Series Expansion

Dexter Kim(jwkonline@gmail.com)

Starting from binomial expansion, some basic techniques of series expansion and notable points are presented in a concise way.

1 The Binomial Expansion

The expansion of $(1+x)^n$ for an integer n is a well-known fact.

$$(1+x)^n = \sum_{i=0}^{n} \frac{n!}{k!(n-k)!} x^k = \sum_{i=0}^{n} \frac{n(n-1)\cdots(n-k+1)}{1\cdot 2\cdots k} x^k = 1 + nx + \frac{n(n-1)}{2} x^2 + \cdots$$

This expansion is called the binomial expansion. One does not require n to be an integer, in principle. For a real number ν one obtains the equivalent expansion as the following, which is also known as Newton's binomial expansion.(Caveat: k runs to infinity)

$$(1+x)^\nu = \sum_{i=0}^{\infty} \frac{\nu(\nu-1)\cdots(\nu-k+1)}{1\cdot 2\cdots k} x^k = 1 + \nu x + \frac{\nu(\nu-1)}{2} x^2 + \cdots$$

This is nothing but a Taylor expansion of x^ν at $x = 1$. Due to uniqueness theorem of Taylor expansions, it is totally fine to do an expansion by this method. However, keep in mind that the radius of convergence is restricted to $|x| < 1$ when ν is not an integer. For applications to approximations, this is a minor issue no one needs to take care about, but it is nevertheless a good decision to know that something awry can happen in some cases. Using gamma functions one can extend the definition to complex numbered ν, but it rarely occurs and since it is readily extendible from real numbers we will not dwell on the possibility.

2 Taylor Expansion using Binomial Expansion

When dealing with complex expressions, obtaining Taylor expansion coefficients by direct differentiation often becomes a daunting task. This is a good, typical example.

$$\frac{1}{\sqrt{a^2+x^2}}$$

Rewrite the expression as follows.

$$\frac{1}{\sqrt{a^2+x^2}} = \frac{1}{a\sqrt{1+(x/a)^2}} = \frac{1}{a}\left[1+\left(\frac{x}{a}\right)^2\right]^{-1/2}$$

This expression is easy to expand, as the following formula is known.

$$\left[1+\left(\frac{x}{a}\right)^2\right]^{-1/2} = 1 - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{1}{2}(-\frac{1}{2}\cdot -\frac{3}{2})\left(\frac{x}{a}\right)^4 + \cdots$$

Substitution yields this result.

$$\frac{1}{\sqrt{a^2+x^2}} = \frac{1}{a}\left[1 - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{3}{8}\left(\frac{x}{a}\right)^4 - \frac{5}{16}\left(\frac{x}{a}\right)^6 + \cdots\right]$$

Tada! One must keep in mind that x was assumed to be small. The actual assumption one used is $|x/a| < 1$. For the opposite limit of $|a/x| < 1$, one can follow the same procedures. When expressions are entangled into a product, multiplication of infinite series will do the job. Since this is nothing but a rote calculation, it will be skipped.

A typical example of the above formula is $\gamma = (1-\beta^2)^{(-1/2)}$ from the theory of special relativity.

$$\gamma = \sqrt{\frac{1}{1+(-\beta^2)}} = 1 - \frac{1}{2}(-\beta^2) + \frac{3}{8}(-\beta^2)^2 - \frac{5}{16}(-\beta^2)^3 + \cdots$$

For sake of completeness, let's obtain an approximation of γ at the other limit. This is the approximation using $\beta = 1-\alpha$.

$$\gamma = \sqrt{\frac{1}{2\alpha-\alpha^2}} = \sqrt{\frac{1}{2\alpha}}\left[1-\frac{\alpha}{2}\right]^{-1/2} = \sqrt{\frac{1}{2\alpha}}\left[1 - \frac{1}{2}\left(-\frac{\alpha}{2}\right) + \frac{3}{8}\left(-\frac{\alpha}{2}\right)^2 - \frac{5}{16}\left(-\frac{\alpha}{2}\right)^3 + \cdots\right]$$

The brute-force method of solving differential equations by expansions of this form is called the Frobenius method. The method absorbes the most singular part to the overall multiplicative factor and keeps subsequent terms regular. It is easily verifiable from the expansion that when α is about 0.2, or β is about 0.8, the relative error is only a mediocre value of 5%($1.66\cdots$ vs $1.58\cdots$; $0.05 \times 1.66 \sim 0.08$). One can also note that when γ exceeds 2, the velocity is greater than 0.9 times the speed of light(the exact value of $\gamma = 2$ occurs around $0.86c$).

One can also exploit multiplication of power series to obtain a Taylor expansion of much complicated expressions. In many cases, this is far faster than obtaining the coefficients by brute-force differentiation. Let's obtain a series expansion of the following expression at $x = 0$.

$$\begin{aligned}\frac{1}{\sqrt{a^2+x^2}}e^{-x^2/b^2} &= \frac{1}{a}\left[1 - \frac{1}{2}\left(\frac{x}{a}\right)^2 + \frac{3}{8}\left(\frac{x}{a}\right)^4 - \frac{5}{16}\left(\frac{x}{a}\right)^6 + \cdots\right] \\ &\times \left[1 + \left(-\frac{x^2}{b^2}\right) + \frac{1}{2}\left(-\frac{x^2}{b^2}\right)^2 + \frac{1}{3!}\left(-\frac{x^2}{b^2}\right)^3 + \cdots\right] \\ = \frac{1}{a}\Bigg[1 &+ \left(-\frac{1}{2a^2} - \frac{1}{b^2}\right)x^2 + \left\{\frac{3}{8a^4} + \frac{1}{2b^4} + \left(-\frac{1}{2a^2}\right)\left(-\frac{1}{b^2}\right)\right\}x^4 \\ + \Bigg\{-\frac{5}{16a^6} - \frac{1}{6b^6} &+ \left(-\frac{1}{2a^2}\right)\left(\frac{1}{2b^4}\right) + \left(\frac{3}{8a^4}\right)\left(-\frac{1}{b^2}\right)\Bigg\}x^6 + \cdots\Bigg]\end{aligned}$$

Obtaining the series expansion from derivatives is not an enjoyable task, at least for the above example.

3 Series Expansion of Inverses

One often finds oneself in need of inverted series expansion, i.e. in place of

$$y = f(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots$$

one seeks for

$$x = f^{-1}(y) = b_0 + b_1 y + b_2 y^2 + b_3 y^3 + \cdots$$

Such problems are frequent in statistical mechanics, especially when one is working on degenerate gases or virial expansions. For sake of simplicity, we will tackle the problem of obtaining the series expansion of the inverse function when the leading term is linear, since one can always redefine, or shift, the function $y = f(x)$ so that the leading term is linear.

$$y = a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots$$

The most primitive method (I believe it was Newton who introduced it) was to iterate repeatedly; setting $x = b_1 y + \delta$

$$\begin{aligned} 0 &= -y + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= -y + a_1(b_1 y + \delta) + a_2(b_1 y + \delta)^2 + \cdots \\ &= (a_1 b_1 - 1)y + \delta + \mathcal{O}(y^2) \\ &\therefore b_1 = 1/a_1 \end{aligned}$$

Next step, set $x = y/a_1 + b_2 y^2 + \delta$

$$\begin{aligned} 0 &= -y + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= -y + a_1(y/a_1 + b_2 y^2 + \delta) + a_2(y/a_1 + b_2 y^2 + \delta)^2 + a_3(y/a_1 + b_2 y^2 + \delta)^3 + \cdots \\ &= (a_1/a_1 - 1)y + (a_1 b_2 + a_2/(a_1)^2)y^2 + \delta + \mathcal{O}(y^3) \\ &\therefore b_2 = -a_2/(a_1)^3 \end{aligned}$$

One uses $x = y/a_1 - [a_2/(a_1)^3]y^2 + b_3 y^3 \delta$ for the next iteration process. This process is extremely slow. Actually, one can speed up everything by using a deceptively simple tactic; just substitute x by series expansions of y!

$$\begin{aligned} 0 &= -y + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + \cdots \\ &= -y + a_1(b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 \cdots) + a_2(b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 \cdots)^2 \\ &\quad + a_3(b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 \cdots)^3 + a_4(b_1 y + b_2 y^2 + b_3 y^3 + b_4 y^4 \cdots)^4 + \cdots \\ &= (a_1 b_1 - 1)y + [a_1 b_2 + a_2(b_1)^2]y^2 + [a_1 b_3 + 2a_2 b_1 b_2 + a_3(b_1)^3]y^3 \\ &\quad + [a_1 b_4 + a_2[(b_2)^2 + 2b_1 b_3] + 3a_3(b_1)^2 b_2 + a_4(b_1)^4]y^4 + \cdots \end{aligned}$$

Equating order by order in y, we solve the following set of coupled equations;

$$\begin{aligned}
0 &= a_1 b_1 - 1 \\
0 &= a_1 b_2 + a_2 (b_1)^2 \\
0 &= a_1 b_3 + 2 a_2 b_1 b_2 + a_3 (b_1)^3 \\
0 &= a_1 b_4 + a_2 [(b_2)^2 + 2 b_1 b_3] + 3 a_3 (b_1)^2 b_2 + a_4 (b_1)^4 \\
&\vdots
\end{aligned}$$

Nothing has changed from the primitive method, but the iteration process has sped up.

$$\begin{aligned}
b_1 &= 1/a_1 \\
b_2 &= -[a_2 (b_1)^2]/a_1 = -a_2/(a_1)^3 \\
b_3 &= -[2 a_2 b_1 b_2 + a_3 (b_1)^3]/a_1 = 2 (a_2)^2/(a_1)^5 - a_3/(a_1)^4 \\
b_4 &= -[a_2 [(b_2)^2 + 2 b_1 b_3] + 3 a_3 (b_1)^2 b_2 + a_4 (b_1)^4]/a_1 = \cdots \\
&\vdots
\end{aligned}$$

If the function $y = f(x)$ is odd, the inverse function will be odd as well. One can exploit this fact and use the series expansion $x = b_1 y + b_3 y^3 + b_5 y^5 + b_7 y^7 + \cdots$ for the iteration ansatz when one wishes to invert an odd power series of x.

4 Geometric Series

Although this is nothing but a special case of binomial expansion, it is convenient to remember the following formula.

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + x^4 + \cdots$$

We will see its usefulness in following sections.

5 Division by Power Series

One is usually taught to divide by a power series using the method employed for polynomials. It works well, but one can also use geometric series to do a series expansion. Let's have a look at series expansion of $\tan x$

$$\tan x = \frac{\sin x}{\cos x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots}$$

To use geometric series, one sums up all remaining terms as one.

$$\begin{aligned}
\tan x &= \frac{\sin x}{\cos x} = \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots}{1 - \frac{1}{2}x^2 + \frac{1}{24}x^4 + \cdots} \\
&= \frac{x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots}{1 - (\frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots)} \\
&= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)\left[1 + \left(\frac{1}{2}x^2 - \frac{1}{24}x^4 + \cdots\right) + \left(\frac{1}{2}x^2 + \cdots\right)^2 + \cdots\right] \\
&= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)\left[1 + \frac{1}{2}x^2 - \frac{1}{24}x^4 + \frac{1}{4}x^4 + \cdots\right] \\
&= \left(x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \cdots\right)\left(1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \cdots\right) \\
&= x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \frac{1}{2}x^3 - \frac{1}{12}x^5 + \frac{5}{24}x^5 + \cdots \\
&= x + \frac{1}{3}x^3 + \frac{16}{120}x^5 + \cdots = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots
\end{aligned}$$

Let's look at another example, the series expansion of Langevin function $L(x) \equiv \coth x - 1/x$ at $x = 0$. This function appears when one calculates magnetisation of gas with classical magnetic moment. Suppose that we know the following formula

$$\tanh x = x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots$$

The unwieldy $\coth x$ can be recast using the procedure explained in this section.

$$\begin{aligned}
\coth x &= \frac{1}{\tanh x} = \frac{1}{x - \frac{1}{3}x^3 + \frac{2}{15}x^5 + \cdots} \\
&= \frac{1}{x}\left(1 - \frac{1}{3}x^2 + \frac{2}{15}x^4 + \cdots\right)^{-1} \\
&= \frac{1}{x}\left(1 - \left[\frac{1}{3}x^2 - \frac{2}{15}x^4 + \cdots\right]\right)^{-1} \\
&= \frac{1}{x}\left(1 + \left[\frac{1}{3}x^2 - \frac{2}{15}x^4 + \cdots\right] + \left[\frac{1}{3}x^2 + \cdots\right]^2 + \cdots\right) \\
&= \frac{1}{x}\left(1 + \frac{1}{3}x^2 - \frac{2}{15}x^4 + \frac{1}{9}x^4 + \cdots\right) \\
&= \frac{1}{x}\left(1 + \frac{1}{3}x^2 - \frac{1}{45}x^4 + \cdots\right) \\
&= \frac{1}{x} + \frac{1}{3}x - \frac{1}{45}x^3 + \cdots
\end{aligned}$$

Thus, one can conclude that series expansion of Langevin function is $L(x) = x/3 - x^3/45 + \cdots$, which one can easily verify by looking up tables.

As one looks for higher orders, using geometric series becomes infeasible due to large powers one needs to compute. This method works best for first few orders.

6 Numerical Approximation using Binomial Expansion: $\sqrt{80}$

How can we obtain the square root of 80 quickly? Start with the binomial expansion.

$$\sqrt{80} = (81-1)^{1/2} = 9\left(1-\frac{1}{81}\right)^{1/2}$$

We obtain the following series expansion.

$$\sqrt{80} = 9\left(1+\frac{1}{2}\left(-\frac{1}{81}\right)-\frac{1}{8}\left(-\frac{1}{81}\right)^2+\frac{1}{16}\left(-\frac{1}{81}\right)^3+\cdots\right)$$

The factor $-1/8$ comes from $\frac{1}{2}\times\frac{1}{2}\times\left(-\frac{1}{2}\right)$, and the factor $\frac{1}{16}$ from $\frac{1}{6}\times\frac{1}{2}\times\left(-\frac{1}{2}\right)\left(-\frac{3}{2}\right)$. Pushing the calculation,

$$\sqrt{80} = 9\left(1-\frac{1}{162}-\frac{1}{52488}-\frac{1}{8503056}+\cdots\right)$$

Everyone does well up to this point. Changing the fractions into decimals, however, is not an easy task. Well, we just expand once more. Geometric series will be needed in this case. Calculating up to sixth decimal places,

$$\begin{aligned}
\sqrt{80} &= 9\left(1-\frac{1}{160+2}-\frac{1}{50000+2488}+\cdots\right)\\
&= 9\left(1-\frac{1}{160}\left(\frac{1}{1+1/80}\right)-\frac{1}{50000}\left(\frac{1}{1+2488/50000}\right)+\cdots\right)\\
&= 9\left(1-(0.00625)\left(\frac{1}{1+0.0125}\right)-(0.00002)\left(\frac{1}{1+0.04976}\right)\cdots\right)\\
&= 9\Big(1-(0.00625)\left(1-0.0125+(0.0125)^2+\cdots\right)\\
&\quad -(0.00002)\left(1-0.04976+(0.04976)^2+\cdots\right)\cdots\Big)\\
&= 9(1-0.006250+0.000078-0.000001\\
&\quad -0.000020+0.000001+\cdots)\\
&= 9(1-0.006173-0.000019+\cdots)\\
&= 9(1-0.006192+\cdots)\\
&= 9-0.05573+\cdots = 8.94427\cdots
\end{aligned}$$

We have written up to five decimal places at the last step(since $9(\sim 10)$ is multiplied, a significant figure is dropped). Entering $\sqrt{80}$ into a calculator returns the value $8.9442719\cdots$, so we can say that the result is exact up to significant digits wanted(Caveat: Size of error differs even if one has the same number of significant digits. The error of 0.01 in 9.57 and the error of 0.01 in 1.01 differs by a magnitude of 10). One more comment; I heared that when electronic calculators were first built, people competed with the calculator on this problem to assess the computational power of it.

Fortunately, it is rare to deal with such many digits and an error of 5% is decent for a hand calculation. Given a good expanding point, it means first order approximation suffices for most cases. Comparing the digits, one can observe that the difference of $\sqrt{80}$ and 9 is less than 1%. One could have already guessed this from the first subleading term, which is around $1/160 \sim 0.006$.

7 Numerical Approximation using Binomial Expansion: α^2

For another excercise, let's try squaring the fine structure constant $\alpha = 1/137$.

$$\begin{aligned}\alpha^2 &= \left(\frac{1}{137}\right)^2 = \left(\frac{1}{140-3}\right)^2 \\ &= \left(\frac{1}{140}\right)^2 \left(1 - \frac{3}{140}\right)^{-2}\end{aligned}$$

One can rewrite this by utilising $14^2 = 196 = 200 - 4$ and series expansion of $(1+x)^{-2}$

$$\begin{aligned}\alpha^2 &= \frac{1}{20000}\left(1 - \frac{4}{200} + \cdots\right)\left(1 + \frac{3}{70} + \cdots\right) \\ &= 0.00005(1 + 1/50 + 1/20 + \cdots) \\ &= 5 \times (1.07) \times 10^{-5} = 5.35 \times 10^{-5}\end{aligned}$$

When going to the second line, use $4/200 = 1/50$ and $70/3 \sim 20(1{+}3/20)$ and erase off subleading corrections. Since all neglected terms are smaller than $1/100$, one can expect the error to be around 1%. This expectation is met when 5.3279E-5 is returned after giving $1/137^2$ as an input to a calculator.

8 Asymptotic Series

Not all power series converge. This is why one is taught convergence tests[1] and method of obtaining the radius of convergence of a power series. In a number of cases, one encounters power series with zero radius of convergence. Are such power series useless?

Let's avert our eyes from this question and consider the following physics problem: Given a harmonic potential with small anharmonic perturbation $V(x) = x^2/2 + \epsilon x^4/4$, calculate the partition function $Z(\beta) = \sum \exp(-\beta E_i)$.

$$Z(\beta) = \frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdp e^{-\beta(p^2/2+V(x))}$$

The standard approach is to treat ϵ as a small variable and obtain an expression that has the form of a power series in ϵ.

$$Z(\beta, \epsilon) = Z_0(\beta) + \epsilon Z_1(\beta) + \epsilon^2 Z_2(\beta) + \cdots$$

[1] Convergence tests are aimed to check whether the 'tail' of the sum is regulated or not. Finite number of terms with finite values, no matter how large they are, add up to a finite value. This means one only needs to assess the convergence of terms that are included after some finite N, say 1 googol=10^{100}.

This is the standard way to effect the integral.

$$
\begin{aligned}
Z(\beta,\epsilon) &= \frac{1}{h}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dxdpe^{-\beta(p^2/2+V(x))} \\
&= \sqrt{\frac{2\pi}{h\beta}}\int_{-\infty}^{\infty} dxe^{-\beta(x^2/2+\epsilon x^4/4)} \\
&= \sqrt{\frac{2\pi}{h\beta}}\int_{-\infty}^{\infty} dxe^{-\beta\epsilon x^4/4}e^{-\beta x^2/2} \\
&= \sqrt{\frac{2\pi}{h\beta}}\int_{-\infty}^{\infty} dx\left(1-\epsilon\beta\frac{x^4}{4}+\epsilon^2\beta^2\frac{x^8}{32}+\cdots\right)e^{-\beta x^2/2} \\
&= \frac{2\pi}{h\beta}\left(1-\epsilon\frac{3}{4\beta}+\epsilon^2\frac{105}{32\beta^2}+\cdots\right)
\end{aligned}
$$

The anharmonic contributions are negligible at low temperatures or large $\beta = 1/kT$, just as expected. Now, we can ask, is this result reliable? The above partition function as a power series of ϵ is in fact an asymptotic series, which means that the radius of convergence is zero. Why? For $\epsilon < 0$, the energy spectrum of the system is unbounded below, thus the partition function simply does no longer exist![2] In some sense, we are commiting an intellectual fraud; we will be jailed in prison *veritas* and sentenced to read Rudin's *analysis* for the rest of our lives.

Or, are we? One must remember that all physics calculations are *models* of the real world. Reality would be somewhat different from the implicit model of our calculations. In our example of calculating the partition function, the potential was modelled as quadratic and small correction of $\epsilon x^4/4$ was added to capture the deviations from our model. When deviations overwhelm the baseline set by the quadratic part, it simply means that the way we modelled the system is just plain wrong. One should never consider the values of x to approach where this happens. Besides, one never has a box of infinite width; the limits of integration as $-\infty$ and ∞ are mere approximations. What we want is the gist of the system to be captured by the model despite these approximations. This justifies our use of asymptotic series in ϵ expansion; we will only use a few lowest orders.

Asymptotic series can be used to approximate values numerically. Examples include evaluation of the error function, which gives the cumulative probability density of normal probability distributions. Check *Arfken* for further details.

[2]The same argument applies to quantum mechanics as well; many perturbative expansions in quantum mechanics are asymptotic series, which means that one does not expect the perturbative expansion to converge. Freeman Dyson is known to have raised this argument against convergence of higher loop corrections in perturbative quantum field theories.