THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

27th Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION
(AIME II)

SOLUTIONS PAMPHLET

Wednesday, April 1, 2009

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:
American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@maa.org

The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:
Steve Blasberg, AIME Chair
San Jose, CA 95129 USA

1. 1,2,3,. . .,4, . . . , 5
   $1, 2, 3, 4, \ldots, 5$

   (Answer: 114)

   Bill must have used $164 - 130 = 34$ more ounces of red paint than blue paint and $188 - 130 = 58$ more ounces of white paint than blue paint. It follows that it took $34 + 58 = 92$ ounces of paint to paint the pink stripe, and therefore 92 ounces to paint each stripe. Thus $130 - 92 = 38$ ounces of blue paint were left, and a total of $3 \cdot 38 = 114$ ounces of paint were left.

2. (Answer: 469)

   It follows from the properties of exponents that

$$\begin{aligned}
&a^{(\log_3 7)^2} + b^{(\log_7 11)^2} + c^{(\log_{11} 25)^2}\\
&= (a^{\log_3 7})^{\log_3 7} + (b^{\log_7 11})^{\log_7 11} + (c^{\log_{11} 25})^{\log_{11} 25}\\
&= 27^{\log_3 7} + 49^{\log_7 11} + \sqrt{11}^{\log_{11} 25}\\
&= 3^{3\log_3 7} + 7^{2\log_7 11} + 11^{\frac{1}{2}\cdot\log_{11} 25}\\
&= 7^3 + 11^2 + \sqrt{25} = 343 + 121 + 5 = 469.
\end{aligned}$$

3. (Answer: 141)

   Because $\angle EBA$ and $\angle ACB$ are both complementary to $\angle EBC$, the angles EBA and ACB are equal, and right triangles BAE and CBA are similar. Thus

$$\frac{AE}{AB} = \frac{AB}{BC} = \frac{AB}{2AE}.$$

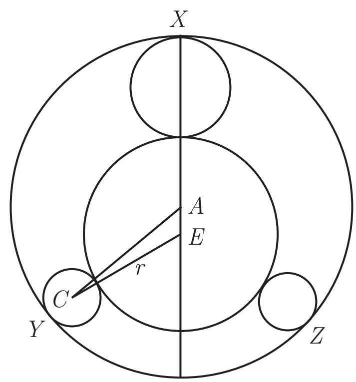
   Hence $2 \cdot AE^2 = AB^2$ and $AD = 2 \cdot AE = 2 \cdot \frac{100}{\sqrt{2}} = 100\sqrt{2}$. Because $141 < 100\sqrt{2} < 142$, the requested answer is 141.

4. (Answer: 089)

   Let c be the number of children in the contest, and let g be the average number of grapes eaten by each contestant. Then g is an integer, and $c \cdot g = 2009 = 7^2 \cdot 41$. Furthermore, the number of grapes eaten by the child in last place is $g - (c - 1) \geq 0$, so $c \leq g + 1$. Therefore the possible choices for the ordered pair (c, g) are $(1, 2009)$, $(7, 287)$, and $(41, 49)$. The value of $n = g + (c - 1)$ is minimized when $(c, g) = (41, 49)$, and the minimum value is $41 + 49 - 1 = 89$.

5. (Answer: 032)

   Let circles B, C, and D be tangent to circle A at points X, Y, and Z, respectively. Circle E has radius r. Because circles C and D are congruent, point E is on the diameter of circle A through point X. Note that $AC = 8$, $AE = r - 4$, $CE = r + 2$, and $\angle CAE = 60°$. Therefore by the Law of Cosines, it follows that $(r-4)^2 + 8^2 - 2(r-4) \cdot 8 \cdot \cos 60° = (r+2)^2$. Expanding and simplifying yields $r^2 - 8r + 16 + 64 - 8(r-4) = r^2 + 4r + 4$. Solving for r gives $r = \frac{27}{5}$. The requested answer is $27 + 5 = 32$.



6. (Answer: 750)

Let A be the number of ways in which 5 distinct numbers can be selected from the set of the first 14 natural numbers, and let B be the number of ways in which 5 distinct numbers, no two of which are consecutive, can be selected from the same set. Then $m = A - B$. Because $A = \binom{14}{5}$, the problem is reduced to finding B.
Consider the natural numbers $1 \le a_1 < a_2 < a_3 < a_4 < a_5 \le 14$. If no two of them are consecutive, the numbers $b_1 = a_1$, $b_2 = a_2 - 1$, $b_3 = a_3 - 2$, $b_4 = a_4 - 3$, and $b_5 = a_5 - 4$ are distinct numbers from the interval $[1, 10]$. Conversely, if $b_1 < b_2 < b_3 < b_4 < b_5$ are distinct natural numbers from the interval $[1, 10]$, then $a_1 = b_1$, $a_2 = b_2 + 1$, $a_3 = b_3 + 2$, $a_4 = b_4 + 3$, and $a_5 = b_5 + 4$ are from the interval $[1, 14]$, and no two of them are consecutive. Therefore counting B is the same as counting the number of ways of choosing 5 distinct numbers from the set of the first 10 natural numbers. Thus $B = \binom{10}{5}$. Hence $m = A - B = \binom{14}{5} - \binom{10}{5} = 2002 - 252 = 1750$ and the answer is 750.

7. (Answer: 401)

The fact that $\binom{2m}{m} = \frac{2^m m!(2m-1)!!}{m!m!} = \frac{2^m(2m-1)!!}{m!}$ is an integer implies that if p^a is a power of an odd prime dividing $(2m)!!$, then p^a divides $m!$ and hence $(2m-1)!!$. Thus any odd prime powers which divide the denominator in any given term in the sum divide the numerator as well. Therefore, when reduced to lowest terms, the denominator of the mth term in the sum is the highest power of 2 that divides $(2m)!!$. Hence $2^a b$ is the highest power of 2 dividing $(2\cdot 2009)!! = 4018!!$. Also note that $b = 1$ because b is odd and $2^a b$ is a power of 2. Because $4018!! = 2^{2009}\cdot 2009!$, the highest power of 2 that divides 4018!! is

$$2009 + \left\lfloor \frac{2009}{2} \right\rfloor + \left\lfloor \frac{2009}{2^2} \right\rfloor + \cdots + \left\lfloor \frac{2009}{2^{10}} \right\rfloor =$$
$$2009 + 1004 + 502 + 251 + 125 + 62 + 31 + 15 + 7 + 3 + 1 = 4010.$$

Thus the requested answer is $\dfrac{4010\cdot 1}{10} = 401.$

8. (Answer: 041)

The probability that Dave or Linda rolls a die k times to get the first six is the probability that there are $k-1$ rolls which are not six followed by one roll of six, which is $p_k = \left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)$. The probability that Dave will need one roll and Linda will need one or two rolls is then $p_1(p_1+p_2)$. The probability that Dave will need $k > 1$ rolls and Linda will need $k-1$, k, or $k+1$ rolls is then $p_k(p_{k-1}+p_k+p_{k+1})$. It follows that the desired probability is $p_1(p_1+p_2) + \displaystyle\sum_{k=2}^{\infty} p_k(p_{k-1}+p_k+p_{k+1})$. This is

$$\begin{aligned}
&\frac{1}{6}\cdot\left(\frac{1}{6}+\frac{5}{6}\cdot\frac{1}{6}\right)\\
&\quad+\sum_{k=2}^{\infty}\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)\left(\left(\frac{5}{6}\right)^{k-2}\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)+\left(\frac{5}{6}\right)^{k}\left(\frac{1}{6}\right)\right)\\
&=\left(\frac{1}{6}\right)\cdot\left(\frac{6}{36}+\frac{5}{36}\right)\\
&\quad+\sum_{k=2}^{\infty}\left(\frac{5}{6}\right)^{k-1}\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{k-2}\left(\frac{1}{6}\right)\left(1+\left(\frac{5}{6}\right)+\left(\frac{5}{6}\right)^2\right)\\
&=\frac{11}{6^3}+\frac{91}{6^4}\cdot\frac{\frac{5}{6}}{1-\left(\frac{5}{6}\right)^2}\\
&=\frac{11}{6^3}+\frac{91}{6^3}\cdot\frac{5}{11}=\frac{576}{6^3\cdot 11}=\frac{8}{33}.
\end{aligned}$$

Thus the final answer is $8+33=41$.

9. (Answer: 000)

Let (a,b,c) be a nonnegative integer solution to $4x+3y+2z = 2000$. Then $(a+1, b+1, c+1)$ is a positive integer solution to $4x+3y+2z = 2009$. Conversely, if (a,b,c) is a positive integer solution to $4x+3y+2z = 2009$, then $(a-1, b-1, c-1)$ is a nonnegative integer solution to $4x+3y+2z=2000$. This establishes a one-to-one correspondence between the nonnegative integer solutions to $4x+3y+2z=2000$ and the positive integer solutions to $4x+3y+2z=2009$. The difference $m-n$ is therefore the number of nonnegative integer solutions to $4x+3y+2z=2000$ in which $xyz=0$. If $x=0$, then $3y=2(1000-z)$ and it follows that z must be one more than a nonnegative multiple of 3, and $z \leq 1000$. Thus the possible values of z are 1, 4,..., 1000, and this is a total of 334 values. Similarly, there are 501 solutions with $y=0$, and 167 solutions with $z=0$. Note that the solutions $(0,0,1000)$ and $(500,0,0)$ are each counted twice, so the total number of nonnegative integer solutions to $4x+3y+2z=2000$ in which $xyz=0$ is $334+501+167-2=1000$, and the requested remainder is 0.

(Note: Using a computer algebra system, one can verify that $m=83834$ and $n=82834$.)

10. (Answer: 096)

Note that $\triangle ABC$ has a right angle at B. Place $\triangle ABC$ on the coordinate plane with A at the origin, B at $(5,0)$, C at $(5,12)$, and D at a point (x,y). Let point E be at $(10,0)$; then $\angle ACB$ equals $\angle ECB$. This means that point D lies on the line segment CE. Then $\tan\angle CEB = \frac{12}{5} = \frac{y}{10-x}$, so $5y = 120-12x$. Thus

$$\tan(\angle BAD) = \tan\left(\frac{\angle CAB}{2}\right) = \frac{\sin(\angle CAB)}{1+\cos(\angle CAB)} = \frac{12/13}{1+5/13} = \frac{12}{18} = \frac{y}{x}.$$

It follows that $12x = 18y$. Combining this with $5y = 120-12x$ yields $y = \frac{120}{23}$ and $x = \frac{180}{23}$. The distance from A to D is then $\sqrt{\left(\frac{180}{23}\right)^2 + \left(\frac{120}{23}\right)^2} = \frac{60}{23}\sqrt{3^2+2^2} = \frac{60\sqrt{13}}{23}$. The requested answer is then $60+23+13=96$.

11. (Answer: 125)

The inequality $|\log m - \log k| < \log n$ is equivalent to $-\log n < \log m - \log k < \log n$, which is equivalent to $\log \frac{m}{n} < \log k < \log mn$. Write $m = nq+r$, where q is a positive integer and r is an integer with $0 \leq r < n$. The inequality then becomes

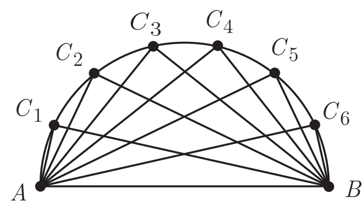
$$\log\left(q+\frac{r}{n}\right) < \log k < \log(n(nq+r)).$$

There are $n(nq+r)-q-1$ possible values of k, namely, $q+1$, $q+2$,..., $n(nq+r)-1$. By the given condition, $n(nq+r)-q-1=50$ or $(n^2-1)q+nr=51$. The potential values of n are 2, 3,..., 7. The only solutions are $(n,q,r)=(2,17,0)$ and $(3,6,1)$. Hence $(m,n)=(34,2)$ or $(19,3)$, and $mn=68$ or 57. Thus the sum is 125.

12. (Answer: 803)

Let the pairs be (a_i, b_i) for $i = 1, 2, 3, \ldots, k$, and set $S = \sum_{i=1}^{k}(a_i + b_i)$. From the given conditions, it follows that $1 + 2 + \cdots + 2k \le S \le 2009 + 2008 + \cdots + (2010 - k)$, giving $k(2k+1) \le \frac{1}{2}k(4019 - k)$. Solving this inequality for k yields $k \le \frac{4017}{5}$, and therefore k cannot exceed 803. This value of 803 can be achieved by choosing pairs $(1, 1207)$, $(2, 1208)$,..., $(401, 1607)$, $(402, 805)$,..., $(803, 1206)$.

13. (Answer: 672)



Draw the semicircular arc in the complex plane so that A is at -2 and B is at 2. This arc is then half the circle of radius 2 centered at 0 and the twelve given chords are congruent to the twelve chords $\overline{AC_1}$, $\overline{AC_2}$, ..., $\overline{AC_6}$, $\overline{AC_7}$,..., $\overline{AC_{11}}$, $\overline{AC_{12}}$, where C_7, C_8,..., C_{12} are the reflections of C_1, C_2,..., C_6 in the real axis. The 14 points A, B, C_1, C_2,..., C_6, C_7, C_8,..., C_{12} are then the 14 fourteenth roots of 2^{14}, all satisfying the equation $z^{14} = 2^{14}$. The chord from point z to 2 has the same length as the modulus of the complex number $w = z - 2$. These complex numbers w each satisfy the equation $(w+2)^{14} = 2^{14}$. The product of the lengths of the original twelve chords and AB is the same as the modulus of the product of the roots of the equation $\dfrac{(w+2)^{14} - 2^{14}}{w} = 0$. The product of the roots is equal to the constant term in the fraction when written as a polynomial. According to the Binomial Theorem, this constant term is $\binom{14}{13}2^{13} = 14 \cdot 2^{13}$. This product equals the required product times AB, which equals 4, so n is $\dfrac{14 \cdot 2^{13}}{4} = 14 \cdot 2^{11} = 28672$, and the requested remainder is 672.

14. (Answer: 983)

The recursion formula is equivalent to $a_{n+1} = 2\left(\frac{4}{5}a_n + \frac{3}{5}\sqrt{4^n - a_n^2}\right)$, which resembles a sum-of-angles trigonometric identity with $a_n = 2^n \sin\alpha$.

Let $\theta = \sin^{-1}(\frac{3}{5})$. Then $\cos\theta = \frac{4}{5}$ and

$$a_{n+1} = 2\left(2^n\cos\theta\sin\alpha + \sin\theta\sqrt{4^n(1-\sin^2\alpha)}\right)$$
$$= \begin{cases} 2^{n+1}\sin(\alpha+\theta), & \text{if } \cos\alpha > 0 \\ 2^{n+1}\sin(\alpha-\theta), & \text{if } \cos\alpha < 0. \end{cases}$$

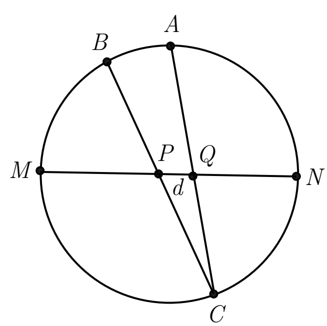
Because $\cos 45° < \frac{4}{5} < \cos 30°$, it follows that $30° < \theta < 45°$. Hence the angle increases by θ until it reaches 3θ, after which it oscillates between 2θ and 3θ. Thus

$$a_n = \begin{cases} 2^n\sin(n\theta), & \text{if } n = 0, 1, 2 \\ 2^n\sin(2\theta), & \text{if } n > 2 \text{ and even} \\ 2^n\sin(3\theta), & \text{if } n > 1 \text{ and odd.} \end{cases}$$

Thus $a_{10} = 2^{10}\sin 2\theta = 1024\cdot\frac{24}{25} = \frac{24576}{25}$, and the requested answer is 983.

15. (Answer: 014)

Let $[XYZ]$ represent the area of $\triangle XYZ$.



Let $\overline{BC}$ and $\overline{AC}$ intersect $\overline{MN}$ at points P and Q respectively, and let $\frac{CM}{CN} = x$. Then

$$\frac{MP}{NP} = \frac{[BMC]}{[BNC]} = \frac{BM\cdot CM\sin\angle BMC}{BN\cdot CN\sin\angle BNC} = \frac{3x}{4}.$$

Because $NP = 1 - MP$, it follows that $MP = \frac{3x}{3x+4}$. Similarly,

$$\frac{MQ}{NQ} = \frac{[AMC]}{[ANC]} = \frac{AM\cdot CM\sin\angle AMC}{AN\cdot CN\sin\angle ANC} = x,$$

giving $MQ = \frac{x}{x+1}$. The fact that $MQ - MP = d$ implies that $\frac{x}{x+1} - \frac{3x}{3x+4} = d$, or equivalently, $3dx^2 + (7d-1)x + 4d = 0$. Because the discriminant of this equation, which is $d^2 - 14d + 1$, must be nonnegative and $0 < d < 1$, the largest value of d is $7 - 4\sqrt{3}$, and $r + s + t = 14$.

The

American Mathematics Competitions

are Sponsored by

The Mathematical Association of America
The Akamai Foundation

Contributors

Academy of Applied Sciences
American Mathematical Association of Two-Year Colleges
American Mathematical Society
American Society of Pension Actuaries
American Statistical Association
Art of Problem Solving
Awesome Math
Canada/USA Mathcamp
Casualty Actuarial Society
Clay Mathematics Institute
IDEA Math
Institute for Operations Research and the Management Sciences
L. G. Balfour Company
Mathpath
Math Zoom Academy
Mu Alpha Theta
National Assessment & Testing
National Council of Teachers of Mathematics
Pedagoguery Software Inc.
Pi Mu Epsilon
Society of Actuaries
U.S.A. Math Talent Search
W. H. Freeman and Company
Wolfram Research Inc.