Some properties on generalized Ahmed's integral

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1 Introduction

The following integral

$$\int_0^1 \frac{\arctan \sqrt{x^2+2}}{\sqrt{x^2+2}} \frac{dx}{x^2+1} = \frac{5\pi^2}{96} \tag{1.1}$$

is called the *Ahmed's integral*, which became famous since it is first discovered by Ahmed in [Ahm02]. Here, we present a formula which generalizes this integral to the considerable extent.

We introduce the three-parameter function $A(p,q,r)$ defined by the following integral representation.

$$A(p,q,r) := pqr \int_0^1 \frac{\arctan q\sqrt{p^2x^2+1}}{q\sqrt{p^2x^2+1}} \frac{dx}{(r^2+1)p^2x^2+1}. \tag{1.2}$$

In view of (1.1), it is of no harm to call it as the *(generalized) Ahmed's integral* of parameter p,q,r. Indeed, we retrieve the original Ahmed's integral with the choice of parameters as $(p,q,r) = (1/\sqrt{2}, \sqrt{2}, 1)$.

In spite of the bizarre and asymmetric formulation in (1.2), however, it turns out that the role of p, q and r are nearly symmetric. To make this claim precise, we introduce the following *modified Ahmed's integral*

$$\tilde{A}(p,q,r) := A(p,q,r) + \arctan\left(r\sqrt{p^2+1}\right)\arctan\left(\frac{pq}{\sqrt{q^2+1}}\right). \tag{1.3}$$

We will check that $\tilde{A}(p,q,r)$ remains invariant under the cyclic change of variable $(p,q,r) \mapsto (q,r,p) \mapsto (r,p,q)$. This will be a direct consequence of our main theorem.

Let us returning to the problem of generalizing the result (1.1). To present our results, we first introduce some notations. For each triple of parameters $p,q,r > 0$, we define the *complementary parameters* as

$$\tilde{p} = (q^2+1)^{1/2} r, \quad \tilde{q} = (r^2+1)^{1/2} p, \quad \tilde{r} = (p^2+1)^{1/2} q, \quad \text{and} \quad k = pqr.$$

We fix these notations throughout this article. In other words, $\tilde{p}$, $\tilde{q}$, $\tilde{r}$ and k are considered as functions of p, q and r. Then our main theorem deals with an integral representation formula of the modified Ahmed's integral.

Theorem 1.1. *For each $p,q,r > 0$ satisfying $pqr \leq 1$, the following integral representation holds.*

$$\tilde{A}(p,q,r) = 2\chi_2(k) + \frac{k}{2}\int_0^1 \frac{1}{1-k^2x^2} \log\left(\frac{1+\tilde{p}^2x^2}{1+\tilde{p}^2} \times \frac{1+\tilde{q}^2x^2}{1+\tilde{q}^2} \times \frac{1+\tilde{r}^2x^2}{1+\tilde{r}^2}\right) dx, \tag{1.4}$$

where $\chi_2(z)$ denotes the Legendre chi function *defined by*

$$\chi_2(z) = \sum_{n=1}^{\infty} \frac{z^{2n+1}}{(2n+1)^2}.$$

2 Preliminary

As a preliminary, we review some background knowledge that will be used throughout the proof. Recall that the following generalized binomial series holds for any $\alpha \in \mathbb{C}$

$$(1+z)^{\alpha} = \sum_{j=0}^{\infty} \binom{\alpha}{j} z^j, \quad |z| < 1, \tag{2.1}$$

where the symbol $\binom{\alpha}{j}$ denotes the generalized binomial coefficient

$$\binom{\alpha}{j} = \frac{\alpha(\alpha-1)\cdots(\alpha-j+1)}{j!}, \quad j \geq 0.$$

Exploiting (2.1), we can check that basic identities for the binomial coefficients remain valid for this generalized one. Those include the Pascal's identity and the Vandermonde's convolution formula.

With $\alpha = n \in \{0, 1, 2, \cdots\}$, this reduces to the classical result of the binomial theorem. Of course, each summand vanishes for $j > n$ in this case. Nevertheless we write it as an infinite series. This is convenient particularly when interchanging the order of summation since we need not make some cumbersome conversion of the range of summation.

Now let $\alpha = -n-1$ for an integer n. By applying some elementary manipulation, we notice that the series expansion of $(1+z)^{-1-n}$ can also be written as

$$\frac{1}{(1+z)^{1+n}} = \sum_{j=0}^{\infty} (-1)^j \binom{n+j}{j} z^j, \quad |z| < 1. \tag{2.2}$$

If $n = 0$, this reduces to the classical formula for the geometric series.

3 Proof of the main theorem

Our strategy toward the main theorem is quite simple; we identify the Taylor series of the modified Ahmed's integral and then evaluate it. To be specific, we first calculate the Taylor expansion of $A(p, q, r)$ and the remaining arctangent term, respectively. Then we merge the Taylor coefficients of these two functions. To this end, we temporarily restrict the range of parameters to $0 < p, q, r < 1$ throughout this section.

Series expansion of the Ahmed's integral Utilizing the Taylor series of the arctangent to the integrand,

$$\begin{aligned} A(p,q,r) &= k \int_0^1 \left(\sum_{m=0}^{\infty} (-1)^m \tilde{q}^{2m} x^{2m} \right) \left(\sum_{b=0}^{\infty} \frac{(-1)^b}{2b+1} q^{2b} (p^2 x^2 + 1)^b \right) dx \\ &= k \sum_{m=0}^{\infty} \sum_{b=0}^{\infty} \frac{(-1)^{m+b}}{2b+1} \tilde{q}^{2m} q^{2b} \int_0^1 x^{2m} \left(\sum_{j=0}^{\infty} \binom{b}{j} p^{2j} x^{2j} \right) dx \\ &= k \sum_{m=0}^{\infty} \sum_{b=0}^{\infty} \sum_{j=0}^{\infty} \frac{(-1)^{m+b}}{(2m+2j+1)(2b+1)} \binom{b}{j} \tilde{q}^{2m} p^{2j} q^{2b}. \end{aligned}$$

Now expanding $\tilde{q}^{2m} = p^{2m}(1+r^2)^m$ using the binomial series (2.1), we notice that

$$A(p,q,r) = k \sum_{m=0}^{\infty} \sum_{b=0}^{\infty} \sum_{j=0}^{\infty} \sum_{c=0}^{\infty} \frac{(-1)^{m+b}}{(2m+2j+1)(2b+1)} \binom{b}{j} \binom{m}{c} p^{2m+2j} q^{2b} r^{2c}.$$

We make the change of variable $a = m + j$, or equivalently, $m = a - j$. Then interchanging the order of summation between j and a, we obtain

$$\begin{aligned} A(p,q,r) &= k\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\sum_{j=0}^{\infty}\sum_{a=j}^{\infty}\frac{(-1)^{a+b+j}}{(2a+1)(2b+1)}\binom{b}{j}\binom{a-j}{c}p^{2a}q^{2b}r^{2c} \\ &= k\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\left[\sum_{j=0}^{a}\frac{(-1)^{a+b+j}}{(2a+1)(2b+1)}\binom{b}{j}\binom{a-j}{c}\right]p^{2a}q^{2b}r^{2c} \end{aligned}$$

To simplify this coefficient further, we adopt the generating function method. That is, we consider the following power series

$$\begin{aligned} \sum_{a=0}^{\infty}\left[\sum_{j=0}^{a}\frac{(-1)^{a+b+j}}{(2a+1)(2b+1)}\binom{b}{j}\binom{a-j}{c}\right]x^a &= \frac{(-1)^b}{(2a+1)(2b+1)}\sum_{a=0}^{\infty}\sum_{j=0}^{a}\binom{b}{j}x^j\times(-1)^{a-j}\binom{a-j}{c}x^{a-j} \\ &= \frac{(-1)^b}{(2a+1)(2b+1)}\left\{\sum_{j=0}^{\infty}\binom{b}{j}x^j\right\}\left\{\sum_{j=0}^{\infty}(-1)^j\binom{j}{c}x^j\right\}. \end{aligned}$$

Exploiting (2.2) it follows that

$$\begin{aligned} \sum_{a=0}^{\infty}\left[\sum_{j=0}^{a}\frac{(-1)^{a+b+j}}{(2a+1)(2b+1)}\binom{b}{j}\binom{a-j}{c}\right]x^a &= \frac{(-1)^b}{(2a+1)(2b+1)}(1+x)^b\left\{\frac{(-1)^c x^c}{(1+x)^{c+1}}\right\} \\ &= \frac{(-1)^{b+c}}{(2a+1)(2b+1)}\frac{x^c}{(1+x)^{c-b+1}}. \end{aligned}$$

Comparing the coefficients of the both sides, it follows that

$$\sum_{j=0}^{a}\frac{(-1)^{a+b+j}}{(2a+1)(2b+1)}\binom{b}{j}\binom{a-j}{c} = \begin{cases}\dfrac{(-1)^{a+b}}{(2a+1)(2b+1)}\dbinom{a-b}{a-c}, & \text{if } a\ge c\\ 0, & \text{otherwise.}\end{cases} \tag{3.1}$$

Series expansion of the arctangent term Now we apply the similar technique to the following arctangent term

$$\arctan\tilde{p}\arctan(k/\tilde{p}) = \arctan\left(r\sqrt{p^2+1}\right)\arctan\left(\frac{pq}{\sqrt{q^2+1}}\right).$$

In this case, the calculation turns out to be much easier.

$$\begin{aligned} \arctan\tilde{p}\arctan(k/\tilde{p}) &= k\left(\sum_{c=0}^{\infty}\frac{(-1)^c}{2c+1}\tilde{p}^{2c}\right)\left(\sum_{a=0}^{\infty}\frac{(-1)^a}{2a+1}\frac{p^{2a}q^{2a}}{(q^2+1)^a}\right) \\ &= k\sum_{a=0}^{\infty}\sum_{c=0}^{\infty}\sum_{j=0}^{\infty}\frac{(-1)^{a+c+j}}{(2a+1)(2c+1)}\binom{a+j-1}{j}\tilde{p}^{2c}p^{2a}q^{2a+2j} \\ &= k\sum_{a=0}^{\infty}\sum_{c=0}^{\infty}\sum_{j=0}^{\infty}\sum_{l=0}^{\infty}\frac{(-1)^{a+c+j}}{(2a+1)(2c+1)}\binom{a+j-1}{j}\binom{c}{l}p^{2a}q^{2a+2l+2j}r^{2c}. \end{aligned}$$

Thus with the change of variable $b = a + l + j$,

$$\arctan\tilde{p}\arctan(k/\tilde{p}) = k\sum_{a=0}^{\infty}\sum_{b=0}^{\infty}\sum_{c=0}^{\infty}\left[\sum_{l=0}^{b-a}\frac{(-1)^{b+c+l}}{(2a+1)(2c+1)}\binom{b-l-1}{b-a-l}\binom{c}{l}\right]p^{2a}q^{2b}r^{2c},$$

where the sum $\sum_{l=0}^{b-a}$ is considered zero whenever $b < a$. Now applying the generating function method as in (3.1) shows that

$$\sum_{l=0}^{b-a} \frac{(-1)^{b+c+l}}{(2a+1)(2c+1)} \binom{b-l-1}{b-a-l} \binom{c}{l} = \begin{cases} \dfrac{(-1)^{a+c}}{(2a+1)(2c+1)} \dbinom{c-a}{b-a}, & \text{if } b \geq a \\ 0, & \text{otherwise.} \end{cases} \tag{3.2}$$

3.1 Series expansion of the modified Ahmed's integral

In order to identify the coefficients $d_{a,b,c}$ in the series expansion of $\tilde{A}(p,q,r)$

$$\tilde{A}(p,q,r) = \sum_{a=0}^{\infty} \sum_{b=0}^{\infty} \sum_{c=0}^{\infty} d_{a,b,c} p^{2a+1} q^{2b+1} r^{2c+1},$$

we have to combine (3.1) and (3.2). (Here, we remark that $\tilde{A}(p,q,r)$ is an odd function with respect to both p, q and r. Thus the above expansion makes sense.) Let us introduce the *discrete step function*

$$H[n] = \begin{cases} 1, & n \geq 0 \\ 0, & n < 0, \end{cases}$$

With aid of this function, we can simply combine (3.1) and (3.2) to obtain

$$d_{a,b,c} = \frac{(-1)^{a-b} H[a-c]}{(2a+1)(2b+1)} \binom{a-b}{a-c} + \frac{(-1)^{a-c} H[b-a]}{(2a+1)(2c+1)} \binom{c-a}{b-a}. \tag{3.3}$$

The following result shows that this seemingly asymmetric formula yields a perfectly symmetric one.

Lemma 3.1. *Suppose that $a, b, c \geq 0$ and that $d_{a,b,c}$ is the coefficient of $p^{2a+1} q^{2b+1} r^{2c+1}$ in the series expansion of $\tilde{A}(p,q,r)$. Also let $\alpha \leq \beta \leq \gamma$ be a reordering of a, b, c. Then $d_{a,b,c} \neq 0$ only if $(\alpha, \beta, \gamma) = (a,b,c)$ or (b,c,a) or (c,a,b), and in this case we have*

$$d_{a,b,c} = \begin{cases} \dfrac{2}{(2\alpha+1)^2}, & \alpha = \beta = \gamma \\ \dfrac{(-1)^{\gamma-\alpha}}{(2\alpha+1)(2\gamma+1)} \dbinom{\gamma-\alpha}{\beta-\alpha}, & \textit{otherwise.} \end{cases} \tag{3.4}$$

Proof of Lemma. The proof adopts the divide and conquer method.

- **Case 1.** $a = b = c$, then (3.3) readily reduces to

$$d_{a,a,a} = \frac{2}{(2a+1)^2}.$$

- **Case 2.** Assume $a \leq b < c$ or $a \leq b < c$. In any cases, we have $c > a$ and $b \geq a$. Thus we can put $(\alpha, \beta, \gamma) = (a,b,c)$ and hence

$$d_{a,b,c} = \frac{(-1)^{c-a}}{(2a+1)(2c+1)} \binom{c-a}{b-a} = \frac{(-1)^{\gamma-\alpha}}{(2\alpha+1)(2\gamma+1)} \binom{\gamma-\alpha}{\beta-\alpha}.$$

- **Case 3.** Assume $b \leq c < a$ or $b < c \leq a$. In any cases, we have $a > b$ and $a \geq c$. Thus we can put $(\alpha, \beta, \gamma) = (b,c,a)$ and

$$d_{a,b,c} = \frac{(-1)^{b-a}}{(2b+1)(2a+1)} \binom{a-b}{a-c} = \frac{(-1)^{\gamma-\alpha}}{(2\alpha+1)(2\gamma+1)} \binom{\gamma-\alpha}{\gamma-\beta}.$$

  Thus (3.4) follows from the fact that $\binom{\gamma-\alpha}{\gamma-\beta} = \binom{\gamma-\alpha}{\beta-\alpha}$.

- **Case 4.** Assume $c \le a < b$ or $c < a \le b$. Then we can put $(\alpha, \beta, \gamma) = (c, a, b)$ and
$$d_{a,b,c} = \frac{(-1)^{\beta-\gamma}}{(2\beta+1)(2\gamma+1)}\binom{\beta-\gamma}{\beta-\alpha} + \frac{(-1)^{\alpha-\beta}}{(2\alpha+1)(2\beta+1)}\binom{\alpha-\beta}{\gamma-\beta}$$
Here, we remark that (3.4) immediately follows if $\alpha = \beta$ or $\beta = \gamma$. So we may assume $\alpha < \beta < \gamma$ further. Using the identity $\binom{-n}{j} = (-1)^j\binom{n+j-1}{j}$, this reduces to
$$d_{a,b,c} = \frac{(-1)^{\gamma-\alpha}}{(2\alpha+1)(2\beta+1)(2\gamma+1)}\left[(2\alpha+1)\binom{\gamma-\alpha-1}{\beta-\alpha} + (2\gamma+1)\binom{\gamma-\alpha-1}{\gamma-\beta}\right].$$
To simplify the parenthesised terms, we use the identity $k\binom{\alpha}{k} = \alpha\binom{\alpha-1}{k-1}$.
$$\begin{aligned}
&(2\alpha+1)\binom{\gamma-\alpha-1}{\beta-\alpha} + (2\gamma+1)\binom{\gamma-\alpha-1}{\gamma-\beta} \\
&= 2(\alpha-\beta)\binom{\gamma-\alpha-1}{\beta-\alpha} + 2(\gamma-\beta)\binom{\gamma-\alpha-1}{\gamma-\beta} + (2\beta+1)\left[\binom{\gamma-\alpha-1}{\gamma-\beta} + \binom{\gamma-\alpha-1}{\beta-\alpha}\right] \\
&= 2(\gamma-\alpha-1)\left[\binom{\gamma-\alpha-2}{\gamma-\beta-1} - \binom{\gamma-\alpha-2}{\beta-\alpha-1}\right] + (2\beta+1)\binom{\gamma-\alpha}{\beta-\alpha} \\
&= (2\beta+1)\binom{\gamma-\alpha}{\beta-\alpha}.
\end{aligned}$$
This proves (3.4).

- **Case 5.** Assume $b < a < c$. Then (3.3) vanishes trivially.

- **Case 6.** Assume $c < b < a$. Then $0 < a - b < a - c$ and $\binom{a-b}{a-c} = 0$. Thus (3.3) vanishes also in this case.

- **Case 7.** Assume $a < c < b$. Then $0 < c - a < b - a$ and $\binom{c-a}{b-a} = 0$. Thus (3.3) vanishes also in this case.

Therefore, combining all these cases proves the lemma. □

The lemma then says that, upon rewriting the indices as $(\alpha, \beta, \gamma) = (n, n+l, n+m)$ with $m, n \ge 0$ and $0 \le l \le m$, the Taylor series of $\tilde{A}(p, q, r)$ is given by
$$\tilde{A}(p,q,r) = 2\sum_{n=0}^{\infty}\frac{k^{2n+1}}{(2n+1)^2} + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\sum_{l=0}^{m}\frac{(-1)^m}{(2n+1)(2n+2m+1)}\binom{m}{l}k^{2n+1}(p^{2l}q^{2m} + q^{2l}r^{2m} + r^{2l}p^{2m}).$$
Now applying the binomial theorem and simplifying further,
$$\begin{aligned}
\tilde{A}(p,q,r) &= 2\chi_2(k) + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m}{(2n+1)(2n+2m+1)}k^{2n+1}(\tilde{p}^{2m} + \tilde{q}^{2m} + \tilde{r}^{2m}) \\
&= 2\chi_2(k) + \frac{1}{2}\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\frac{(-1)^m}{m}k^{2n+1}(\tilde{p}^{2m} + \tilde{q}^{2m} + \tilde{r}^{2m})\int_0^1 x^{2n}(1-x^{2m})\,dx \\
&= 2\chi_2(k) + \frac{1}{2}\sum_{n=0}^{\infty}k^{2n+1}\int_0^1 x^{2n}\log\left(\frac{1+\tilde{p}^2x^2}{1+\tilde{p}^2}\cdot\frac{1+\tilde{q}^2x^2}{1+\tilde{q}^2}\cdot\frac{1+\tilde{r}^2x^2}{1+\tilde{r}^2}\right)dx \\
&= 2\chi_2(k) + \frac{k}{2}\int_0^1 \frac{1}{1-k^2x^2}\log\left(\frac{1+\tilde{p}^2x^2}{1+\tilde{p}^2}\cdot\frac{1+\tilde{q}^2x^2}{1+\tilde{q}^2}\cdot\frac{1+\tilde{r}^2x^2}{1+\tilde{r}^2}\right)dx
\end{aligned}$$
as desired, proving Theorem 1.1 for $0 < p, q, r < 1$. Now the general result follows from the principle of analytic continuation.

4 Properties of the Ahmed's integral

In this section we consider various properties that are satisfied by the generalized Ahmed's integrals.

4.1 Consequences of the main theorem

Here we want to discuss some consequences of our main theorem. Our first concern is that (1.4) works only for $k \leq 1$ since both $\chi_2(k)$ and the integral term has logarithmic singularity at $k = 1$. But it is evident from the left-hand side of (1.4) that this is indeed an artifact: they must cancel out each other. So we want to modify (1.4) so that the resulting identity works for *any* $p, q, r > 0$. To this end, we exploit the following surprising identity

$$\chi_2\left(\frac{1-z}{1+z}\right) + \chi_2(z) = \frac{\pi^2}{8} + \log z \operatorname{artanh} z.$$

Thus together with the following another unexpected observation

$$\frac{1+(\tilde{p}/k)^2}{1+\tilde{p}^2} \times \frac{1+(\tilde{q}/k)^2}{1+\tilde{q}^2} \times \frac{1+(\tilde{r}/k)^2}{1+\tilde{r}^2} = \frac{1}{k^4}, \tag{4.1}$$

we immediately obtain the following corollary.

Corollary 4.1. *For each $p, q, r > 0$, the following integral representation holds.*

$$\tilde{A}(p,q,r) = \frac{\pi^2}{4} + 2\chi_2\left(\frac{k-1}{k+1}\right) + \frac{k}{2}\int_0^1 \frac{1}{1-k^2x^2} \log\left(k^4 \times \frac{1+\tilde{p}^2x^2}{1+\tilde{p}^2} \times \frac{1+\tilde{q}^2x^2}{1+\tilde{q}^2} \times \frac{1+\tilde{r}^2x^2}{1+\tilde{r}^2}\right) dx. \tag{4.2}$$

Now we want to simply this further. The result (4.2) shows that it is of the key importance to resolve the following integral

$$F(k,p) = \frac{k}{2}\int_0^1 \frac{1}{1-k^2x^2} \log\left(\frac{1+p^2x^2}{1+p^2}\right) dt.$$

To this end, we temporarily assume $k < 1$ and use the technique of differentiation under integration. Indeed, it is clear that $F(k, 0) = 0$. Then simple calculation using partial fractional decomposition shows that

$$\frac{\partial}{\partial p} F(k,p) = \frac{(1-k^2)p \operatorname{artanh} k}{(1+p^2)(k^2+p^2)} - \frac{k \arctan p}{k^2+p^2}.$$

Integrating with respect to p, it follows that the former term reduces to a nice function

$$F(k,p) = \frac{1}{2}\operatorname{artanh}(k) \log\left(\frac{1+(p/k)^2}{1+p^2}\right) - \int_0^p \frac{k \arctan t}{k^2+t^2}\, dt.$$

Thus the problem reduces to that of investigating the properties of the following integral

$$\tilde{F}(k,p) = \int_0^p \frac{k \arctan t}{k^2+t^2}\, dt. \tag{4.3}$$

We first make a brief remark. In view of our remarkable identity (4.1), the identity (4.2) now reduces to

$$\tilde{A}(p,q,r) = \frac{\pi^2}{4} + 2\chi_2\left(\frac{k-1}{k+1}\right) - \tilde{F}(k,\tilde{p}) - \tilde{F}(k,\tilde{q}) - \tilde{F}(k,\tilde{r}). \tag{4.4}$$

Each term involved in this formula now defines analytic function near the domain $\{(p,q,r) \in \mathbb{C}^3 : p,q,r > 0\}$. Thus we can expect a better behavior compared to (1.4) or (4.2). Indeed, applying integration by parts, we find that $\tilde{F}(k,p)$ satisfies the following formula

$$\tilde{F}(k,p) = \arctan(p) \arctan\left(\frac{p}{k}\right) - \tilde{F}\left(\frac{1}{k}, \frac{p}{k}\right). \tag{4.5}$$

In particular, if we plug $k = 1$ then it immediately follows that $\tilde{F}(1, p) = \frac{1}{2}\arctan^2 p$. Thus by (4.4), we obtain

$$A(p, q, r) = \frac{\pi^2}{4} - \frac{1}{2}\left\{\arctan^2\tilde{p} + \arctan^2\tilde{q} + \arctan^2\tilde{r}\right\} - \arctan\tilde{p}\arctan(1/\tilde{p}).$$

Then utilizing the simple property $\arctan(1/\tilde{p}) = \frac{\pi}{2} - \arctan\tilde{p}$, we can simply this further and hence obtain the following corollary.

Corollary 4.2. *For any $p, q, r > 0$ satisfying $pqr = 1$, the following identity holds*

$$A(p, q, r) = \frac{\pi^2}{8} + \frac{1}{2}\left\{\arctan^2\left(\frac{1}{\tilde{p}}\right) - \arctan^2(\tilde{q}) - \arctan^2(\tilde{r})\right\}. \tag{4.6}$$

4.2 Inversion of parameters and special values at infinity

Let us consider the following involution

$$\tau(p, q, r) = \left(\frac{1}{q}, \frac{1}{p}, \frac{1}{r}\right).$$

Under this special change of variables, we easily check that

$$\tilde{p}\circ\tau = \frac{\tilde{r}}{k}, \quad \tilde{q}\circ\tau = \frac{\tilde{q}}{k}, \quad \tilde{r}\circ\tau = \frac{\tilde{p}}{k}, \quad \text{and} \quad k\circ\tau = \frac{1}{k}.$$

Plugging this relation to (4.5), we have

$$\begin{aligned}\tilde{F}(k, \tilde{p}) + \tilde{F}(k\circ\tau, \tilde{r}\circ\tau) &= \arctan(\tilde{p})\arctan(\tilde{r}\circ\tau),\\ \tilde{F}(k, \tilde{q}) + \tilde{F}(k\circ\tau, \tilde{q}\circ\tau) &= \arctan(\tilde{q})\arctan(\tilde{q}\circ\tau),\\ \tilde{F}(k, \tilde{r}) + \tilde{F}(k\circ\tau, \tilde{p}\circ\tau) &= \arctan(\tilde{r})\arctan(\tilde{p}\circ\tau).\end{aligned}$$

Plugging this to $\tilde{A} + \tilde{A}\circ\tau$ and utilizing (4.4), we obtain

$$\tilde{A} + \tilde{A}\circ\tau = \frac{\pi^2}{2} - \arctan(\tilde{p})\arctan(\tilde{r}\circ\tau) - \arctan(\tilde{q})\arctan(\tilde{q}\circ\tau) - \arctan(\tilde{r})\arctan(\tilde{p}\circ\tau).$$

This shows that $\tilde{A}$ is invariant under the involution τ up to elementary functions. Simplifying further, we obtain the following proposition.

Proposition 4.3. *The following identity holds.*

$$A(p, q, r) + (A\circ\tau)(p, q, r) = \frac{\pi}{2}\left\{\arctan\left(\frac{1}{\tilde{p}}\right) + \arctan\left(\frac{1}{\tilde{p}\circ\tau}\right)\right\} - \arctan(\tilde{q})\arctan(\tilde{q}\circ\tau). \tag{4.7}$$

The identity (4.7) itself gives special values of A for values (p, q, r) satisfying $(p, q, r) = \tau(p, q, r)$. This is equivalent to $pq = 1$ and $r = 1$. Then $k = 1$ and we easily check that both Corollary 4.2 and Proposition 4.3 give the same result. Thus the idea of using this proposition to obtain a special value ends up with no fruitful result, except when $A\circ\tau$ vanishes. This happens when one of the parameters goes to infinity. Indeed, if one onf p, q or r tends to infinity, we have $A\circ\tau \to 0$. Thus

$$\begin{aligned}A(\infty, q, r) &= \frac{\pi}{2}\left\{\arctan\left(\frac{1}{r\sqrt{q^2+1}}\right) + \arctan r - \arctan\left(\frac{\sqrt{r^2+1}}{qr}\right)\right\}\\ A(p, \infty, r) &= \frac{\pi}{2}\arctan\left(\frac{pr}{\sqrt{p^2+1}}\right),\\ A(p, q, \infty) &= \frac{\pi}{2}\arctan q.\end{aligned}$$

This completes the calculation of $A(p, q, r)$ when $k = \infty$.

4.3 Application to the Coxeter's integrals

In this section we consider another special family of integrals called the *Coxeter's integral*. This integral is originated from the problem posed by Prof. Coxeter in [Cox88].

Theorem 4.4. *Suppose that $a \geq |b|$ and that $a\cos\theta + b > 0$ for $\theta \in (0, \alpha)$. Then we have*

$$\int_0^\alpha \arctan\sqrt{\frac{\cos\theta + 1}{a\cos\theta + b}}\, d\theta = 2A\left(\sqrt{\frac{a-b}{2}\cdot\frac{1-\cos\alpha}{a\cos\alpha + b}}, \sqrt{\frac{2}{a+b}}, \sqrt{\frac{a+b}{a-b}}\right). \tag{4.8}$$

Proof of Theorem. We make use of the simple change of variable $\theta \mapsto 2\theta$ to obtain

$$\begin{aligned}\int_0^\alpha \arctan\sqrt{\frac{\cos\theta + 1}{a\cos\theta + b}}\, d\theta &= \int_0^\alpha \arctan\sqrt{\frac{2\cos^2(\theta/2)}{a + b - 2a\sin^2(\theta/2)}}\, d\theta \\ &= 2\int_0^{\alpha/2} \arctan\left(\frac{\cos\theta}{\sqrt{(a+b)/2 - a\sin^2\theta}}\right) d\theta.\end{aligned}$$

This shows that it suffices to consider the integral of the form

$$I(p, q, \phi) = \int_0^\phi \arctan\left(\frac{\cos\theta}{p\sqrt{q^2 - \sin^2\theta}}\right) d\theta,$$

where parameters satisfy $p > 0$, $\phi \in [0, \frac{\pi}{2}]$ and $\sin\phi \leq q < 1$. With this notation we readily identify that

$$\int_0^\alpha \arctan\sqrt{\frac{\cos\theta + 1}{a\cos\theta + b}}\, d\theta = 2I\left(\sqrt{a}, \sqrt{\frac{a+b}{2a}}, \frac{\alpha}{2}\right).$$

Then the substitution $\sin\theta \mapsto q\sin\theta$ together with $\tilde{\phi} = \arcsin(\sin\phi/q)$ yields

$$I(p, q, \phi) = \int_0^{\tilde{\phi}} \arctan\left(\frac{\sqrt{1 - q^2\sin^2\theta}}{pq\cos\theta}\right)\left(\frac{q\cos\theta}{\sqrt{1 - q^2\sin^2\theta}}\right) d\theta.$$

Applying the substitution $\tan\theta \mapsto t\tan\tilde{\phi}$, this further simplifies

$$I(p, q, \phi) = \int_0^1 \arctan\left(\frac{\sqrt{1 + t^2(1-q^2)\tan^2\tilde{\phi}}}{pq}\right)\left(\frac{q}{\sqrt{1 + t^2(1-q^2)\tan^2\tilde{\phi}}}\right) \frac{\tan\tilde{\phi}\, dt}{1 + t^2\tan^2\tilde{\phi}}.$$

Then by comparison with the definition (1.2), it follows that

$$I(p, q, \phi) = A\left(\sqrt{1-q^2}\tan\tilde{\phi}, \frac{1}{pq}, \frac{q}{\sqrt{1-q^2}}\right).$$

Therefore, after some simple algebraic manipulations, we obtain the desired result. □

Now let's consider the following special cases of the Coxeter's integrals:

$$\begin{aligned}
I_1 &= \int_0^{\frac{\pi}{2}} \arccos\left(\frac{\cos\theta}{1 + 2\cos\theta}\right) d\theta, & I_2 &= \int_0^{\frac{\pi}{3}} \arccos\left(\frac{\cos\theta}{1 + 2\cos\theta}\right) d\theta, \\
I_3 &= \int_0^{\frac{\pi}{2}} \arccos\left(\frac{1}{1 + 2\cos\theta}\right) d\theta, & I_4 &= \int_0^{\frac{\pi}{3}} \arccos\left(\frac{1}{1 + 2\cos\theta}\right) d\theta, \\
I_5 &= \int_0^{\cos^{-1}(1/3)} \arccos\left(\frac{1 - \cos\theta}{2\cos\theta}\right) d\theta, & I_6 &= \int_0^{\frac{\pi}{3}} \arccos\left(\frac{1 - \cos\theta}{2\cos\theta}\right) d\theta, \\
I_7 &= \int_0^{\frac{\pi}{2}} \arccos\sqrt{\frac{\cos\theta}{1 + 2\cos\theta}}\, d\theta, & I_8 &= \int_0^{\frac{\pi}{3}} \arccos\sqrt{\frac{\cos\theta}{1 + 2\cos\theta}}\, d\theta.
\end{aligned}$$

By the half-angle formula for the cosine, for $|A| < 1$,

$$\arccos A = 2\arccos\sqrt{\frac{A+1}{2}} = 2\arctan\sqrt{\frac{1-A}{1+A}}.$$

Applying this to reduce the arc-cosine integrand to the corresponding arctangent integrand, we find that

$$\int_0^\alpha \arccos\left(\frac{\cos\theta}{1+2\cos\theta}\right) d\theta = 2\int_0^\alpha \arctan\sqrt{\frac{\cos\theta+1}{3\cos\theta+1}}\, d\theta = 4A\left(\sqrt{\frac{1-\cos\alpha}{3\cos\alpha+1}}, \frac{1}{\sqrt{2}}, \sqrt{2}\right).$$

This allows us to write I_1 and I_2 in terms of the Ahmed's integral. Likewise,

$$\int_0^\alpha \arccos\left(\frac{1}{1+2\cos\theta}\right) d\theta = 2\int_0^\alpha \arctan\sqrt{\frac{\cos\theta}{\cos\theta+1}}\, d\theta = \pi\alpha - 4A\left(\sqrt{\frac{1-\cos\alpha}{2\cos\alpha}}, \sqrt{2}, 1\right)$$

for I_3 and I_4,

$$\int_0^\alpha \arccos\left(\frac{1-\cos\theta}{2\cos\theta}\right) d\theta = 2\int_0^\alpha \arctan\sqrt{\frac{3\cos\theta-1}{\cos\theta+1}}\, d\theta = \pi\alpha - 4A\left(\sqrt{\frac{2(1-\cos\alpha)}{3\cos\alpha-1}}, 1, \frac{1}{\sqrt{2}}\right)$$

for I_5 and I_6, and finally

$$\int_0^\alpha \arccos\sqrt{\frac{\cos\theta}{2\cos\theta+1}}\, d\theta = \int_0^\alpha \arctan\sqrt{\frac{\cos\theta+1}{\cos\theta}}\, d\theta = 2A\left(\sqrt{\frac{1-\cos\alpha}{2\cos\alpha}}, \sqrt{2}, 1\right)$$

for I_7 and I_8. Applying these identities shows that all of I_1 to I_8, except for I_2, reduce either to the case $k = 1$ or to the case $k = \infty$. So we obtain the closed form for them.

Now we have only one remaining case I_2, which is also directly related to the original Coxeter's integral in [Cox88].

References

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[Cox88] H. S. M. Coxeter. A challenging definite integral. *Am. Math. Mon.*, 95:330, 1988.