{2}{3^2} + \sum_{n=3}^{\infty} 0,$$

$$\frac{7}{9} = \frac{2}{3} + \frac{0}{3^2} + \sum_{n=3}^{\infty} \frac{2}{3^n}, \quad \frac{8}{9} = \frac{2}{3} + \frac{2}{3^2} + \sum_{n=3}^{\infty} 0.$$

$$\Rightarrow f\left(\frac{1}{3}\right) = \frac{0}{2^2} + \sum_{n=2}^{\infty} \frac{2}{2^{n+1}} = \frac{\frac{2}{2^3}}{1 - \frac{1}{2}} = \frac{2}{2^2} = f\left(\frac{2}{3}\right),$$

$$f\left(\frac{1}{9}\right) = \frac{0}{2^2} + \frac{0}{2^3} + \sum_{n=3}^{\infty} \frac{2}{2^{n+1}} = \frac{\frac{2}{2^4}}{1 - \frac{1}{2}} = \frac{2}{2^3} = f\left(\frac{2}{9}\right),$$

$$f\left(\frac{7}{9}\right) = \frac{2}{2^2} + \frac{0}{2^3} + \sum_{n=3}^{\infty} \frac{2}{2^{n+1}} = \frac{2}{2^2} + \frac{\frac{2}{2^4}}{1 - \frac{1}{2}} = \frac{2}{2^2} + \frac{2}{2^3} = f\left(\frac{8}{9}\right).$$
(4) \( f(a_n) = f(b_n) \):

Let \( a_{n_i}, b_{n_i} \) be the end points of one of the middle third intervals deleted from \( F_n \) in the construction of the Cantor set \( C \).

\[
\Rightarrow a_{1_1} = \frac{1}{3} = \sum_{n=2}^{\infty} \frac{2}{3^n}, \ b_{1_1} = 1 - \frac{1}{3} = \frac{2}{3},
\]

\[
a_{2_1} = a_{1_1} - \frac{2}{3^2} = \sum_{n=3}^{\infty} \frac{2}{3^n}, \ b_{2_1} = a_{1_1} - \frac{1}{3^2} = a_{1_1} - \sum_{n=3}^{\infty} \frac{2}{3^n} = \frac{2}{3^2},
\]

\[
a_{2_2} = b_{1_1} + \frac{1}{3^2} = \frac{2}{3} + \sum_{n=3}^{\infty} \frac{2}{3^n}, \ b_{2_2} = b_{1_1} + \frac{2}{3^2} = \frac{2}{3} + \frac{2}{3^2},
\]

\[
a_{3_1} = a_{2_1} - \frac{2}{3^3} = \sum_{n=4}^{\infty} \frac{2}{3^n}, \ b_{3_1} = a_{2_1} - \frac{1}{3^3} = a_{2_1} - \sum_{n=4}^{\infty} \frac{2}{3^n} = \frac{2}{3^3},
\]

\[
a_{3_2} = b_{2_1} + \frac{1}{3^3} = \frac{2}{3} + \sum_{n=4}^{\infty} \frac{2}{3^n}, \ b_{3_2} = b_{2_1} + \frac{2}{3^3} = \frac{2}{3} + \frac{2}{3^3},
\]

\[
a_{3_3} = a_{2_2} - \frac{2}{3^3} = \frac{2}{3} + \sum_{n=4}^{\infty} \frac{2}{3^n}, \ b_{3_3} = a_{2_2} - \frac{1}{3^3} = \frac{2}{3} + \frac{2}{3^3} + \frac{2}{3^4},
\]

\[
a_{3_4} = b_{2_2} + \frac{1}{3^3} = \frac{2}{3} + \frac{2}{3^2} + \sum_{n=4}^{\infty} \frac{2}{3^n}, \ b_{3_4} = b_{2_2} + \frac{2}{3^3} = \frac{2}{3} + \frac{2}{3^2} + \frac{2}{3^3},
\]
\begin{align*}
a_{41} &= a_{31} - \frac{2}{3^4}, \quad b_{41} = a_{31} - \frac{1}{3^4}, \quad a_{42} = b_{31} + \frac{1}{3^4}, \quad b_{42} = b_{31} + \frac{2}{3^4}, \\
a_{43} &= a_{32} - \frac{2}{3^4}, \quad b_{43} = a_{32} - \frac{1}{3^4}, \quad a_{44} = b_{32} + \frac{1}{3^4}, \quad b_{44} = b_{32} + \frac{2}{3^4}, \\
a_{45} &= a_{33} - \frac{2}{3^4}, \quad b_{45} = a_{33} - \frac{1}{3^4}, \quad a_{46} = b_{33} + \frac{1}{3^4}, \quad b_{46} = b_{33} + \frac{2}{3^4}, \\
a_{47} &= a_{34} - \frac{2}{3^4}, \quad b_{47} = a_{34} - \frac{1}{3^4}, \quad a_{48} = b_{34} + \frac{1}{3^4}, \quad b_{48} = b_{34} + \frac{2}{3^4}, \\
\vdots \\
a_{n_{2i-1}} &= a_{(n-1)i} - \frac{2}{3^n}, \quad b_{n_{2i-1}} = a_{(n-1)i} - \frac{1}{3^n} = a_{(n-1)i} - \sum_{k=n+1}^{\infty} \frac{2}{3^k}, \\
a_{n_{2i}} &= b_{(n-1)i} + \frac{1}{3^n} = b_{(n-1)i} + \sum_{k=n+1}^{\infty} \frac{2}{3^k}, \quad b_{n_{2i}} = b_{(n-1)i} + \frac{2}{3^n}, \\
\text{where } a_{(n-1)i} = \sum_{k=1}^{\infty} \frac{x_k}{3^k} \text{ and } b_{(n-1)i} = \sum_{k=1}^{\infty} \frac{y_k}{3^k} \text{ are expressed with } x_n, y_n \in \{0, 2\}, \text{ for each } i = 1, 2, \ldots, 2^{n-1}. \\
\Rightarrow f(a_{n_{2i-1}}) - f(b_{n_{2i-1}}) &= f(a_{(n-1)i} - \frac{2}{3^n}) - f(a_{(n-1)i} - \sum_{k=n+1}^{\infty} \frac{2}{3^k})
\end{align*}
\[
= -\frac{2}{2^{n+1}} + \sum_{k=n+1}^{\infty} \frac{2}{2^{k+1}} = -\frac{1}{2^n} + \frac{2}{2^{n+2}} = -\frac{1}{2^n} + \frac{1}{2^n} = 0 \text{ and}
\]
\[
f(a_{n,2}) - f(b_{n,2}) = f(b_{(n-1),i} + \sum_{k=n+1}^{\infty} \frac{2}{3^k}) - f(b_{(n-1),i} + \frac{2}{3^n})
\]
\[
= \sum_{k=n+1}^{\infty} \frac{2}{2^{k+1}} - \frac{2}{2^{n+1}} = \frac{2}{2^{n+2}} \cdot \frac{2}{1 - \frac{1}{2}} - \frac{1}{2^n} = \frac{1}{2^n} - \frac{1}{2^n} = 0.
\]

(5) **The continuity of the Cantor function and its graph:**

Since \( f : C \rightarrow [0, 1] \) is a non-decreasing surjective function
by (1) and (2) above, \( F : [0, 1] \rightarrow [0, 1] \) is also a non-decreasing surjective function.

Also note that \( \mathcal{S} = \{[0, b), (b, 1] \mid b \in [0, 1]\} \) is a subbasis of \([0, 1]\).

(i) First we will show that \( F^{-1}([0, b)) \) is open in \([0, 1] \forall b \in [0, 1]; \)

Let \( b \in [0, 1] \) and let \( x \in F^{-1}([0, b)). \)

\[ \Rightarrow 0 \leq F(x) < b. \]

Choose \( r \in \mathbb{R} \) such that \( F(x) < r < b. \)

\[ \Rightarrow \text{Since } F : [0, 1] \rightarrow [0, 1] \text{ is surjective, there exists } a, s \in [0, 1] \]
such that \( b = F(a) \) and \( r = F(s). \)
⇒ Since $0 \leq F(x) < r = F(s)$ and $F$ is non-decreasing, $0 \leq x < s$. 
⇒ Since $F$ is non-decreasing, $F([0, s)) \subset [0, r]$.
⇒ $x \in [0, s) \subset F^{-1}([0, r]) \subset F^{-1}([0, b])$.
⇒ Since $[0, s)$ is open in $[0, 1]$, $F^{-1}([0, b))$ is open in $[0, 1]$.
(ii) Similarly, $F^{-1}((b, 1])$ is open in $[0, 1]$ for any $b \in [0, 1]$.
⇒ $F^{-1}(S)$ is open in $[0, 1]$ for every $S \in \mathcal{S}$ by (i) and (ii).
⇒ Since $\mathcal{S}$ is a subbasis for $[0, 1]$, $F : [0, 1] \to [0, 1]$ is continuous by Theorem 4.39.
The sketch of the graph of $F : [0, 1] \to [0, 1]$ is as in Figure 10.
Figure 11: The graph of the Cantor function
Suggestions for Further Reading

Most textbooks on point-set and general topology include compactness and related properties. For a readable account somewhat more advanced than in this text, *General Topology* by Kelley and *Topology: A First Course* by Munkres are recommended. For a proof that every compact, perfect, totally disconnected, metric space is homeomorphic to the Cantor set, see *Topology* by Hocking and Young or *General Topology* by Willard. These texts also contain accessible proofs of the Hahn-Mazurkiewicz Theorem.

Historical Notes for Chapter 6

The term *compact* was introduced in 1904 by Maurice Fréchet to describe those spaces in which every sequence has a convergent
subsequence. Such spaces are now called *sequentially compact*. Sequential compactness is equivalent to compactness in metric spaces. The property of compactness has a long and complicated history. Basically, the purpose of defining compactness was to generalize the properties of closed and bounded intervals to general topological spaces. The early attempts to achieve this goal included sequential compactness, countable compactness, the Bolzano-Weierstrass property, and, finally, the modern property of compactness, which was introduced by P. S. Alexander and Paul Urysohn in 1923.

The first theorems on compactness was the Heine-Borel Theorem (Theorem 2.41), which states that a closed and bounded interval $[a, b]$ is compact. Actually, Emile Borel proved in his doctoral thesis in 1894 that every countable open cover of $[a, b]$ has a finite subcover; in other words, that $[a, b]$ is countably compact. The extension to arbitrary open covers was made possible by the work of Ernst Lindelöf (1870 - 1946), who showed that every open cover of $[a, b]$ has
a countable subcover. Eduard Heine (1821 - 1881), whose name appears in the Heine-Borel Theorem, was not involved in its discovery. Heine’s primary mathematical contribution was to prove in 1872 that every continuous real-valued function defined on \([a, b]\) is uniformly continuous. A. M. Schoenflies (1858 - 1923), in reading Heine’s proof, noted a relation to Borel’s theorem and gave Borel’s result its present name, the Heine-Borel Theorem. It is doubtful that Heine would have claimed any credit for the famous theorem named in his honor.

The Heine-Borel Theorem was easily extended from closed and bounded intervals to closed and bounded subsets of \(\mathbb{R}\). W. H. Young (1863 - 1942) extended the theorem to \(\mathbb{R}^2\) in 1902 by proving that every open cover of a closed and bounded subset of \(\mathbb{R}^2\) has a finite subcover. Henri Lesbegue (1875 - 1941) published the same result in 1904 and claimed to have known the extension to \(\mathbb{R}^n\) as early as 1898. Consideration of compactness via the finite intersection property is
due to Frigyes Riesz in 1908. Felix Hausdorff showed in *Grundzüge der Mengenlehre* in 1914 that sequential compactness, countable compactness, the Bolzano-Weierstrass property, and compactness are all equivalent in metric spaces. The equivalence of sequential compactness and compactness in the metric case was shown earlier by Fréchet.

The crucial step in the characterization of compact subsets of $\mathbb{R}^n$ (Theorem 28) can be traced to Weierstrass, who proved that closed and bounded subsets of $\mathbb{R}^2$ have the Bolzano-Weierstrass property.

Lindelöf spaces were first considered by Ernst Lindelöf, who proved the Lindelöf Theorem (Theorem 32) and that every subspaces of $\mathbb{R}^n$ has the Lindelöf property in 1903. The term *Lindelöf space* was coined by K. Kuratowski and W. Sierpinski, who initiated the formal study of these spaces in 1921. The prototype of Theorem 26 on uniform continuity is, of course, Heine’s Theorem of 1872 which showed that every real-valued continuous function with domain a
closed and bounded interval is uniformly continuous. Camille Jordan proved in 1893 the precursors of Theorems 17 and 20 for compact subsets of $\mathbb{R}^n$.

As mentioned earlier, the definition of compactness in use today was proposed in 1923 by the Russian mathematicians P. S. Alexander and Paul Urysohn. They called the property \textit{bicompactness} and developed its properties in a series of papers published in 1923, 1924, and 1929. Their results included Theorems 11, 12, and 14. Alexander proved Theorems 17, 19, and 20 in 1927. An equivalent definition of compactness was given in 1921 by Leopold Vietoris, whose results included Theorems 6, 11, and 12.

Local compactness was introduced independently by Heinrich Tietze and Alexandorff. The one-point compactification and Theorem 51 are due to Alexandorff. The Stone-Čech compactification was developed in 1937 independently by Eduard Čech and M. H. Stone.

Of the other ideas of Chapter 6, Lebesgue numbers were first used by
Henri Lebesgue, and total boundedness was defined by Hausdorff in *Grundzüge der Mengenlehre*. The Cantor set was studied by Cantor in 1883, and the extension of Cantor’s Nested Intervals Theorem (Theorem 2.39) to Cantor’s Theorem of Deduction (Theorem 8) was also made by Cantor. As was noted in the text, space-filling curves were first considered by Guiseppe Peano, and Peano spaces are named in his honor. The Hahn-Mazurkiewicz Theorem is due to Hans Hahn and Stephan Mazurkiewicz.
References


