THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

23rd Annual *(Alternate)*

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION

(AIME)

Tuesday, March 22, 2005

1. DO NOT OPEN THIS BOOKLET UNTIL YOUR PROCTOR GIVES THE SIGNAL TO BEGIN.

2. This is a 15-question, 3-hour examination. All answers are integers ranging from 000 to 999, inclusive. Your score will be the number of correct answers; i.e., there is neither partial credit nor a penalty for wrong answers.

3. No aids other than scratch paper, graph paper, ruler, compass, and protractor are permitted. In particular, **calculators and computers are not permitted.**

4. A combination of the AIME and the American Mathematics Contest 10 or the American Mathematics Contest 12 scores are used to determine eligibility for participation in the U.S.A. Mathematical Olympiad (USAMO). The USAMO will be given on TUESDAY and WEDNESDAY, April 19 & 20, 2005.

5. Record all of your answers, and certain other information, on the AIME answer form. Only the answer form will be collected from you.

The publication, reproduction or communication of the problems or solutions of the AIME during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Duplication at any time via copier, telephone, email, world wide web, or media of any type is a violation of the competition rules.

1. A game uses a deck of n different cards, where n is an integer and $n \geq 6$. The number of possible sets of 6 cards that can be drawn from the deck is 6 times the number of possible sets of 3 cards that can be drawn. Find n.

2. A hotel packed a breakfast for each of three guests. Each breakfast should have consisted of three types of rolls, one each of nut, cheese, and fruit rolls. The preparer wrapped each of the nine rolls, and, once they were wrapped, the rolls were indistinguishable from one another. She then randomly put three rolls in a bag for each of the guests. Given that the probability that each guest got one roll of each type is m/n, where m and n are relatively prime positive integers, find $m + n$.

3. An infinite geometric series has sum 2005. A new series, obtained by squaring each term of the original series, has sum 10 times the sum of the original series. The common ratio of the original series is m/n, where m and n are relatively prime positive integers. Find $m + n$.

4. Find the number of positive integers that are divisors of at least one of $10^{10}, 15^7, 18^{11}$.

5. Determine the number of ordered pairs (a, b) of integers such that $\log_a b + 6\log_b a = 5$, $2 \leq a \leq 2005$, and $2 \leq b \leq 2005$.

6. The cards in a stack of $2n$ cards are numbered consecutively from 1 through $2n$ from top to bottom. The top n cards are removed, kept in order, and form pile A. The remaining cards form pile B. The cards are now restacked into a single stack by taking cards alternately from the tops of pile B and pile A, respectively. In this process, card number $(n + 1)$ is the bottom card of the new stack, card number 1 is on top of this card, and so on, until piles A and B are exhausted. If, after the restacking process, at least one card from each pile occupies the same position that it occupied in the original stack, the stack is called *magical.* For example, eight cards form a magical stack because cards number 3 and number 6 retain their original positions. Find the number of cards in the magical stack in which card number 131 retains its original position.

7. Let
$$x = \frac{4}{(\sqrt{5}+1)(\sqrt[4]{5}+1)(\sqrt[8]{5}+1)(\sqrt[16]{5}+1)}.$$
Find $(x + 1)^{48}$.

8. Circles $\mathcal{C}_1$ and $\mathcal{C}_2$ are externally tangent, and they are both internally tangent to circle $\mathcal{C}_3$. The radii of $\mathcal{C}_1$ and $\mathcal{C}_2$ are 4 and 10, respectively, and the centers of the three circles are collinear. A chord of $\mathcal{C}_3$ is also a common external tangent of $\mathcal{C}_1$ and $\mathcal{C}_2$. Given that the length of the chord is $m\sqrt{n}/p$, where m, n, and p are positive integers, m and p are relatively prime, and n is not divisible by the square of any prime, find $m+n+p$.

9. For how many positive integers n less than or equal to 1000 is

$$(\sin t + i\cos t)^n = \sin nt + i\cos nt$$

true for all real t?

10. Given that $\mathcal{O}$ is a regular octahedron, that $\mathcal{C}$ is the cube whose vertices are the centers of the faces of $\mathcal{O}$, and that the ratio of the volume of $\mathcal{O}$ to that of $\mathcal{C}$ is m/n, where m and n are relatively prime positive integers, find $m+n$.

11. Let m be a positive integer, and let $a_0, a_1, \ldots, a_m$ be a sequence of real numbers such that $a_0 = 37$, $a_1 = 72$, $a_m = 0$, and

$$a_{k+1} = a_{k-1} - \frac{3}{a_k}$$

for $k = 1, 2, \ldots, m-1$. Find m.

12. Square $ABCD$ has center O, $AB = 900$, E and F are on $\overline{AB}$ with $AE < BF$ and E between A and F, $m\angle EOF = 45^\circ$, and $EF = 400$. Given that $BF = p + q\sqrt{r}$, where p, q, and r are positive integers and r is not divisible by the square of any prime, find $p+q+r$.

13. Let $P(x)$ be a polynomial with integer coefficients that satisfies $P(17) = 10$ and $P(24) = 17$. Given that the equation $P(n) = n+3$ has two distinct integer solutions n_1 and n_2, find the product $n_1 \cdot n_2$.

14. In $\triangle ABC$, $AB = 13$, $BC = 15$, and $CA = 14$. Point D is on $\overline{BC}$ with $CD = 6$. Point E is on $\overline{BC}$ such that $\angle BAE \cong \angle CAD$. Given that $BE = p/q$, where p and q are relatively prime positive integers, find q.

15. Let ω_1 and ω_2 denote the circles $x^2 + y^2 + 10x - 24y - 87 = 0$ and $x^2 + y^2 - 10x - 24y + 153 = 0$, respectively. Let m be the smallest positive value of a for which the line $y = ax$ contains the center of a circle that is internally tangent to ω_1 and externally tangent to ω_2. Given that $m^2 = p/q$, where p and q are relatively prime positive integers, find $p+q$.

Your Exam Manager will receive a copy of the 2005 AIME Solution Pamphlet with the scores.

CONTACT US -- *Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:*

American Mathematics Competitions
University of Nebraska, P.O. Box 81606
Lincoln, NE 68501-1606
Phone: 402-472-2257; Fax: 402-472-6087; email: amcinfo@unl.edu

The problems and solutions for this AIME were prepared by the MAA's Committee on the AIME under the direction of:

David Hankin, AIME Chair
207 Corbin Place, Brooklyn, NY 11235 USA

2005 USAMO -- THE USA MATHEMATICAL OLYMPIAD (USAMO) is a 6-question, 9-hour, essay-type examination. The USAMO will be held in your school on Tuesday, April 19 & Wednesday, April 20. Your teacher has more details on who qualifies for the USAMO in the AMC 10/12 and AIME Teachers' Manuals. The best way to prepare for the USAMO is to study previous years of these exams, the World Olympiad Problems/Solutions and review the contents of the Arbelos. Copies may be ordered from the web sites indicated below.

PUBLICATIONS -- For a complete listing of available publications please visit the following web sites:

AMC - - http://www.unl.edu/amc/d-publication/publication.html
MAA -- https://enterprise.maa.org/ecomtpro/timssnet/common/tnt_frontpage.cfm