□ APPENDIX

G Complex Numbers

1. $(5-6i)+(3+2i)=(5+3)+(-6+2)i=8+(-4)i=8-4i$

2. $\left(4-\frac{1}{2}i\right)-\left(9+\frac{5}{2}i\right)=(4-9)+\left(-\frac{1}{2}-\frac{5}{2}\right)i=-5+(-3)i=-5-3i$

3. $(2+5i)(4-i)=2(4)+2(-i)+(5i)(4)+(5i)(-i)=8-2i+20i-5i^2$
$=8+18i-5(-1)=8+18i+5=13+18i$

4. $(1-2i)(8-3i)=8-3i-16i+6(-1)=2-19i$

5. $\overline{12+7i}=12-7i$

6. $2i\left(\frac{1}{2}-i\right)=i-2(-1)=2+i \quad\Rightarrow\quad \overline{2i\left(\frac{1}{2}-i\right)}=\overline{2+i}=2-i$

7. $\dfrac{1+4i}{3+2i}=\dfrac{1+4i}{3+2i}\cdot\dfrac{3-2i}{3-2i}=\dfrac{3-2i+12i-8(-1)}{3^2+2^2}=\dfrac{11+10i}{13}=\dfrac{11}{13}+\dfrac{10}{13}i$

8. $\dfrac{3+2i}{1-4i}=\dfrac{3+2i}{1-4i}\cdot\dfrac{1+4i}{1+4i}=\dfrac{3+12i+2i+8(-1)}{1^2+4^2}=\dfrac{-5+14i}{17}=-\dfrac{5}{17}+\dfrac{14}{17}i$

9. $\dfrac{1}{1+i}=\dfrac{1}{1+i}\cdot\dfrac{1-i}{1-i}=\dfrac{1-i}{1-(-1)}=\dfrac{1-i}{2}=\dfrac{1}{2}-\dfrac{1}{2}i$

10. $\dfrac{3}{4-3i}=\dfrac{3}{4-3i}\cdot\dfrac{4+3i}{4+3i}=\dfrac{12+9i}{16-9(-1)}=\dfrac{12}{25}+\dfrac{9}{25}i$

11. $i^3=i^2\cdot i=(-1)i=-i$

12. $i^{100}=\left(i^2\right)^{50}=(-1)^{50}=1$

13. $\sqrt{-25}=\sqrt{25}\,i=5i$

14. $\sqrt{-3}\sqrt{-12}=\sqrt{3}\,i\sqrt{12}\,i=\sqrt{3\cdot 12}\,i^2=\sqrt{36}\,(-1)=-6$

15. $\overline{12-5i}=12+15i$ and $|12-15i|=\sqrt{12^2+(-5)^2}=\sqrt{144+25}=\sqrt{169}=13$

16. $\overline{-1+2\sqrt{2}\,i}=-1-2\sqrt{2}\,i$ and $\left|-1+2\sqrt{2}\,i\right|=\sqrt{(-1)^2+\left(2\sqrt{2}\right)^2}=\sqrt{1+8}=\sqrt{9}=3$

17. $\overline{-4i}=\overline{0-4i}=0+4i=4i$ and $|-4i|=\sqrt{0^2+(-4)^2}=\sqrt{16}=4$

18. Let $z=a+bi$, $w=c+di$.

(a) $\overline{z+w}=\overline{(a+bi)+(c+di)}=\overline{(a+c)+(b+d)i}$
$=(a+c)-(b+d)i=(a-bi)+(c-di)=\overline{z}+\overline{w}$

(b) $\overline{zw}=\overline{(a+bi)(c+di)}=\overline{(ac-bd)+(ad+bc)i}=(ac-bd)-(ad+bc)i$.
On the other hand, $\overline{z}\,\overline{w}=(a-bi)(c-di)=(ac-bd)-(ad+bc)i=\overline{zw}$.

(c) Use mathematical induction and part (b): Let S_n be the statement that $\overline{z^n} = \overline{z}^{\,n}$.
S_1 is true because $\overline{z^1} = \overline{z} = \overline{z}^{\,1}$. Assume S_k is true, that is $\overline{z^k} = \overline{z}^{\,k}$. Then
$\overline{z^{k+1}} = \overline{z^{1+k}} = \overline{zz^k} = \overline{z}\,\overline{z^k}$ [part (b) with $w = z^k$] $= \overline{z}^{\,1}\overline{z}^{\,k} = \overline{z}^{\,1+k} = \overline{z}^{\,k+1}$, which shows that S_{k+1} is true.
Therefore, by mathematical induction, $\overline{z^n} = \overline{z}^{\,n}$ for every positive integer n.
Another proof: Use part (b) with $w = z$, and mathematical induction.

19. $4x^2 + 9 = 0 \;\Leftrightarrow\; 4x^2 = -9 \;\Leftrightarrow\; x^2 = -\frac{9}{4} \;\Leftrightarrow\; x = \pm\sqrt{-\frac{9}{4}} = \pm\sqrt{\frac{9}{4}}\,i = \pm\frac{3}{2}i.$

20. $x^4 = 1 \;\Leftrightarrow\; x^4 - 1 = 0 \;\Leftrightarrow\; (x^2 - 1)(x^2 + 1) = 0 \;\Leftrightarrow\; x^2 - 1 = 0$ or $x^2 + 1 = 0 \;\Leftrightarrow$
$x = \pm 1$ or $x = \pm i$.

21. $x^2 + 2x + 5 = 0 \;\Leftrightarrow\; x = \dfrac{-2 \pm \sqrt{2^2 - 4 \cdot 1 \cdot 5}}{2 \cdot 1} = \dfrac{-2 \pm \sqrt{-16}}{2} = \dfrac{-2 \pm 4i}{2} = -1 \pm 2i$

22. $2x^2 - 2x + 1 = 0 \;\Leftrightarrow\; x = \dfrac{-(-2) \pm \sqrt{(-2)^2 - 4 \cdot 2 \cdot 1}}{2 \cdot 2} = \dfrac{2 \pm \sqrt{-4}}{4} = \dfrac{2 \pm 2i}{4} = \dfrac{1}{2} \pm \dfrac{1}{2}i$

23. By the quadratic formula, $z^2 + z + 2 = 0 \;\Leftrightarrow\; z = \dfrac{-1 \pm \sqrt{1^2 - 4(1)(2)}}{2(1)} = \dfrac{-1 \pm \sqrt{-7}}{2} = -\dfrac{1}{2} \pm \dfrac{\sqrt{7}}{2}i.$

24. $z^2 + \frac{1}{2}z + \frac{1}{4} = 0 \;\Leftrightarrow\; 4z^2 + 2z + 1 = 0 \;\Leftrightarrow$
$z = \dfrac{-2 \pm \sqrt{2^2 - 4(4)(1)}}{2(4)} = \dfrac{-2 \pm \sqrt{-12}}{8} = \dfrac{-2 \pm 2\sqrt{3}\,i}{8} = -\dfrac{1}{4} \pm \dfrac{\sqrt{3}}{4}i$

25. For $z = -3 + 3i$, $r = \sqrt{(-3)^2 + 3^2} = 3\sqrt{2}$ and $\tan\theta = \frac{3}{-3} = -1 \;\Rightarrow\; \theta = \frac{3\pi}{4}$ (since z lies in the second quadrant). Therefore, $-3 + 3i = 3\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$.

26. For $z = 1 - \sqrt{3}\,i$, $r = \sqrt{1^2 + \left(-\sqrt{3}\right)^2} = 2$ and $\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \;\Rightarrow\; \theta = \frac{5\pi}{3}$ (since z lies in the fourth quadrant). Therefore, $1 - \sqrt{3}\,i = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$.

27. For $z = 3 + 4i$, $r = \sqrt{3^2 + 4^2} = 5$ and $\tan\theta = \frac{4}{3} \;\Rightarrow\; \theta = \tan^{-1}\left(\frac{4}{3}\right)$ (since z lies in the first quadrant).
Therefore, $3 + 4i = 5\left[\cos\left(\tan^{-1}\frac{4}{3}\right) + i\sin\left(\tan^{-1}\frac{4}{3}\right)\right]$.

28. For $z = 8i$, $r = \sqrt{0^2 + 8^2} = 8$ and $\tan\theta = \frac{8}{0}$ is undefined, so $\theta = \frac{\pi}{2}$ (since z lies on the positive imaginary axis).
Therefore, $8i = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$.

29. For $z = \sqrt{3} + i$, $r = \sqrt{\left(\sqrt{3}\right)^2 + 1^2} = 2$ and $\tan\theta = \frac{1}{\sqrt{3}} \;\Rightarrow\; \theta = \frac{\pi}{6} \;\Rightarrow\; z = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.
For $w = 1 + \sqrt{3}\,i$, $r = 2$ and $\tan\theta = \sqrt{3} \;\Rightarrow\; \theta = \frac{\pi}{3} \;\Rightarrow\; w = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$.
Therefore, $zw = 2 \cdot 2\left[\cos\left(\frac{\pi}{6} + \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{\pi}{3}\right)\right] = 4\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$,
$z/w = \frac{2}{2}\left[\cos\left(\frac{\pi}{6} - \frac{\pi}{3}\right) + i\sin\left(\frac{\pi}{6} - \frac{\pi}{3}\right)\right] = \cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)$, and $1 = 1 + 0i = 1(\cos 0 + i\sin 0) \;\Rightarrow$
$1/z = \frac{1}{2}\left[\cos\left(0 - \frac{\pi}{6}\right) + i\sin\left(0 - \frac{\pi}{6}\right)\right] = \frac{1}{2}\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$. For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$.

30. For $z = 4\sqrt{3} - 4i$, $r = \sqrt{(4\sqrt{3})^2 + (-4)^2} = \sqrt{64} = 8$ and $\tan\theta = \frac{-4}{4\sqrt{3}} = -\frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{11\pi}{6} \Rightarrow$ $z = 8\left(\cos\frac{11\pi}{6} + i\sin\frac{11\pi}{6}\right)$. For $w = 8i$, $r = \sqrt{0^2 + 8^2} = 8$ and $\tan\theta = \frac{8}{0}$ is undefined, so $\theta = \frac{\pi}{2} \Rightarrow$ $w = 8\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. Therefore, $zw = 8 \cdot 8\left[\cos\left(\frac{11\pi}{6} + \frac{\pi}{2}\right) + i\sin\left(\frac{11\pi}{6} + \frac{\pi}{2}\right)\right] = 64\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, $z/w = \frac{8}{8}\left[\cos\left(\frac{11\pi}{6} - \frac{\pi}{2}\right) + i\sin\left(\frac{11\pi}{6} - \frac{\pi}{2}\right)\right] = \cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}$, and $1 = 1 + 0i = 1(\cos 0 + i\sin 0) \Rightarrow 1/z = \frac{1}{8}\left[\cos\left(0 - \frac{11\pi}{6}\right) + i\sin\left(0 - \frac{11\pi}{6}\right)\right] = \frac{1}{8}\left[\cos\left(\frac{\pi}{6}\right) + i\sin\left(\frac{\pi}{6}\right)\right]$. For $1/z$, we could also use the formula that precedes Example 5 to obtain $1/z = \frac{1}{8}\left(\cos\frac{11\pi}{6} - i\sin\frac{11\pi}{6}\right)$.

31. For $z = 2\sqrt{3} - 2i$, $r = \sqrt{(2\sqrt{3})^2 + (-2)^2} = 4$ and $\tan\theta = \frac{-2}{2\sqrt{3}} = -\frac{1}{\sqrt{3}}$ $\Rightarrow \theta = -\frac{\pi}{6} \Rightarrow z = 4\left[\cos\left(-\frac{\pi}{6}\right) + i\sin\left(-\frac{\pi}{6}\right)\right]$. For $w = -1 + i$, $r = \sqrt{2}$, $\tan\theta = \frac{1}{-1} = -1 \Rightarrow \theta = \frac{3\pi}{4} \Rightarrow w = \sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$. Therefore,

$zw = 4\sqrt{2}\left[\cos\left(-\frac{\pi}{6} + \frac{3\pi}{4}\right) + i\sin\left(-\frac{\pi}{6} + \frac{3\pi}{4}\right)\right] = 4\sqrt{2}\left(\cos\frac{7\pi}{12} + i\sin\frac{7\pi}{12}\right)$,

$z/w = \frac{4}{\sqrt{2}}\left[\cos\left(-\frac{\pi}{6} - \frac{3\pi}{4}\right) + i\sin\left(-\frac{\pi}{6} - \frac{3\pi}{4}\right)\right] = \frac{4}{\sqrt{2}}\left[\cos\left(-\frac{11\pi}{12}\right) + i\sin\left(-\frac{11\pi}{12}\right)\right]$

$= 2\sqrt{2}\left(\cos\frac{13\pi}{12} + i\sin\frac{13\pi}{12}\right)$, and

$1/z = \frac{1}{4}\left[\cos\left(-\frac{\pi}{6}\right) - i\sin\left(-\frac{\pi}{6}\right)\right] = \frac{1}{4}\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$.

32. For $z = 4(\sqrt{3} + i) = 4\sqrt{3} + 4i$, $r = \sqrt{(4\sqrt{3})^2 + 4^2} = \sqrt{64} = 8$ and $\tan\theta = \frac{4}{4\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow$ $z = 8\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. For $w = -3 - 3i$, $r = \sqrt{(-3)^2 + (-3)^2} = \sqrt{18} = 3\sqrt{2}$ and $\tan\theta = \frac{-3}{-3} = 1 \Rightarrow \theta = \frac{5\pi}{4} \Rightarrow w = 3\sqrt{2}\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right)$. Therefore,

$zw = 8 \cdot 3\sqrt{2}\left[\cos\left(\frac{\pi}{6} + \frac{5\pi}{4}\right) + i\sin\left(\frac{\pi}{6} + \frac{5\pi}{4}\right)\right] = 24\sqrt{2}\left(\cos\frac{17\pi}{12} + i\sin\frac{17\pi}{12}\right)$,

$z/w = \frac{8}{3\sqrt{2}}\left[\cos\left(\frac{\pi}{6} - \frac{5\pi}{4}\right) + i\sin\left(\frac{\pi}{6} - \frac{5\pi}{4}\right)\right] = \frac{4\sqrt{2}}{3}\left[\cos\left(-\frac{13\pi}{12}\right) + i\sin\left(-\frac{13\pi}{12}\right)\right]$, and

$1/z = \frac{1}{8}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$.

33. For $z = 1 + i$, $r = \sqrt{2}$ and $\tan\theta = \frac{1}{1} = 1 \Rightarrow \theta = \frac{\pi}{4} \Rightarrow z = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. So by De Moivre's Theorem,

$$(1+i)^{20} = \left[\sqrt{2}\left(\cos\tfrac{\pi}{4} + i\sin\tfrac{\pi}{4}\right)\right]^{20} = \left(2^{1/2}\right)^{20}\left(\cos\tfrac{20\cdot\pi}{4} + i\sin\tfrac{20\cdot\pi}{4}\right)$$
$$= 2^{10}(\cos 5\pi + i\sin 5\pi) = 2^{10}[-1 + i(0)] = -2^{10} = -1024$$

34. For $z = 1 - \sqrt{3}i$, $r = \sqrt{1^2 + (-\sqrt{3})^2} = 2$ and $\tan\theta = \frac{-\sqrt{3}}{1} = -\sqrt{3} \Rightarrow \theta = \frac{5\pi}{3} \Rightarrow$ $z = 2\left(\cos\frac{5\pi}{3} + i\sin\frac{5\pi}{3}\right)$. So by De Moivre's Theorem,

$$\left(1 - \sqrt{3}\,i\right)^5 = \left[2\left(\cos\tfrac{5\pi}{3} + i\sin\tfrac{5\pi}{3}\right)\right]^5 = 2^5\left(\cos\tfrac{5\cdot 5\pi}{3} + i\sin\tfrac{5\cdot 5\pi}{3}\right)$$
$$= 2^5\left(\cos\tfrac{\pi}{3} + i\sin\tfrac{\pi}{3}\right) = 32\left(\tfrac{1}{2} + \tfrac{\sqrt{3}}{2}i\right) = 16 + 16\sqrt{3}\,i$$

35. For $z = 2\sqrt{3} + 2i$, $r = \sqrt{(2\sqrt{3})^2 + 2^2} = \sqrt{16} = 4$ and $\tan\theta = \frac{2}{2\sqrt{3}} = \frac{1}{\sqrt{3}} \Rightarrow \theta = \frac{\pi}{6} \Rightarrow$

$z = 4\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$. So by De Moivre's Theorem,

$$\left(2\sqrt{3} + 2i\right)^5 = \left[4\left(\cos\tfrac{\pi}{6} + i\sin\tfrac{\pi}{6}\right)\right]^5 = 4^5\left(\cos\tfrac{5\pi}{6} + i\sin\tfrac{5\pi}{6}\right) = 1024\left[-\tfrac{\sqrt{3}}{2} + \tfrac{1}{2}i\right] = -512\sqrt{3} + 512i$$

36. For $z = 1 - i$, $r = \sqrt{2}$ and $\tan\theta = \frac{-1}{1} = -1 \Rightarrow \theta = \frac{7\pi}{4} \Rightarrow z = \sqrt{2}\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) \Rightarrow$

$$(1-i)^8 = \left[\sqrt{2}\left(\cos\tfrac{7\pi}{4} + i\sin\tfrac{7\pi}{4}\right)\right]^8 = 2^4\left(\cos\tfrac{8\cdot 7\pi}{4} + i\sin\tfrac{8\cdot 7\pi}{4}\right)$$

$$= 16(\cos 14\pi + i\sin 14\pi) = 16(1 + 0i) = 16$$

37. $1 = 1 + 0i = 1(\cos 0 + i\sin 0)$. Using Equation 3 with $r = 1$, $n = 8$, and $\theta = 0$, we have

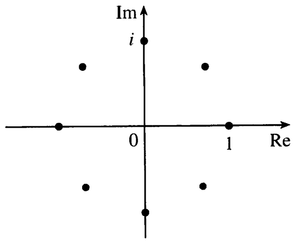
$$w_k = 1^{1/8}\left[\cos\left(\frac{0 + 2k\pi}{8}\right) + i\sin\left(\frac{0 + 2k\pi}{8}\right)\right] = \cos\frac{k\pi}{4} + i\sin\frac{k\pi}{4}, \text{ where } k = 0, 1, 2, \ldots, 7.$$

$w_0 = 1(\cos 0 + i\sin 0) = 1$, $w_1 = 1\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

$w_2 = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right) = i$, $w_3 = 1\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right) = -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i$,

$w_4 = 1(\cos\pi + i\sin\pi) = -1$, $w_5 = 1\left(\cos\frac{5\pi}{4} + i\sin\frac{5\pi}{4}\right) = -\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$,

$w_6 = 1\left(\cos\frac{3\pi}{2} + i\sin\frac{3\pi}{2}\right) = -i$, $w_7 = 1\left(\cos\frac{7\pi}{4} + i\sin\frac{7\pi}{4}\right) = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i$



38. $32 = 32 + 0i = 32(\cos 0 + i\sin 0)$. Using Equation 3 with $r = 32$, $n = 5$, and $\theta = 0$, we have

$$w_k = 32^{1/5}\left[\cos\left(\frac{0 + 2k\pi}{5}\right) + i\sin\left(\frac{0 + 2k\pi}{5}\right)\right] = 2\left(\cos\tfrac{2}{5}\pi k + i\sin\tfrac{2}{5}\pi k\right), \text{ where } k = 0, 1, 2, 3, 4.$$

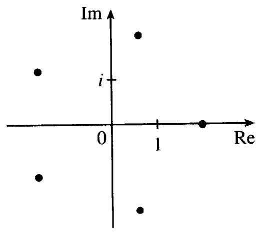
$w_0 = 2(\cos 0 + i\sin 0) = 2$

$w_1 = 2\left(\cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}\right)$

$w_2 = 2\left(\cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5}\right)$

$w_3 = 2\left(\cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5}\right)$

$w_4 = 2\left(\cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5}\right)$



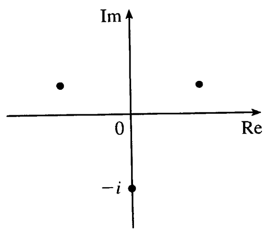
39. $i = 0 + i = 1\left(\cos\frac{\pi}{2} + i\sin\frac{\pi}{2}\right)$. Using Equation 3 with $r = 1$, $n = 3$, and $\theta = \frac{\pi}{2}$, we have

$$w_k = 1^{1/3}\left[\cos\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{2} + 2k\pi}{3}\right)\right], \text{ where } k = 0, 1, 2.$$

$w_0 = \left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_1 = \left(\cos\frac{5\pi}{6} + i\sin\frac{5\pi}{6}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$

$w_2 = \left(\cos\frac{9\pi}{6} + i\sin\frac{9\pi}{6}\right) = -i$



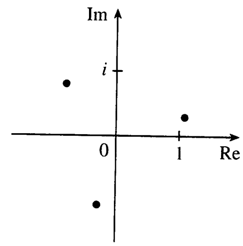
40. $1 + i = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$. Using Equation 3 with $r = \sqrt{2}$, $n = 3$, and $\theta = \frac{\pi}{4}$, we have

$$w_k = \left(\sqrt{2}\right)^{1/3}\left[\cos\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right) + i\sin\left(\frac{\frac{\pi}{4} + 2k\pi}{3}\right)\right], \text{ where } k = 0, 1, 2.$$

$$w_0 = 2^{1/6}\left(\cos\tfrac{\pi}{12} + i\sin\tfrac{\pi}{12}\right)$$

$$w_1 = 2^{1/6}\left(\cos\tfrac{3\pi}{4} + i\sin\tfrac{3\pi}{4}\right) = 2^{1/6}\left(-\tfrac{1}{\sqrt{2}} + \tfrac{1}{\sqrt{2}}i\right) = -2^{-1/3} + 2^{-1/3}i$$

$$w_2 = 2^{1/6}\left(\cos\tfrac{17\pi}{12} + i\sin\tfrac{17\pi}{12}\right)$$



41. Using Euler's formula (6) with $y = \frac{\pi}{2}$, we have $e^{i\pi/2} = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = 0 + 1i = i$.

42. Using Euler's formula (6) with $y = 2\pi$, we have $e^{2\pi i} = \cos 2\pi + i\sin 2\pi = 1$.

43. Using Euler's formula (6) with $y = \dfrac{\pi}{3}$, we have $e^{i\pi/3} = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + \dfrac{\sqrt{3}}{2}i$.

44. Using Euler's formula (6) with $y = -\pi$, we have $e^{-i\pi} = \cos(-\pi) + i\sin(-\pi) = -1$.

45. Using Equation 7 with $x = 2$ and $y = \pi$, we have $e^{2+i\pi} = e^2 e^{i\pi} = e^2(\cos\pi + i\sin\pi) = e^2(-1+0) = -e^2$.

46. Using Equation 7 with $x = \pi$ and $y = 1$, we have $e^{\pi+i} = e^{\pi}\cdot e^{1i} = e^{\pi}(\cos 1 + i\sin 1) = e^{\pi}\cos 1 + (e^{\pi}\sin 1)i$.

47. $F(x) = e^{rx} = e^{(a+bi)x} = e^{ax+bxi} = e^{ax}(\cos bx + i\sin bx) = e^{ax}\cos bx + i(e^{ax}\sin bx) \quad\Rightarrow$

$$\begin{aligned} F'(x) &= (e^{ax}\cos bx)' + i(e^{ax}\sin bx)' = (ae^{ax}\cos bx - be^{ax}\sin bx) + i(ae^{ax}\sin bx + be^{ax}\cos bx) \\ &= a\left[e^{ax}(\cos bx + i\sin bx)\right] + b\left[e^{ax}(-\sin bx + i\cos bx)\right] = ae^{rx} + b\left[e^{ax}(i^2\sin bx + i\cos bx)\right] \\ &= ae^{rx} + bi\left[e^{ax}(\cos bx + i\sin bx)\right] = ae^{rx} + bie^{rx} = (a+bi)e^{rx} = re^{rx} \end{aligned}$$

48. (a) From Exercise 47, $F(x) = e^{(1+i)x} \quad\Rightarrow\quad F'(x) = (1+i)e^{(1+i)x}$. So

$$\int e^{(1+i)x}\,dx = \frac{1}{1+i}\int F'(x)\,dx = \frac{1}{1+i}F(x) + C = \frac{1-i}{2}F(x) + C = \frac{1-i}{2}e^{(1+i)x} + C$$

(b) $\int e^{(1+i)x}\,dx = \int e^x e^{ix}\,dx = \int e^x(\cos x + i\sin x)\,dx = \int e^x\cos x\,dx + i\int e^x\sin x\,dx$ **(1)**.

Also,

$$\begin{aligned} \frac{1-i}{2}e^{(1+i)x} &= \tfrac{1}{2}e^{(1+i)x} - \tfrac{1}{2}ie^{(1+i)x} = \tfrac{1}{2}e^{x+ix} - \tfrac{1}{2}ie^{x+ix} \\ &= \tfrac{1}{2}e^x(\cos x + i\sin x) - \tfrac{1}{2}ie^x(\cos x + i\sin x) \\ &= \tfrac{1}{2}e^x\cos x + \tfrac{1}{2}e^x\sin x + \tfrac{1}{2}ie^x\sin x - \tfrac{1}{2}ie^x\cos x \\ &= \tfrac{1}{2}e^x(\cos x + \sin x) + i\left[\tfrac{1}{2}e^x(\sin x - \cos x)\right] \quad \textbf{(2)} \end{aligned}$$

Equating the real and imaginary parts in **(1)** and **(2)**, we see that $\int e^x\cos x\,dx = \frac{1}{2}e^x(\cos x + \sin x) + C$ and $\int e^x\sin x\,dx = \frac{1}{2}e^x(\sin x - \cos x) + C$.