Notion of Particles in Curved Space

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This manuscript is intended for a better conceptual understanding of particles in high energy physics. Since this note only aims to give a brief overview, most of the details are left to references.

1 Particles and quantum field theory ...

1.1 Particle classification à la Wigner

What is the meaning of the statement that a system possesses a symmetry? A system is descibed by its time-evolution rules (dynamics) and its current state. Suppose that we have a choreography of the states that the system has evolved through the given time-evolution rules from some initial state. Also, suppose that there is an operation that changes every state recorded on the choreograph to some other state. When the change in the choreography induced by this operation does not change the time-evolution rules, i.e. if it is impossible to discern any difference between time-evolution rules of the two choreographs, then the operation is called a symmetry operation and the system is said to possess a symmetry.

Two symmetry operations in succession is also a symmetry operation. A symmetry operation can always be undone. A mathematical structure that captures these features is called a group[1]; this is why physicists study groups to study symmetries. In many cases the terms symmetry and symmetry group are used interchangeably, because there is no need to keep distinction between them. The same convention will be employed in this manuscript as well. Since there is a plethora of texts on group theory, technical details will be omitted.

A mathematical object can be studied by how it 'acts' on other mathematical objects. This is the idea behind group representation theory; a group can be studied by its action on vector spaces. Note that the name 'representation' is quite adequate since we are not studying the mathematical object itself, but how it can be represented. We are usually interested in unitary representations of symmetries, meaning that symmetries are modelled by unitary transformations on a vector space. This is due to Wigner's theorem. Wigner's theorem states that to make sense of probability interpretation of a quantum mechanical system with symmetries, then the symmetries must be represented as unitary transformations or anti-unitary transformations acting on the vectors of the Hilbert space.

[1] A little caveat; groups require associativity of operations. Breaking of associativity results in a mathematical structure called a *quasigroup*.

In some cases, a representation of a symmetry group acting on a vector space can be split into representations that act on a subspace of that vector space. Such a representation is called reducible. If the representation is not reducible, then it is called irreducible. Irreducible representations work as building blocks to other representations; any representation can be decomposed into a sum of irreducible representations. This is why most studies on group representation theory intended for applications to physics focus on unitary irreducible representations.

So, what is all this stuff about symmetries and their modelling got to do with particles? Wigner also came up with the idea that *particles can be classified by unitary irreducible representations of the symmetries of Minkowski space*. The symmetry group of Minkowski space, $ISO(3,1)$, is also called the Poincaré group[2]. There are two sets of generators for the Poincaré group.

P_μ	translation generators, non-compact
$M_{\mu\nu} = -M_{\nu\mu}$	rotation generators, compact & non-compact

Their commutation relations are expressed schematically as $[P, P] = 0$, $[P, M] \sim \eta P$, and $[M, M] \sim \eta M$. There are two invariants that can be formed from the generators of this group.

$$P^2 = -M^2 \qquad \text{mass} \tag{1}$$

$$W^2 \sim (\epsilon M P)^2 \qquad \text{spin} \tag{2}$$

$W_\mu = \frac{1}{2}\epsilon_{\mu\nu\rho\lambda} M^{\nu\rho} P^\lambda$ is called the Pauli-Lubanski (pseudo)vector. We still have a huge degeneracy of states; those two numbers are not specific enough. Another label that can be used is eigenvalues of P. The states are now distinguished by quantum numbers P, P^2, and W^2. Still, this is not enough to classify all states. Let's write other possible quantum numbers collectively as Z.

$$|\psi\rangle = \left|P, P^2, W^2, Z\right\rangle \tag{3}$$

What can be said about the as-of-yet unspecified quantum numbers Z? There are subgroups of $ISO(3,1)$ that keeps the momentum P unchanged; such a group is called the *little group* or a *stabiliser subgroup*. The states with quantum numbers specified by P, P^2, and W^2 will still have some representation with respect to the little group, so the little group can be used to classify the states. Since any momentum P can be related to another momentum P' with the same invariant mass $M^2 = -P^2 = -P'^2$, the space of states with momentum label P and that of P' will have the same structure, i.e. isomorphic. Therefore, studying the subspace with quantum numbers P, P^2, and W^2 specified is enough to determine the possible particle states. There are two cases which are physically realisable.

1. Massive states; $M^2 > 0$. We can set the momentum to be at rest, i.e. $P = (m, \vec{0})$. The little

[2] The group $ISO(3,1)$ consists of four connected pieces; proper and orthochronous, improper and orthochronous, improper and non-orthochronous, and proper and non-orthochronous. Improper means space parity is reversed, while non-orthochronous means the direction of future time is reversed.

group is $SO(3)$, so spin and angular momentum classification we have learned in quantum mechanics courses can be used to classify the particle states.

2. Massless states; $M^2 = 0$. The momentum can be set to the value $P = (E, E\hat{z})$. The little group turns out to be $ISO(2)$. The compact subgroup[3] of this little group is $SO(2) = U(1)$, and a representation of this group is either one-dimensional (spin zero), or two-dimensional (positive/negative helicity).

This is the essence of Wigner's classification scheme. For details consult references 2 and 7. This method of finding unitary irreducible representations to construct particle states also works for anti-de Sitter space, which has a large set of symmetries. Check reference 8 and references therein. However, such a fortunate coincidence does not occur for general manifolds that describe some spacetime of interest. The method that can be generalised to describe particles on such spacetimes will be the subject of the next section.

1.2 Weinbergian approach to particles

Steven Weinberg put forward the view that a theory of particles with localness (in the sense that the S-matrix obtained from the theory obeys cluster decomposition) should necessarily be decribed by fields, in his book on quantum field theory (in)famous for its abstruseness and thoroughness. Note the difference of this viewpoint from that of Wilson's; Wilsonian viewpoint of quantum field theory is based on the observation that the large-scale behaviour of a system usually do not depend on the microscopic details, so it can be effectively described by fields.

To have a theory of particles, we should understand what a particle is. What is the distinguishing feature of particles? In other words, by what property we call a particle-like nature, in contrast to a wave-like nature? It is enumerability; particles come in discrete numbers[4]. Such a feature can be captured by algebra of simple harmonic oscillators a and $a^\dagger$.

$$[a, a] = [a^\dagger, a^\dagger] = 0 \qquad\qquad [a, a^\dagger] = 1 \tag{4}$$

The square brackets are to be interpreted as commutators for bosonic particles and bosonic oscillators, and as anti-commutators for fermionic particles and fermionic oscillators. Conforming to the literature, we will call a the annihilation operator and $a^\dagger$ the creation operator. The operators come with labels that denote what kind of particle state it creates and annihilates; momentum labels, particle species label, internal state label, etc.

The necessity of fields is invoked when we start to consider experiments that can be done with particles. One of the most widely studied class of experiments done with particles is scattering experiments. The information of such a scattering experiment is encoded in the object called a S-matrix. It returns the vector corresponding to the expected final state of the system when acted on the vector corresponding to the initial state of the system. Since particles are modelled as

[3] This is because a unitary representation of a non-compact group cannot be finite dimensional, and we expect that number of states distinguished by Z to be finite. If this is not the case then we can inscribe an infinite amount of information onto a single particle, which seems an absurd thing to do.

[4] The distinguishing feature of waves is that they are subject to interference and superposition.

representations of the algebra of simple harmonic oscillators, the S-matrix is written as a sum of products of them. Given the *interaction* Hamiltonian H_I, the S-matrix can be written as a Dyson series in H_I.

$$\mathbf{S} = \mathcal{T} \exp \left[-i \int_{-\infty}^{+\infty} dt \ H_I(t) \right] \tag{5}$$

Obviously, the S-matrix must have Lorentz covariance built in it; that is what relativity is all about. Secondly, the S-matrix must have a kind of 'localness' built in it. By localness we mean experiments done in a laboratory are not affected by the outcomes of experiments done in some laboratory far away. When particles belonging to set A are subject to experiments conducted in laboratory a and particles belonging to set B are subject to experiments conducted in laboratory b which is far away from laboratory a, then the S-matrix should be written as a direct product of S-matrices of the set A and the set B. This is realised as cluster decomposition in a S-matrix. Suppose that particles of initial state can be divided into two groups A and B. Cluster decomposition means that scattering outcomes of $A \cup B$ have a contribution from scattering consisting solely of A and solely of B.

$$\begin{aligned}\{\text{scattering events of set } A \cup B\} = &\{\text{scattering events of set } A\} \times \{\text{scattering events of set } B\} \\ &+ \{\text{scattering events not captured by above}\}\end{aligned}$$

The 'scattering events not captured by above' is called the connected part of the S-matrix. This part describes the scattering events that can occur only if all particles in the set A and B participate in the event.

An S-matrix that obeys Lorentz covariance and has cluster decomposition property can be constructed from an interaction Hamiltonian obtained from a Hamiltonian density $\mathcal{H}(x)$ with the following property.

$$H_I(t) = \int d^3\mathbf{x} \ \mathcal{H}(\mathbf{x}, t) \qquad\qquad [\mathcal{H}(x), \mathcal{H}(x')] = 0 \text{ for space-like separation} \tag{6}$$

The most easy way to build such an Hamiltonian density is to assemble creation and annihilation operators (that fall in the same representation) into fields that carry information of locality, i.e. the information of position x, and use the fields to construct it.

$$\psi_l^+ \sim \sum u_l(x; \mathbf{p}, \sigma, n) \ a(\mathbf{p}, \sigma, n) \tag{7}$$

$$\psi_l^- \sim \sum v_l(x; \mathbf{p}, \sigma, n) \ a^\dagger(\mathbf{p}, \sigma, n) \tag{8}$$

The index l refers to field type, $\mathbf{p}$ to momentum, n to particle species, and σ to others (usually spin). To have a right representation under translations, the mode functions u and v should have a Fourier-mode like part.

$$u_l \to e^{ipx} u_l \qquad\qquad v_l \to e^{-ipx} v_l \tag{9}$$

Imposing commutativity of the Hamiltonian density at space-like separation requires more work. To do this, one should combine positive frequency fields ψ_l^+ and negative frequency fields ψ_l^- into a suitable linear combination that inherits the (anti-)commutativity at space-like separation.

$$\psi_l = A_l \psi_l^+ + B_l \psi_l^- \text{ such that } [\psi_l(x), \psi_{l'}(x')] = [\psi_l(x), \psi_{l'}^\dagger(x')] = 0 \text{ for space-like separation}$$

The fields constructed this way satisfies the Klein-Gordon equation $(\Box - m^2)\psi_l = 0$ due to (9), and imposing right transformation rules under the Lorentz group one can obtain other conditions such as the Dirac equation $(\gamma^\mu \partial_\mu + m)\psi = 0$ for spin $\frac{1}{2}$ fields. For details check chapter 5 of reference 2.

In effect, the final result is not very different from how the whole subject started in the first place; solve the wave equation and promote coefficients of the wave modes to operators, i.e. second quantise.

2 ... on a curved spacetime

2.1 Einsteinian relativity

Nature does not come with grids. Relativity is a manifestation of this fact; all coordinate systems are artificial. On the other hand, it is almost impossible to do any meaningful calculation without any coordinate systems. This is why relativity deals with coordinate transformations in practice. The role of relativity is to introduce consistent rules that matches experimental outcomes of one oberserver to another, which are dependent on the coordinate systems the observers use.

Einstein's observation, the celebrated principle of equivalence, can be succintly summarised as follows; all experiments done in a sufficiently small free-falling box (in other words, local experiments) always yields the same outcomes that only depend on the settings inside the box. The interior of the small box can be approximated to a region of flat spacetime; this is the physical interpretation of Fermi normal coordinates.

Another consequence of principle of equivalence is that notions relevant to flat spacetime can be extended to curved spacetimes. A manifold is described by coordinate patches that cover the whole manifold when patched together. Taking those patches to be sufficiently small, i.e. taking them to describe physics inside some sufficiently small boxes that can be considered flat, we can extend what we know on flat spacetime to curved spacetime.

Extending what we know on flat spacetime to curved spacetime by appropriate patching, the model of particles become quantised fields that obey an extended version of the wave equation applicable to curved manifolds. Some care must be taken in the process, however. Although fields relevant to particle physics are well-defined on $\mathbb{R}^n$, some of them are not well-defined on arbitrary manifolds. For example, a vector field on S^{2n} must have at least one point where the field vanishes, which is also known as the hairy ball theorem. Another example is spin structures; fermionic particles cannot be modelled on arbitrary manifolds. Such complications will be largely ignored in this manuscript. Chapter 12 of reference 5 serves as a good overview on this subject.

2.2 Mode operators on curved spacetime

In second quantisation the field is decomposed into a sum of wave modes, and the coefficients become operators. The problem of obtaining creation and annihilation operators reduce to the problem of reading out the coefficients of mode functions. This was a trivial task in flat spacetime since the mode functions were just Fourier modes; just inverse Fourier transform the fields. On curved spacetime mode functions in general are not Fourier modes, which complicates matters.

On the other hand, mode functions are solutions to the wave equation. Wave equations are linear PDEs; their solution set is a vector space. This means that standard techniques for manipulating vector spaces can be used to obtain coefficients of the wave equations. Define the following sequilinear operation on solutions to the Klein-Gordon equation as the inner product[5].

$$(f|g)_\Sigma \equiv i \int_\Sigma d^d y \sqrt{|h|} n^\mu \left(f^* \nabla_\mu g - g \nabla_\mu f^*\right) \tag{10}$$

Σ denotes the spacelike surface on which the inner product is evaluated, y^a is the coordinate system that covers the submanifold Σ, n^μ is the unit future-directed normal($n^\mu n_\mu = -1$) to Σ, and h is the determinant of the induced metric $h_{ab} = \frac{\partial x^\mu}{\partial y^a}\frac{\partial x^\nu}{\partial y^b} g_{\mu\nu}$. Note that the inner product satisfies the following relations.

$$[(f|g)_\Sigma]^* = (g|f)_\Sigma = -(f^*|g^*)_\Sigma \tag{11}$$

Find a complete set of solutions f_k and f_k^* that satisfy the following condition.

$$(f_k|f_{k'})_\Sigma = -(f_k^*|f_{k'}^*)_\Sigma = \delta_{kk'} \tag{12}$$

$$(f_k|f_{k'}^*)_\Sigma = 0 \tag{13}$$

The functions f_k and f_k^* will be used as mode functions.

Given the inner product and the complete set of basis f_k and f_k^* with standard normalisation (12) and (13), the creation and annihilation operators $a_k^\dagger$ and a_k can be read out from the real scalar field ϕ[6] by the inner product.

$$\phi(t,x) = \sum_k a_k f_k(t,x) + a_k^\dagger f_k^*(t,x) \tag{14}$$

$$a_k = (f_k|\phi)_\Sigma \tag{15}$$

$$a_k^\dagger = -(f_k^*|\phi)_\Sigma \tag{16}$$

The following relation among mode operators are consistent with the canonical quantisation con-

[5] Standard notation is (f,g), not the Dirac bra-ket notation used here. The reason for adopting such an unconventional notation is because this notation makes some calculations more intuitive. If you are interested in more standard stuff, check references 3 and 4.

[6] Only real scalar fields will be considered in this section.

dition $[\phi(\vec{x},t),\pi_\phi(\vec{y},t)] = i\delta^{n-1}(\vec{x}-\vec{y})$, π_ϕ denoting the momentum conjugate $\delta S/\delta\dot{\phi}$.

$$[a_k, a_{k'}^\dagger] = \delta_{kk'} \tag{17}$$

$$[a_k, a_{k'}] = [a_k^\dagger, a_{k'}^\dagger] = 0 \tag{18}$$

2.3 Bogoliubov transformations and the Unruh effect

In general, mode expansions are not unique. Suppose that the scalar field ϕ admits the following two mode expansions, each set of mode functions subject to standard normalisation (12) and (13).

$$\phi(t,x) = \sum_k a_k f_k(t,x) + a_k^\dagger f_k^*(t,x) \tag{19}$$

$$= \sum_k b_k g_k(t,x) + b_k^\dagger g_k^*(t,x) \tag{20}$$

Note that the sum $\sum |f_k)(f_k| - |f_k^*)(f_k^*|$ over an index of a complete set k is nothing but the identity. This fact can be used to relate mode functions of the different sets.

$$|g_k) = \left[\sum_{k'} |f_{k'})(f_{k'}| - |f_{k'}^*)(f_{k'}^*|\right] |g_k) \tag{21}$$

$$= \sum_{k'} (f_{k'}|g_k)|f_{k'}) - (f_{k'}^*|g_k)|f_{k'}^*) \tag{22}$$

$$= \sum_{k'} \alpha_{kk'}|f_{k'}) + \beta_{kk'}|f_{k'}^*) \tag{23}$$

The relation $|f_k) = \sum \alpha_{kk'}^*|g_{k'}) - \beta_{kk'}|g_{k'}^*)$ can be shown in the same way. The mode operators a_k and b_k can be related to each other by using the inner product defined above.

$$a_k = (f_k|\phi) = \sum_{k'} b_{k'}(f_k|g_{k'}) + b_{k'}^\dagger (f_k|g_{k'}^*) \tag{24}$$

$$= \sum_{k'} \alpha_{k'k} b_{k'} + \beta_{k'k}^* b_{k'}^\dagger \tag{25}$$

An analoguous computation leads to the relation $b_k = \sum \alpha_{kk'}^* a_{k'} - \beta_{kk'}^* a_{k'}^\dagger$. This is the Bogoliubov transformation. The coefficients obey the following unitarity condition.

$$\sum_k (\alpha_{ik}\alpha_{jk}^* - \beta_{ik}\beta_{jk}^*) = \delta_{ij} \tag{26}$$

$$\sum_k (\alpha_{ik}\beta_{jk} - \beta_{ik}\alpha_{jk}) = 0 \tag{27}$$

What does Bogoliubov transformation tell us about particles? It tells us that the notion of particles is an observer-dependent concept. Suppose that we have a vacuum defined by mode operators $a_k|0\rangle = 0$. The expectation value of the number operator for b_k for this vacuum is *not* zero in general.

$$\langle n_{b,k} \rangle = \langle 0| b_k^\dagger b_k |0\rangle = \sum_{k'} |\beta_{kk'}|^2 \neq 0 \tag{28}$$

This effect is not observed in usual Minkowski modes, however. This is because $\beta_{kk'} = 0$ for standard mode functions in flat space; positive frequency modes of an observer are linear combinations of positive frequency modes of another. In this sense, the vacuum of Minkowski space is unique. Observers related to each other by Lorentz boosts see the same vacuum.

A nontrivial $\beta_{kk'} \neq 0$ occurs when considering wave equation solutions to Rindler wedge in 1+1 dimensions. Let's consider a massless scalar field for sake of convenience. In 1+1 dimensions a massless free scalar is conformal. This means any mode function found in Minkowski space can be used in the Rindler wedge as well. The following parametrisation is a conformally flat coordinate patch of right Rindler wedge

$$t = be^{\chi_R} \sinh\tau \tag{29}$$

$$x = be^{\chi_R} \cosh\tau \tag{30}$$

$$-dt^2 + dx^2 = b^2 e^{2\chi_R}(-d\tau^2 + d\chi_R^2) \tag{31}$$

and the left.

$$t = be^{-\chi_L} \sinh\tau \tag{32}$$

$$x = -be^{-\chi_L} \cosh\tau \tag{33}$$

$$-dt^2 + dx^2 = b^2 e^{-2\chi_L}(-d\tau^2 + d\chi_L^2) \tag{34}$$

Let's write the mode expansion of a free scalar ϕ explicitly. The following expansion applies for the whole Minkowski space.

$$f_k(x,t) \equiv \frac{1}{\sqrt{2|k|}} \exp\left(-i|k|t + ikx\right) \tag{35}$$

$$\phi(x,t) = \int \frac{dk}{2\pi} \left(a_k f_k(x,t) + a_k^\dagger f_k^*(x,t) \right) \tag{36}$$

$$a_k = (f_k(x,t)|\phi)_\Sigma \tag{37}$$

On the other hand, the following expansion applies for the left or the right Rindler wedge. The label α is used to distinguish between the left and the right.

$$f_q(\chi,\tau) \equiv \frac{1}{\sqrt{2|q|}} \exp\left(-i|q|\tau + iq\chi\right) \tag{38}$$

$$\phi(\chi,\tau)\big|_\alpha = \int \frac{dq}{2\pi} \left(b_{\alpha,q} f_q(\chi_\alpha,\tau) + b_{\alpha,q}^\dagger f_q^*(\chi_\alpha,\tau) \right) \tag{39}$$

$$b_{\alpha,q} = (f_q(\chi_\alpha,\tau)|\phi)_{\Sigma_\alpha} \tag{40}$$

Some computation shows that (Σ_α is the surface $\tau = 0$)[7]

$$\alpha(k|\alpha, q) \equiv (f_k(x,t)|f_q(\chi_\alpha, \tau))_{\Sigma_\alpha} = \frac{1}{2}\left[\sqrt{\left|\frac{q}{k}\right|} + \frac{q}{k}\sqrt{\left|\frac{k}{q}\right|}\right]\int_{-\infty}^{\infty} d\chi_\alpha e^{iq\chi_\alpha - ikx} \tag{41}$$

$$\beta(k|\alpha, q) \equiv -(f_k(x,t)|f_q^*(\chi_\alpha, \tau))_{\Sigma_\alpha} = \frac{1}{2}\left[\sqrt{\left|\frac{q}{k}\right|} + \frac{q}{k}\sqrt{\left|\frac{k}{q}\right|}\right]\int_{-\infty}^{\infty} d\chi_\alpha e^{-iq\chi_\alpha - ikx} \tag{42}$$

which implies the following relation.

$$b_{\alpha,q} = \int \frac{dk}{2\pi}\left(\alpha^*(k|\alpha,q)a_k + \beta(k|\alpha,q)a_k^\dagger\right) \tag{43}$$

$$b_{\alpha,q}^\dagger = \int \frac{dk}{2\pi}\left(\beta^*(k|\alpha,q)a_k + \alpha(k|\alpha,q)a_k^\dagger\right) \tag{44}$$

Note that $\alpha(k|\alpha, q)$ and $\beta(k|\alpha, q)$ are nonzero only if $kq > 0$, which means that left movers ($k < 0$ and $q < 0$) and right movers ($k > 0$ and $q > 0$) decouple. Also, the following relations hold for $\alpha(k|\alpha, q)$ and $\beta(k|\alpha, q)$.

$$\alpha^*(k|\alpha, q) = \alpha(-k|\alpha, -q) \tag{45}$$

$$\beta^*(k|\alpha, q) = \beta(-k|\alpha, -q) \tag{46}$$

Unitarity condition becomes the following relations.

$$2\pi\delta(k - k') = \sum_\alpha \int \frac{dq}{2\pi}\left[\alpha(k|\alpha,q)\alpha^*(k'|\alpha,q) - \beta(k|\alpha,q)\beta^*(k'|\alpha,q)\right] \tag{47}$$

$$2\pi\delta_{\alpha\beta}\delta(q - q') = \int \frac{dk}{2\pi}\left[\alpha^*(k|\alpha,q)\alpha(k|\beta,q') - \beta(k|\alpha,q)\beta^*(k|\beta,q')\right] \tag{48}$$

The Unruh effect can be obtained by looking at the expectation value of occupation numbers with respect to the standard vacuum, the Minkowski vacuum. The Minkowski vacuum $|0\rangle$ is defined by the relation $\forall k, a_k |0\rangle = 0$. This is given as follows.

$$\langle n_{b,\alpha,q}\rangle = \langle 0| b_{\alpha,q}^\dagger b_{\alpha,q} |0\rangle = \int \frac{dk}{2\pi}\, |\beta(k|\alpha,q)|^2 \tag{49}$$

This has a closed form expression. Suppose $\alpha = R$, $k > 0$, and $q > 0$. Then,

$$\beta(k|R, q) = \sqrt{\frac{q}{k}}\int_{-\infty}^{\infty} d\chi_R e^{-iq\chi_R - ikx} \tag{50}$$

$$= \sqrt{\frac{k}{q}}\int_0^{\infty} dx e^{-iq\chi_R - ikx} \tag{51}$$

$$= -i\sqrt{\frac{1}{qk}}\int_0^{\infty} ikdx \left(\frac{ikx}{ikb}\right)^{-iq} e^{-ikx} \tag{52}$$

$$= -i\sqrt{\frac{1}{qk}}\,(ikb)^{iq}\,\Gamma(1 - iq) \tag{53}$$

[7]This computation was copied from an earlier note I've made. Sorry for some inconsistent use of notation compared to first half of this subsection.

Analoguously,

$$\alpha(k|R,q) = -i\sqrt{\frac{1}{qk}}\,(ikb)^{-iq}\,\Gamma(1+iq) \tag{54}$$

We can determine $|\alpha(k|R,q)|^2$ and $|\beta(k|R,q)|^2$ as follows.

$$|\beta(k|R,q)|^2 = \frac{e^{-\pi q}}{kq}|\Gamma(1-iq)|^2 \tag{55}$$

$$|\alpha(k|R,q)|^2 = \frac{e^{\pi q}}{kq}|\Gamma(1-iq)|^2 = e^{2\pi q}|\beta(k|R,q)|^2 \tag{56}$$

Thus, using the unitarity condition,

$$\langle n_{b,R,q}\rangle = \int \frac{dk}{2\pi}\ |\beta(k|R,q)|^2 \tag{57}$$

$$= \frac{1}{e^{2\pi q}-1}\int \frac{dk}{2\pi}\left[|\alpha(k|R,q)|^2 - |\beta(k|R,q)|^2\right] \tag{58}$$

$$= \frac{2\pi\delta(0)}{e^{2\pi q}-1} \tag{59}$$

which is a thermal distribution. The IR divergence $V = 2\pi\delta(0)$ just reflects the fact that the system was not put in a box of finite size. To give dimensions to q, define $\Omega = qa$. a has dimensions of energy, or *acceleration*[8]. This gives the following formula.

$$\frac{\langle n_{b,R,\Omega}\rangle}{V} = \frac{1}{e^{\Omega/(a/2\pi)}-1} \tag{60}$$

Thus, the distribution is the Bose-Einstein distribution in temperature $T_U = a/2\pi$. This is called the Unruh temperature.

2.4 Reality of Unruh effect and the Unruh-DeWitt detector model

Up to now there is no reason to believe that the Unruh temperature $T_U = a/2\pi$ derived in the previous subsection to be physical; after all, Bogoliubov transformation is just a mathematical manipulation. To assert that an observer in a motion of uniform acceleration will pick up a thermal atmosphere of particles, we need a (realistic) model of particle detector. Let's review the model of a detector devised by Unruh and DeWitt, which is the model considered in reference 3.

The detector moves along the worldline $x^\mu(s)$ parametrised by s, the detector's proper time. Interaction Lagrangian of the detector and massless scalar field ϕ is given as follows.

$$\mathcal{L}_{\text{int}} = cm(s)\phi[x(s)] \tag{61}$$

c is the small coupling constant and $m(s)$ is the detector's monopole moment operator. Suppose that detector is prepared in its ground state E_0, while the state of the field is given as Minkowski

[8] This is a natural thing to do. q is the dual variable to τ, which is also dimensionless. On a curve of uniform acceleration (χ_R = const) the proper time is measured as $ds^2 = -a^{-2}d\tau^2$, where a is the proper acceleration. The meaning of this is that dimensionless proper time τ is a quantity measured in units of a^{-1}; τ corresponds to 2.54 when measuring an inch in units of centimetres. Since q is the dual variable, q is measured in units of a.

vacuum $|0\rangle$. When the detector is put into motion, the detector's state will undergo transition from its ground state E_0 to an excited state $E > E_0$, changing the field's state to $|\psi\rangle$. The amplitude for this transition at small c can be obtained by first order perturbation theory.

$$ic \langle E, \psi| \int_{-\infty}^{\infty} m(s)\phi[x(s)]ds \, |0, E_0\rangle \tag{62}$$

Using time evolution equation for $m(s)$, $m(s) = e^{iH_0 s}m(0)e^{-iH_0 s}$, the amplitude factorises.

$$ic \langle E| \, m(0) \, |E_0\rangle \int_{-\infty}^{\infty} e^{i(E-E_0)s} \langle\psi| \, \phi[x(s)] \, |0\rangle \, ds \tag{63}$$

Probability is related to the absolute square of its corresponding amplitude. Thus, the following formula is relevant for computing the probability that the detector starts out in its ground state E_0 and ends in some other excited state $E > E_0$.

$$c^2 |\langle E| \, m(0) \, |E_0\rangle|^2 \sum_{|\psi\rangle} \int_{-\infty}^{\infty} e^{-i(E-E_0)s_1} \langle 0| \, \phi[x(s_1)] \, |\psi\rangle \, ds_1 \int_{-\infty}^{\infty} e^{i(E-E_0)s_2} \langle\psi| \, \phi[x(s_2)] \, |0\rangle \, ds_2 \tag{64}$$

The second term is called the *response function* and can be simplified.

$$\mathcal{F}(E) \equiv \sum_{|\psi\rangle} \int_{-\infty}^{\infty} e^{-iEs_1} \langle 0| \, \phi[x(s_1)] \, |\psi\rangle \, ds_1 \int_{-\infty}^{\infty} e^{iEs_2} \langle\psi| \, \phi[x(s_2)] \, |0\rangle \, ds_2 \tag{65}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iE(s_1-s_2)} \langle 0| \, \phi[x(s_1)]\phi[x(s_2)] \, |0\rangle \, ds_1 ds_2 \tag{66}$$

$$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{-iE(s_1-s_2)} G^+(x(s_1), x(s_2)) ds_1 ds_2 \tag{67}$$

$G^+(x', x) \equiv \langle\phi(x')\phi(x)\rangle$ is called the positive frequency Wightman function. Note that response function is completely irrelevant to inner workings of the detector; the details of the detector are captured by the term $c^2 |\langle E| \, m(0) \, |E_0\rangle|^2$.

The response function defined above can be simplified in some cases. Rewriting the integral,

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} ds_2 \int_{-\infty}^{\infty} e^{-iE\Delta s} G^+(x(s_2 + \Delta s), x(s_2)) d\Delta s \tag{68}$$

and supposing that the positive frequency Wightman function $G^+(x(s_1), x(s_2))$ only depends on the proper time separation $\Delta s = s_1 - s_2$,

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} ds \int_{-\infty}^{\infty} e^{-iE\Delta s} G^+(x(\Delta s), x(0)) d\Delta s \tag{69}$$

we see that a more appropriate quantity would be the response function divided by the length of whole proper time interval, which is related to transition probability per unit proper time.

$$\mathcal{F}'(E) = \int_{-\infty}^{\infty} e^{-iE\Delta s} G^+(x(\Delta s), x(0)) d\Delta s \tag{70}$$

Although computation of the response function of free massless scalar in $1 + 1$ dimensions do not give a proper thermal distribution, we will stick to it for consistency[9]. Computing the positive frequency Wightman function (exploiting translational invariance),

$$G^+(x,0) = \langle \phi(x')\phi(x) \rangle = \int \frac{dk}{2\pi} f_k(x) f_k^*(0) \tag{71}$$
$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \frac{1}{2|k|} \exp\left(-i|k|t + ikx\right) \tag{72}$$
$$= \int_{0}^{\infty} \frac{dk}{2\pi} \frac{1}{2k} \left(\exp\left[-ik(t-x)\right] + \exp\left[-ik(t+x)\right]\right) \tag{73}$$
$$= \int_{-\infty}^{\infty} \frac{dk}{2\pi} \left(\frac{1}{4k} + \frac{1}{4|k|}\right) \left(\exp\left[-ik(t-x)\right] + \exp\left[-ik(t+x)\right]\right) \tag{74}$$

we find that we need to evaluate the Fourier transform of k^{-1} and $|k|^{-1}$. Although this is not a well-defined integral, this is a well-defined distribution operation; it can be calculated. The details of the calculation will be given in the appendix, following the procedures of reference 6. The positive frequency Wightman function is given as follows.

$$G^+(x,0) = -\frac{1}{8\pi}\left(i\pi \ \mathrm{sign}(t-x) + i\pi \ \mathrm{sign}(t+x) + 2\log|t-x| + 2\log|t+x|\right) + \mathrm{const.} \tag{75}$$
$$= -\frac{i}{8}\left[\mathrm{sign}(t-x) + \mathrm{sign}(t+x)\right] - \frac{1}{8\pi}\log\left[(t^2-x^2)^2\right] + \mathrm{const.} \tag{76}$$

The trajectory of uniform acceleration a is

$$x = \frac{1}{a}\left[\cosh as - 1\right] \tag{77}$$
$$t = \frac{1}{a}\sinh as \tag{78}$$

so the modified response function reduces to

$$\mathcal{F}'(E) = \int_{-\infty}^{\infty} e^{-iEs}\left\{-\frac{i}{8}\left[\mathrm{sign}(t-x) + \mathrm{sign}(t+x)\right] - \frac{1}{4\pi}\log\left[t^2 - x^2\right]\right\} ds + \mathrm{irr.} \tag{79}$$
$$= \int_{-\infty}^{\infty} e^{-iEs}\left\{-\frac{i}{4}\mathrm{sign}(s) - \frac{1}{4\pi}\log\left[\frac{2}{a^2}(\cosh as - 1)\right]\right\} ds + \mathrm{irr.} \tag{80}$$
$$= -\frac{1}{4\pi E} - \frac{1}{4\pi}\int_{-\infty}^{\infty} e^{-iEs}\log\sinh^2(as/2) ds + \mathrm{irr.} \tag{81}$$

where irr. denotes the term proportional to $\delta(E)$, which is irrelevant for considering transitions between different energy levels of the detector. The second integral (which is a Fourier transform) can be computed by following manipulation.

$$\int_{-\infty}^{\infty} e^{-iEs}\log\sinh^2(as/2) ds = \int_{-\infty}^{\infty}\left[\frac{1}{-iE}e^{-iEs}\right]'\log\sinh^2(as/2) ds \tag{82}$$
$$= \int_{-\infty}^{\infty} e^{-iEs}\frac{1}{iE}\left[\log\sinh^2(as/2)\right]' ds \tag{83}$$
$$= \frac{-ia}{E}\int_{-\infty}^{\infty} e^{-iEs}\coth(as/2) \ ds \tag{84}$$

[9]The computation done in chapter 3.3 of reference 3 is in spacetime of dimensions $3+1$ yields the correct thermal distribution.

$$\therefore \int_{-\infty}^{\infty} e^{-iEs} \log \sinh^2(as/2) ds = \frac{-2i}{E} \int_{-\infty}^{\infty} e^{-i(2E/a)x} \coth x \ dx \tag{85}$$

Fourier transform of $\coth x$ can be computed by sum of residues; $\tanh(x + in\pi) \sim x$ for $x \ll 1$, so points of $x = in\pi$, $n \in \mathbb{Z}$, act as simple poles for $\coth x$. For $E > 0$[10],

$$\int_{-\infty}^{\infty} e^{-iyx} \coth x \ dx = -2\pi i \sum_{n \in \mathbb{Z}^+} e^{-iy(-in\pi)} - \pi i \tag{86}$$

$$= \frac{-2\pi i e^{-y\pi}}{1 - e^{-y\pi}} - \pi i \tag{87}$$

$$= -\pi i \coth(y\pi) = -\pi i \coth\left(\frac{E}{a/2\pi}\right) \tag{88}$$

and for $E < 0$,

$$\int_{-\infty}^{\infty} e^{-iyx} \coth x \ dx = 2\pi i \sum_{n \in \mathbb{Z}^+} e^{-iy(in\pi)} + \pi i \tag{89}$$

$$= \frac{2\pi i e^{y\pi}}{1 - e^{y\pi}} + \pi i \tag{90}$$

$$= -\pi i \coth(y\pi) = -\pi i \coth\left(\frac{E}{a/2\pi}\right) \tag{91}$$

yielding

$$\mathcal{F}'(E) = -\frac{1}{4\pi E} + \frac{1}{2E} \coth\left(\frac{E}{a/2\pi}\right) + \text{irr.} \neq 0 \tag{92}$$

as the final result. Thus, *a uniformly accelerating detector picks up particles in standard vacuum.*

3 Appendix: Computation of Fourier transformations

We adopt the following definitions for Fourier transform $\mathcal{F}[f]$ and inverse Fourier transform $\tilde{\mathcal{F}}[\tilde{f}]$ to treat them in a symmetrical manner.

$$\mathcal{F}[f(x)](k) = \int e^{2\pi ikx} f(x) dx \tag{93}$$

$$\tilde{\mathcal{F}}[\tilde{f}(k)](x) = \int e^{-2\pi ikx} \tilde{f}(k) dk \tag{94}$$

Apart from linearity, the transforms satisfy the following relations.

$$\mathcal{F}[\frac{d}{dx} f(x)](k) = -2\pi ik \mathcal{F}[f(x)](k) \tag{95}$$

$$\tilde{\mathcal{F}}[\frac{d}{dk} \tilde{f}(k)](x) = 2\pi ix \tilde{\mathcal{F}}[\tilde{f}(k)](x) \tag{96}$$

$$\frac{d}{dk} \mathcal{F}[f(x)](k) = \mathcal{F}[2\pi ix f(x)](k) \tag{97}$$

$$\frac{d}{dx} \tilde{\mathcal{F}}[\tilde{f}(k)](x) = \tilde{\mathcal{F}}[-2\pi ik \tilde{f}(k)](x) \tag{98}$$

[10]The residue of the origin contributes half as much since contour passes right through it. Cauchy principal value was used in its evaluation.

3.1 Computation of Fourier transform of x^{-1}

Define $F(k)$ as Fourier transform of x^{-1}.

$$F(k) = \mathcal{F}[x^{-1}] = \int \frac{1}{x} e^{2\pi ikx} dx \tag{99}$$

Note that $F(k) = -F(-k)$, $F'(k) = 2\pi i\delta(k)$. Thus, the only function that $F(k)$ can be is;

$$\mathcal{F}[x^{-1}] = i\pi \ \mathrm{sign}(k) \tag{100}$$

Equivalently, $\tilde{\mathcal{F}}[k^{-1}] = -i\pi \ \mathrm{sign}(x)$ or $\mathrm{sign}(x) = i\pi^{-1}\tilde{\mathcal{F}}[k^{-1}]$.

3.2 Computation of Fourier transform of $|x|^{-1}$

Define $G(k)$ as Fourier transform of $|x|^{-1}$.

$$G(k) = \mathcal{F}[|x|^{-1}] = \int \frac{1}{|x|} e^{2\pi ikx} dx \tag{101}$$

Note that $x|x|^{-1} = \mathrm{sign}(x)$.

$$\frac{d}{dk}G(k) = 2\pi i\mathcal{F}[x|x|^{-1}] = 2\pi i\mathcal{F}[\mathrm{sign}(x)] = -2\mathcal{F}[\tilde{\mathcal{F}}[k^{-1}]] = -\frac{2}{k} \tag{102}$$

Solving the differential equation gives

$$G(k) = \mathcal{F}[|x|^{-1}] = -2\log|k| + C \tag{103}$$

where the integration constant $C = -2(\gamma + \log 2\pi)$. Equivalently, $\tilde{\mathcal{F}}[|k|^{-1}] = -2\log|x| + C$. To understand how integration constant C is calculated, check reference 6.

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