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SOLUTIONS PAMPHLET

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This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

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1. (Answer: 592)

   Without loss of generality, let the radius of the circle be 2. The radii to the endpoints of the chord, along with the chord, form an isosceles triangle with vertex angle 120°. The area of the larger of the two regions is thus 2/3 that of the circle plus the area of the isosceles triangle, and the area of the smaller of the two regions is thus 1/3 that of the circle minus the area of the isosceles triangle. The requested ratio is therefore $\dfrac{\frac{2}{3}\cdot 4\pi+\sqrt{3}}{\frac{1}{3}\cdot 4\pi-\sqrt{3}}=\dfrac{8\pi+3\sqrt{3}}{4\pi-3\sqrt{3}}$, so $abcdef=8\cdot 3\cdot 3\cdot 4\cdot 3\cdot 3=2592$, and the requested remainder is 592.

2. (Answer: 441)

   In order for Terry and Mary to get the same color combination, they must select all red candies or all blue candies, or they must each select one of each color. The probability of getting all red candies is $\dfrac{\binom{10}{2}\binom{8}{2}}{\binom{20}{2}\binom{18}{2}}=\dfrac{10\cdot 9\cdot 8\cdot 7}{20\cdot 19\cdot 18\cdot 17}$. The probability of getting all blue candies is the same. The probability that they each select one of each color is $\dfrac{10^2\cdot 9^2}{\binom{20}{2}\binom{18}{2}}=\dfrac{10^2\cdot 9^2\cdot 4}{20\cdot 19\cdot 18\cdot 17}$. Thus the probability of getting the same combination is

$$2\cdot\frac{10\cdot 9\cdot 8\cdot 7}{20\cdot 19\cdot 18\cdot 17}+\frac{10^2\cdot 9^2\cdot 4}{20\cdot 19\cdot 18\cdot 17}=\frac{10\cdot 9\cdot 8\cdot(14+45)}{20\cdot 19\cdot 18\cdot 17}=\frac{2\cdot 59}{19\cdot 17}=\frac{118}{323},$$

   and $m+n=441$.

3. (Answer: 384)

   Let the dimensions of the block be p cm by q cm by r cm. The invisible cubes form a rectangular solid whose dimensions are $p-1$, $q-1$, and $r-1$. Thus $(p-1)(q-1)(r-1)=231$. There are only five ways to write 231 as a product of three positive integers:

$$231=3\cdot 7\cdot 11=1\cdot 3\cdot 77=1\cdot 7\cdot 33=1\cdot 11\cdot 21=1\cdot 1\cdot 231$$

   The corresponding blocks are $4\times 8\times 12$, $2\times 4\times 78$, $2\times 8\times 34$, $2\times 12\times 22$, and $2\times 2\times 232$. Their volumes are 384, 624, 544, 528, and 928, respectively. Thus the smallest possible value of N is 384.

4. (Answer: 927)

   There are 99 numbers with the desired property that are less than 100. Three-digit numbers with the property must have decimal representations of the form aaa, aab, aba, or abb, where a and b are digits with $a\geq 1$ and $a\neq b$. There are

9 of the first type and $9 \cdot 9 = 81$ of each of the other three. Four-digit numbers with the property must have decimal representations of the form *aaaa*, *aaab*, *aaba*, *aabb*, *abaa*, *abab*, *abba*, or *abbb*. There are 9 of the first type and 81 of each of the other seven. Thus there are a total of $99+9+3\cdot 81+9+7\cdot 81 = 927$ numbers with the desired property.

OR

Count the number of positive integers less than 10,000 that contain at least 3 distinct digits. There are $9 \cdot 9 \cdot 8$ such 3-digit integers. The number of 4-digit integers that contain exactly 3 distinct digits is $\binom{4}{2} \cdot 9 \cdot 9 \cdot 8$ because there are $\binom{4}{2}$ choices for the positions of the two digits that are the same, 9 choices for the digit that appears in the first place, and 9 and 8 choices for the two other digits, respectively. The number of 4-digit integers that contain exactly 4 distinct digits is $9\cdot 9\cdot 8\cdot 7$. Thus there are $9\cdot 9\cdot 8+6\cdot 9\cdot 9\cdot 8+9\cdot 9\cdot 8\cdot 7 = 14\cdot 9\cdot 9\cdot 8 = 9072$ positive integers less than 10,000 that contain at least 3 distinct digits, and there are $9999 - 9072 = 927$ integers with the desired property.

5. (Answer: 766)

Choose a unit of time so that the job is scheduled to be completed in 4 of these units. The first quarter was completed in 1 time unit. For the second quarter of the work, there were only $9/10$ as many workers as in the first quarter, so it was completed in $10/9$ units. For the third quarter, there were only $8/10$ as many workers as in the first quarter, so it was completed in $5/4$ units. This leaves $4-(1+10/9+5/4) = 23/36$ units to complete the final quarter. To finish the job on schedule, the number of workers that are needed is at least $36/23$ of the number of workers needed in the first quarter, or $(36/23)1000$ which is between 1565 and 1566. There are 800 workers at the end of the third quarter, so a minimum of $1566 - 800 = 766$ additional workers must be hired.

6. (Answer: 408)

Let the first monkey take $8x$ bananas from the pile, keeping $6x$ and giving x to each of the others. Let the second monkey take $8y$ bananas from the pile, keeping $2y$ and giving $3y$ to each of the others. Let the third monkey take $24z$ bananas from the pile, keeping $2z$ and giving $11z$ to each of the others. The total number of bananas is $8x + 8y + 24z$. The given ratios imply that $6x + 3y + 11z = 3(x + 3y + 2z)$ and $x + 2y + 11z = 2(x + 3y + 2z)$. Simplify these equations to obtain $3x + 5z = 6y$ and $7z = x + 4y$. Eliminate x to obtain $9y = 13z$. Then $y = 13n$ and $z = 9n$, where n is a positive integer. Substitute to find that $x = 11n$. Thus, the least possible values for x, y and z are 11, 13 and 9, respectively, and the least possible total is $8 \cdot 11 + 8 \cdot 13 + 24 \cdot 9 = 408$.

7. (Answer: 293)

Let $\overline{B'C'}$ and $\overline{CD}$ intersect at H. Note that $B'E = BE = 17$. Apply the Pythagorean Theorem to $\triangle EAB'$ to obtain $AB' = 15$. Because $\angle C'$ and $\angle C'B'E$ are right angles, $\triangle B'AE \sim \triangle HDB' \sim \triangle HC'F$, so the lengths of the sides of each triangle are in the ratio $8 : 15 : 17$. Now $C'F = CF = 3$ implies that $FH = (17/8)3 = 51/8$ and $DH = 25 - (3 + 51/8) = 125/8$. Then $B'D = (8/15)(125/8) = 25/3$. Thus $AD = 70/3$, and the perimeter of $ABCD$ is

$$2 \cdot 25 + 2 \cdot \frac{70}{3} = \frac{290}{3},$$

so $m + n = 290 + 3 = 293$.

OR

Notice first that $B'E = BE = 17$. Apply the Pythagorean Theorem to $\triangle EAB'$ to obtain $AB' = 15$. Draw $\overline{FG}$ parallel to $\overline{CB}$, with G on $\overline{AB}$. Notice that $GE = 17 - 3 = 14$. Because points on the crease $\overline{EF}$ are equidistant from B and B', it follows that $\overline{EF}$ is perpendicular to $\overline{BB'}$, and hence that triangles EGF and $B'AB$ are similar. In particular, $\dfrac{FG}{BA} = \dfrac{GE}{AB'}$. This yields $FG = 70/3$, and the perimeter of $ABCD$ is therefore $290/3$.

8. (Answer: 054)

A positive integer N is a divisor of 2004^{2004} if and only if $N = 2^i 3^j 167^k$ with $0 \le i \le 4008$, $0 \le j \le 2004$, and $0 \le k \le 2004$. Such a number has exactly 2004 positive integer divisors if and only if $(i + 1)(j + 1)(k + 1) = 2004$. Thus the number of values of N meeting the required conditions is equal to the number of ordered triples of positive integers whose product is 2004. Each of the unordered triples $\{1002, 2, 1\}$, $\{668, 3, 1\}$, $\{501, 4, 1\}$, $\{334, 6, 1\}$, $\{334, 3, 2\}$, $\{167, 12, 1\}$, $\{167, 6, 2\}$, and $\{167, 4, 3\}$ can be ordered in 6 possible ways, and the triples $\{2004, 1, 1\}$ and $\{501, 2, 2\}$ can each be ordered in 3 possible ways, so the total is $8 \cdot 6 + 2 \cdot 3 = 54$.

OR

Begin as above. Then, to find the number of ordered triples of positive integers whose product is 2004, represent the triples as $(2^{a_1} \cdot 3^{b_1} \cdot 167^{c_1}, 2^{a_2} \cdot 3^{b_2} \cdot 167^{c_2}, 2^{a_3} \cdot 3^{b_3} \cdot 167^{c_3})$, where $a_1 + a_2 + a_3 = 2$, $b_1 + b_2 + b_3 = 1$, and $c_1 + c_2 + c_3 = 1$, and the a_i's, b_i's, and c_i's are nonnegative integers. The number of solutions of $a_1 + a_2 + a_3 = 2$ is $\binom{4}{2}$ because each solution corresponds to an arrangement of two objects and two dividers. Similarly, the number of solutions of both $b_1 + b_2 + b_3 = 1$ and $c_1 + c_2 + c_3 = 1$ is $\binom{3}{1}$, so the total number of triples is $\binom{4}{2}\binom{3}{1}\binom{3}{1} = 6 \cdot 3 \cdot 3 = 54$.

9. (Answer: 973)

The terms in the sequence are $1, r, r^2, r(2r-1), (2r-1)^2, (2r-1)(3r-2), (3r-2)^2, \ldots$. Assuming that the pattern continues, the ninth term is $(4r-3)^2$ and the tenth term is $(4r-3)(5r-4)$. Thus $(4r-3)^2+(4r-3)(5r-4) = 646$. This leads to $(36r+125)(r-5) = 0$. Because the terms are positive, $r = 5$. Substitute to find that $a_n = (2n-1)^2$ when n is odd, and that $a_n = (2n-3)(2n+1)$ when n is even. The least odd-numbered term greater than 1000 is therefore $a_{17} = 33^2 = 1089$, and $a_{16} = 29 \cdot 33 = 957 < 1000$. The desired value of $n + a_n$ is $957 + 16 = 973$.

The pattern referred to above is

$$\begin{aligned} a_{2n} &= [(n-1)r - (n-2)][nr - (n-1)], \\ a_{2n+1} &= [nr - (n-1)]^2. \end{aligned}$$

This pattern has been verified for the first few positive integral values of n. The above equations imply that

$$\begin{aligned} a_{2n+2} &= 2[nr-(n-1)]^2 - [(n-1)r-(n-2)][nr-(n-1)] \\ &= [nr-(n-1)][2nr-2(n-1)-(n-1)r+(n-2)] \\ &= [nr-(n-1)][(n+1)r-n], \quad \text{and} \\ a_{2n+3} &= \frac{[nr-(n-1)]^2[(n+1)r-n]^2}{[nr-(n-1)]^2} \\ &= [(n+1)r-n]^2. \end{aligned}$$

The above argument, along with the fact that the pattern holds for $n = 1$ and $n = 2$, implies that it holds for all positive integers n.

10. (Answer: 913)

An element of $\mathcal{S}$ has the form 2^a+2^b, where $0 \le a \le 39$, $0 \le b \le 39$, and $a \ne b$, so $\mathcal{S}$ has $\binom{40}{2} = 780$ elements. Without loss of generality, assume $a < b$. Note that 9 divides $2^a + 2^b = 2^a(2^{b-a}+1)$ if and only if 9 divides $2^{b-a}+1$, that is, when $2^{b-a} \equiv 8 \pmod 9$. Because $2^1, 2^2, 2^3, 2^4, 2^5, 2^6,$ and $2^7 \equiv 2, 4, 8, 7, 5, 1,$ and $2 \pmod 9$, respectively, conclude that $2^{b-a} \equiv 8 \pmod 9$ when $b - a = 6k - 3$ for positive integers k. But $b - a = 6k - 3$ implies that $0 \le a \le 39 - (6k - 3)$, so there are $40 - (6k-3) = 43 - 6k$ ordered pairs (a, b) that satisfy $b - a = 6k - 3$. Because $6k - 3 \le 39$, conclude that $1 \le k \le 7$. The number of multiples of 9 in $\mathcal{S}$ is therefore $\sum_{k=1}^{7}(43-6k) = 7 \cdot 43 - 6 \cdot 7 \cdot 8/2 = 7(43-24) = 133$. Thus the probability that an element of $\mathcal{S}$ is divisible by 9 is $133/780$, so $p + q = 913$.

OR

There are $\binom{40}{2} = 780$ such numbers, which can be viewed as strings of length 40 containing 38 0's and two 1's. Express these strings in base-8 by partitioning them into 13 groups of 3 starting from the right and one group of 1, and then expressing each 3-digit binary group as a digit between 0 and 7. Because $9 = 11_8$, a divisibility test similar to the one for 11 in base 10 can be used. Let $(a_d a_{d-1} \ldots a_0)_b$ represent a $(d+1)$-digit number in base b. Then

$$\begin{aligned}(a_d a_{d-1} \ldots a_0)_b &= \sum_{k=0}^{d} a_k b^k \\ &\equiv \sum_{k=0}^{d} a_k (-1)^k \pmod{b+1}.\end{aligned}$$

Thus a base-8 number is divisible by 9 if and only if the sum of the digits in the even-numbered positions differs from the sum of the digits in the odd-numbered positions by a multiple of 9. Because the given base-8 string has at most two nonzero digits, and its greatest digit is at most 110_2 or 6, the two sums must differ by 0. There are two cases. If the first (leftmost) digit in the base-2 string is 1, then the other 1 must be in an odd-numbered group of 3 that has the form 001. There are seven such numbers. In the second case, if the first digit in the base-2 string is 0, one of the 1's must be in an even-numbered group of 3, the other must be in an odd-numbered group, and they must be in the same relative position within each group, that is, both 100, both 010, or both 001. There are $7 \cdot 6 \cdot 3 = 126$ such numbers. Thus the required probability is $(7 + 126)/780 = 133/780$.

11. (Answer: 625)

Use the Pythagorean Theorem to conclude that the distance from the vertex of the cone to a point on the circumference of the base is

$$\sqrt{\left(200\sqrt{7}\right)^2 + 600^2} = 200\sqrt{\left(\sqrt{7}\right)^2 + 3^2} = 800.$$

Cut the cone along the line through the vertex of the cone and the starting point of the fly, and flatten the resulting figure into a sector of a circle with radius 800. Because the circumference of this circle is 1600π and the length of the sector's arc is 1200π, the measure of the sector's central angle is $270°$. The angle determined by the radius of the sector on which the fly starts and the radius on which the fly stops is $135°$. Use the Law of Cosines to conclude that the least distance the fly could have crawled between the start and end positions is

$$\begin{aligned}&\sqrt{\left(375\sqrt{2}\right)^2 + 125^2 - 2 \cdot 375\sqrt{2} \cdot 125 \cdot \cos 135°} \\ &\qquad = 125\sqrt{\left(3\sqrt{2}\right)^2 + 1 + 2 \cdot 3\sqrt{2} \cdot \left(\sqrt{2}/2\right)}\end{aligned}$$

which can be simplified to $125\sqrt{25} = 625$.

12. (Answer: 134)

Let E be the midpoint of $\overline{AB}$, F be the midpoint of $\overline{CD}$, x be the radius of the inner circle, and G be the center of that circle. Then $\overline{GE}\perp\overline{AB}$. Because the point of tangency of the circles centered at A and G is on $\overline{AG}$, $AG = x + 3$. Use the Pythagorean Theorem in $\triangle AEG$ to obtain $GE = \sqrt{x^2 + 6x}$. Similarly, find that $FG = \sqrt{x^2 + 4x}$. Because the height of the trapezoid is $\sqrt{24}$, conclude that

$$\begin{aligned} \sqrt{x^2+6x} + \sqrt{x^2+4x} &= \sqrt{24}, \quad \text{so} \\ \sqrt{x^2+6x} &= \sqrt{24} - \sqrt{x^2+4x}, \\ x^2 + 6x &= 24 + x^2 + 4x - 2\sqrt{24}\sqrt{x^2+4x}, \quad \text{and} \\ \sqrt{24(x^2+4x)} &= 12 - x. \end{aligned}$$

This yields $23x^2 + 120x - 144 = 0$, whose positive root is $x = \dfrac{-60 + 48\sqrt{3}}{23}$.

Thus $k + m + n + p = 60 + 48 + 3 + 23 = 134$.

13. (Answer: 484)

Let $\overline{AD}$ intersect $\overline{CE}$ at F. Extend $\overline{BA}$ through A to R so that $\overline{BR} \cong \overline{CE}$, and extend $\overline{BC}$ through C to P so that $\overline{BP} \cong \overline{AD}$. Then create parallelogram $PBRQ$ by drawing lines through D and E parallel to $\overline{AB}$ and $\overline{BC}$, respectively, with Q the intersection of the two lines. Apply the Law of Cosines to triangle ABC to obtain $AC = 7$. Now

$$\frac{RA}{AB} = \frac{EF}{FC} = \frac{DE}{AC} = \frac{15}{7}, \quad \text{so} \quad \frac{RB}{AB} = \frac{22}{7}.$$

Similarly, $\dfrac{PB}{CB} = \dfrac{22}{7}$. Thus parallelogram $ABCF$ is similar to parallelogram $RBPQ$. Let $K = [ABCF]$. Then $[RBPQ] = (22/7)^2 K$. Also,

$$\begin{aligned} [EBD] &= [RBPQ] - \frac{1}{2}\left([BCER] + [ABPD] + [DFEQ]\right) \\ &= [RBPQ] - \frac{1}{2}\left([RBPQ] + [ABCF]\right) \\ &= \frac{1}{2}\left([RBPQ] - [ABCF]\right). \end{aligned}$$

Thus

$$\frac{[ABC]}{[EBD]} = \frac{\frac{1}{2}K}{\frac{1}{2}\left[(\frac{22}{7})^2 K - K\right]} = \frac{1}{\frac{484}{49} - 1} = \frac{49}{435},$$

and $m+n=484$.

OR

Apply the Law of Cosines to triangle ABC to obtain $AC=7$. Let $\overline{AD}$ intersect $\overline{CE}$ at F. Then $ABCF$ is a parallelogram, which implies that $[ABC]=[BCF]=[CFA]=[FAB]$. Let $ED/AC=r=15/7$. Since $\overline{AC}\parallel\overline{DE}$, conclude $EF/FC=FD/AF=r$. Hence

$$\begin{aligned}\frac{[EBD]}{[ABC]}&=\frac{[BFE]}{[ABC]}+\frac{[EFD]}{[ABC]}+\frac{[DFB]}{[ABC]}=\frac{[BFE]}{[BCF]}+\frac{[EFD]}{[CFA]}+\frac{[DFB]}{[FAB]}\\&=r+r^2+r=r^2+2r=435/49,\end{aligned}$$

implying that $m/n=49/435$ and $m+n=484$.

OR

Apply the Law of Cosines to triangle ABC to obtain $AC=7$. Let $\overline{AD}$ and $\overline{CE}$ intersect at F. Then $ABCF$ is a parallelogram, which implies that $\triangle ABC\cong\triangle CFA$. Note that triangles AFC and DFE are similar. Let the altitudes from B and F to $\overline{AC}$ each have length h. Then the length of the altitude from F to $\overline{ED}$ is $15h/7$. Thus

$$\frac{[ABC]}{[EBD]}=\frac{\frac{1}{2}\cdot 7h}{\frac{1}{2}\cdot 15(h+h+\frac{15}{7}h)}=\frac{7\cdot 7}{15\cdot 29}=\frac{49}{435},$$

so $m+n=484$.

14. (Answer: 108)

To simplify, replace all the 7's with 1's, that is, divide all the numbers in the sum by 7. The desired values of n are the same as the values of n for which + signs can be inserted in a string of n 1's to obtain a sum of 1000. The result will be a sum of x 1's, y 11's, and z 111's, where x, y, and z are nonnegative integers, $x+11y+111z=1000$, and $x+2y+3z=n$. Subtract to find that $9y+108z=1000-n$, so $n=1000-9(y+12z)$. There cannot be more than 1000 1's in a string whose sum is 1000, and the least number of 1's occurs when the string consists of nine 111's and one 1. Therefore $28\le n\le 1000$, and so $0\le y+12z\le 108$. Thus the number of values of n is the number of possible integer values of $y+12z$ between 0 and 108, inclusive, subject to the condition that

$$11y+111z\le 1000. \qquad (1)$$

Note that (1) is equivalent to $11y\le 990-110z+10-z$, and therefore to $y\le 90-10z+(10-z)/11$. Then use the fact that $0\le z\le 9$ to conclude that

(1) is equivalent to $y \le 90 - 10z$, and therefore to $y + 12z \le 90 + 2z$. The fact that y is nonnegative means that $y + 12z \ge 12z$. Thus an ordered pair (y, z) of integers satisfies

$$12z \le y + 12z \le 90 + 2z \qquad (2)$$

if and only if it satisfies (1) and $y \ge 0$. Hence when $z = 0$, $y + 12z$ can have any integer value between 0 and 90, inclusive; when $z = 7$, $y + 12z$ can have any integer value between 84 and 104, inclusive; and when $z = 8$, $y + 12z$ can have any integer value between 96 and 106, inclusive. But $y + 12z > 106$ only if $90 + 2z > 106$, that is, when $z = 9$; and when $z = 9$, (2) implies that $y + 12z = 108$. Thus for nonnegative integers y and z, $y + 12z$ can have any integer value between 0 and 108 except for 107, so there are 108 possible values for n.

OR

To simplify, replace all the 7's with 1's, that is, divide all the numbers in the sum by 7. The desired values of n are the same as the values of n for which + signs can be inserted in a string of n 1's to obtain a sum of 1000. Because the sum is congruent to n modulo 9 and $1000 \equiv 1 \pmod 9$, it follows that $n \equiv 1 \pmod 9$. Also, $n \le 1000$. There are $\lfloor 1000/9 \rfloor + 1 = 112$ positive integers that satisfy both conditions, namely, 1, 10, 19, 28, 37, 46, ... , 1000. For $n = 1$, 10, or 19, the greatest sum that is less than or equal to 1000 is $6 \cdot 111 + 1 = 677$. Thus $n \ge 28$, so there are at most $112 - 3 = 109$ possible values of n, and these values are contained in $\mathcal{S} = \{28, 37, 46, \ldots, 1000\}$. It will be shown that all elements of $\mathcal{S}$ except 37 are possible.

First note that 28 is possible because $9 \cdot 111 + 1 \cdot 1 = 1000$, while 37 is not possible because when $n = 37$, the greatest sum that is at most 1000 is $8 \cdot 111 + 6 \cdot 11 + 1 \cdot 1 = 955$. All other elements of $\mathcal{S}$ are possible because if any element n of $\mathcal{S}$ between 46 and 991 is possible, then $(n + 9)$ must be too. To see this, consider two cases, the case where the sum has no 11's and the case where the sum has at least one 11.

If the sum has no 11's, it must have at least one 1. If it has exactly one 1, there must be nine 111's and $n = 28$. Thus, for $n \ge 46$, the sum has more than one 1, so it must have at least $1000 - 8 \cdot 111 = 112$ 1's, and, for $n < 1000$, at least one 111. To show that if n is possible, then $(n + 9)$ is possible, replace a 111 with $1 + 1 + 1$, replace eleven $(1 + 1)$'s with eleven 11's, and include nine new 1's as +1's. The sum remains 1000.

If the sum has at least one 11, replace an 11 with $1 + 1$, and include nine new 1's as +1's.

Now note that 46 is possible because $8 \cdot 111 + 10 \cdot 11 + 2 \cdot 1 = 1000$, and so all elements of $\mathcal{S}$ except 37 are possible. Thus there are 108 possible values for n.

15. (Answer: 593)

Number the squares from left to right, starting with 0 for the leftmost square, and ending with 1023 for the rightmost square. The 942nd square is thus initially numbered 941. Represent the position of a square after f folds as an ordered triple (p, h, f), where p is the position of the square starting from the left, starting with 0 as the leftmost position, h is the number of paper levels below the square, and f is the number of folds that have been made. For example, the ordered triple that initially describes square number 941 is $(941, 0, 0)$. The first 0 indicates that at the start there are no squares under this one, and the second 0 indicates that no folds have been made.

Note that the function F, defined below, describes the position of a square after $(f + 1)$ folds:

$$F(p, h, f) = \begin{cases} (p, h, f+1) & \text{for } 0 \le p \le 2^{10-f-1} - 1 \\ (2^{10-f} - 1 - p, 2^f + (2^f - 1 - h), f+1) & \\ & \text{for } 2^{10-f-1} \le p \le 2^{10-f} - 1 \end{cases}$$

The top line in the definition indicates that squares on the left half of the strip do not change their position or height as a result of a fold. The second line indicates that, as a result of a fold, the position of a square on the right half of the strip is reflected about the center line of the strip, and that the stack of squares in that position is inverted and placed on the top of the stack that was already in that position's reflection.

Because of the powers of 2 in the definition of F, evaluating F can be made easier if the position and height are expressed in base two. In particular, after f folds, the strip has length 2^{10-f}, so the positions 0 through $2^{10-f} - 1$ are represented by all possible binary strings of $10 - f$ digits. In this representation, $0 \le p \le 2^{10-f-1} - 1$ if and only if the leading digit is 0, and $2^{10-f-1} \le p \le 2^{10-f} - 1$ if and only if the leading digit is 1. In the former case, the new position, now represented by the string of length $10-f-1$, is obtained by deleting the leading 0. In the latter case, the new position $2^{10-f} - 1 - p$ is obtained by truncating the leading 1 and for the remaining digits, changing each 0 to a 1 and each 1 to a 0. Likewise, in this latter case, the new height is $2^f + (2^f - 1 - h)$. When $f \ge 1$, the new height is obtained in the case $2^{10-f-1} \le p \le 2^{10-f} - 1$ by taking the f-digit binary string representing the height, changing each 1 to a 0 and each 0 to a 1, and then appending a 1 on the left. In the case $0 \le p \le 2^{10-f-1} - 1$, the new $(f + 1)$-digit string representing the new height is obtained by appending a 0 to the left of the string.

With these conditions, square number 941 is initially described by $(1110101101, 0, 0)$. In the display below, an arrow is used to denote an application of F. For the first fold

$$(1110101101, 0, 0) \to (001010010, 1, 1),$$

indicating that after the first fold, square 941 is in position $001010010_2 = 82$, and there is one layer under this square. Continue to obtain $(001010010, 1, 1) \to (01010010, 01, 2) \to (1010010, 001, 3) \to (101101, 1110, 4) \to (10010, 10001, 5) \to (1101, 101110, 6) \to (010, 1010001, 7) \to (10, 01010001, 8) \to (1, 110101110, 9) \to (0, 1001010001, 10)$. After 10 folds, the number of layers under square 941 is $1001010001_2 = 593$.

OR

If a square is to the left of the center after n folds, its positions counting from the left and the bottom do not change after $(n+1)$ folds. Otherwise, its positions counting from the right and bottom after n folds become its positions counting from the left and top after $(n+1)$ folds. Also, after n folds the sum of the positions of each square counting from the left and right is $2^{10-n}+1$, and the sum of the positions counting from the bottom and top is 2^n+1. The position of the 942nd square can be described in the table below.

Folds	Position Counting From Left	Position Counting From Right	Position Counting From Bottom	Position Counting From Top
0	942	83	1	1
1	83	430	2	1
2	83	174	2	3
3	83	46	2	7
4	46	19	15	2
5	19	14	18	15
6	14	3	47	18
7	3	6	82	47
8	3	2	82	175
9	2	1	431	82
10	1	1	594	431

Thus there are 593 squares below the final position of the 942nd square.

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