Fibonacci Overview
Callie Wurtz

1 Motivation

Consider the following: Suppose a newly-born pair of rabbits: one male, one female, are put in a field. These rabbits are able to mate at the age of one month and they give birth to a male-female pair in the following month. In other words, two months after a pair of rabbits is born, another pair is produced. Assuming no rabbits die, how many pairs will there be in one year?

This problem was first posed by Leonardo of Pisa in his work, Liber Abaci ("The Book of the Abacus"), which was published in 1202. Leonardo Pisano is better known as Fibonacci, but this nickname didn't appear until the 19th Century. The name Fibonacci is a shortening of filius Bonacci, which means “son of Bonaccio” (like the common English surname John-son). Though he never referred to himself as Fibonacci, it is this name that is attached to the famous number sequence that answers the rabbit problem [Kno05]. Fibonacci numbers are widely used in many different fields of mathematics. Countless books, websites, and even a journal, Fibonacci Quarterly, are devoted entirely to the study of Fibonacci numbers [Uni05]. Not surprisingly, this famous sequence of numbers is the foundation for multiple identities. While these identities are often proved using mathematical induction, Fibonacci numbers have an elegant combinatorial interpretation that allows for counting proofs.

2 Preliminary Ideas

2.1 Common Definitions

For each of the Fibonacci identities, it is assumed that you are familiar with the definitions in the “Common Definitions” file. Multiple identities implicitly use the Rule of Sum and Rule of Product counting principles as well as the terms mutually exclusive and independent. Summation notation is used in Fibonacci Identities 1, 2, and 4. Familiarity with binomial coefficients and the floor function is also necessary for “Fibonacci Identity 1.”

2.2 Fibonacci Numbers Defined

Definition 1 Fibonacci Numbers

The Fibonacci numbers are defined recursively by \( f_0 = 1, f_1 = 1 \), and for \( n \geq 2 \), \( f_n = f_{n-1} + f_{n-2} \). The initial numbers of the Fibonacci sequence are 1, 1, 2, 3, 5, 8, 13, 21, . . . .

The Fibonacci sequence is one famous example of a recurrence relation. In general, a recurrence relation is a formula that expresses a sequence of numbers, where each number in the sequence is defined in terms of one or more previous numbers in the sequence. In the case of the Fibonacci sequence, finding the term \( f_n \) is dependent on knowing both \( f_{n-1} \) and \( f_{n-2} \). As you may have already noticed, it is necessary to have a starting point. Recurrence relations are defined with at least one base case in order to uniquely specify the sequence. In the Fibonacci sequence the base cases are \( f_0 = 1 \) and \( f_1 = 1 \). Consequently, the relation \( f_n = f_{n-1} + f_{n-2} \) with the base conditions given in the previous sentence uniquely defines \( f_n \) for \( n \geq 2 \).

Return for a moment to the rabbit problem. This problem allowed Fibonacci to investigate and simplify a complex sequence. Table 1 depicts the rabbit population each month for a year [Gos03, p. 333].
<table>
<thead>
<tr>
<th>Month</th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>Baby Pairs</td>
<td>1</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
</tr>
<tr>
<td>Mature Pairs</td>
<td>0</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
</tr>
<tr>
<td>Total Pairs</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>3</td>
<td>5</td>
<td>8</td>
<td>13</td>
<td>21</td>
<td>34</td>
<td>55</td>
<td>89</td>
<td>144</td>
<td>233</td>
</tr>
</tbody>
</table>

Table 1: Fibonacci's Rabbit Population

Though the table does not look complicated, it is beneficial to consider the relationships present. Notice that after the first month, every mature pair is creating a baby pair, so the number of mature pairs in one month becomes the number of baby pairs in the next. Also notice that there is a similar correspondence between the second and third rows. After the first month, the total number of pairs in a month becomes the number of mature pairs in the next month.

We conclude, as Fibonacci did, that after one year, the field contains $f_{12} = 233$ rabbits.

### 2.3 A Visual Representation

For combinatorial purposes, a more visual representation of $f_n$ is needed. This interpretation has been taken from *Proofs that Really Count* by Dr. Arthur T. Benjamin and Dr. Jennifer J. Quinn [BQ03].

Picture a checkerboard of dimension $1 \times n$. This board is composed of unit cells numbered 1 through $n$ and said to have length $n$. A board of length $n$ is also referred to as an $n$-board. We will use squares (covering 1 cell) and dominoes (covering 2 cells) to tile the board.

![Figure 1: An arbitrary tiling of a 6-board.](image)

**Theorem 1** $t_n$ is a Recurrence Relation

Let $t_n$ denote the number of distinct tilings of a board of length $n$ using squares and dominoes. Then $t_n$ is a recurrence relation.

**Proof:**

Consider first a board of length 0. Out of convention, we say that there is one way to tile a board with no cells: it is to do nothing. In other words, $t_0 = 1$. Now consider a 1-board. Clearly, the only way to tile this board is with one square. So, $t_1 = 1$. We move on to a 2-board. This board can be tiled with 2 squares or 1 domino, giving a total of 2 tilings.

![The base cases: $t_1 = 1$ and $t_2 = 2$](image)

Suppose we have a board of length $n$ for $n > 2$. Note that any tiling of a board of length $n$ must begin with either a square or a domino. If it begins with a square, we are left with a board of length $n - 1$, which can be tiled in $t_{n-1}$ ways by definition. If the $n$-board begins with a domino, the rest of the board can be tiled in $t_{n-2}$ ways. Since the board begins with either a square or a domino, these two options are mutually exclusive. Thus, $t_n = t_{n-1} + t_{n-2}$. 

$\square$
**Corollary 1** \( t_n \) is identical to \( f_n \)

The number of ways to tile a board of length \( n \) with squares and dominoes is \( f_n \).

**Proof:**
As can be seen in the proof of Theorem 1, the base cases for \( t_n \) are \( t_0 = 1 \) and \( t_1 = 1 \). (It was also explicitly shown that \( t_2 = 2 \), though only two base cases are really necessary.) The recurrence relation was given as \( t_n = t_{n-1} + t_{n-2} \). One can readily see that this relation is the same as that of the Fibonacci sequence. Since the Fibonacci sequence is defined to have the same base cases and the same recurrence relation as \( t_n \), we can conclude that the sequence given by \( t_n \) is identical to the sequence given by \( f_n \).

\( \square \)

Based on the above corollary, we can replace the \( t_n \) notation with standard Fibonacci notation. All of the Fibonacci identities, written in terms of \( f_n \), will be proved combinatorially using this visual interpretation.

The following example further illustrates the idea of our visual interpretation.

**Example 1**
Consider a board of length 4. There are \( f_4 = 5 \) ways to tile this board. They are:

![Five tilings of a 4-board using squares and dominoes.](image)

Figure 2: The five tilings of a 4-board using squares and dominoes.

Before using this combinatorial representation for Fibonacci numbers, it is necessary to introduce a few more definitions that relate to tiling boards.

**Definition 2** *Breakable*
A tiling of an \( n \)-board is *breakable* at cell \( k \) if the tiling can be decomposed into two tilings: one covering cells 1 through \( k \) and the other covering cells \( k + 1 \) through \( n \).

This definition can be clearly seen in the following figure.

![10-board tiling using 3 dominoes and 4 squares.](image)

Figure 3: A 10-board tiling using 3 dominoes and 4 squares.

This board of length 10 is breakable at 2, 3, 5, 7, 8, 9, and 10. Notice that a tiling over a board of length \( n \) is always breakable at \( n \).

The definition of unbreakable is as expected.

**Definition 3** *Unbreakable*
A tiling is *unbreakable* at cell \( k \) if a domino occupies cells \( k \) and \( k + 1 \).

The tiling in Figure 3 is unbreakable at cells 1, 4, and 6.
2.4 Pairs of Tilings

Definition 4 Fault

Given a board of length \( n \) placed above a board of length \( m \), we say that there is a fault at cell \( i \), where \( 1 \leq i \leq n \) and \( 1 \leq i \leq m \), if both tilings are breakable in that place. In the case that the first cells of the tilings are aligned, there is a fault at “cell 0.”

Consider the following placement of a 10-board tiling and an 8-board tiling:

![Diagram of tiling pair with faults marked]

Figure 4: This pair of tilings has 4 faults.

This pair of tilings in this particular placement has 4 faults, which are denoted by the thin lines. Since both boards begin at the same place, there is a fault at cell 0 by definition. There are also faults at cells 3, 5, and 8. It is worth remembering that faults can exist at both the beginning and the end of a board (as in the case of this 8-board).

3 The Problem Presented

Each identity for the Fibonacci sequence, along with The Solution by Counting and corresponding Visual Example, is in a separate file labelled “Fibonacci Identity (#).” Though the identities can be viewed in any order, they are numbered according to my perception of their difficulty level.

References


Fibonacci Identity 1
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The Motivation for the Fibonacci sequence along with its Preliminary Ideas can be found in the “Fibonacci Overview” file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

1 The Problem Presented

<table>
<thead>
<tr>
<th>Theorem 1</th>
</tr>
</thead>
</table>
| \[
\sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} = f_n
\] |

If the notation on the left-hand side of this identity is unfamiliar to you, please consult the “Common Definitions” file.

2 The Solution by Counting

From the right-hand side, we already know that we are counting the number of tilings of an n-board. By our interpretation of the Fibonacci numbers, this number is \( f_n \).

Now we are left to show that \( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \) also counts this number. Notice that we can determine the tilings of a board of length \( n \) by the number of dominoes that the tiling contains. Suppose a tiling contains \( i \) dominoes. Because a domino takes up two cells, \( 0 \leq i \leq \frac{n}{2} \). Also, since there are \( i \) dominoes for \( n \) cells, the tiling must contain \( n-2i \) squares. Thus, this tiling has a total of \( i + (n-2i) = n-i \) tiles. How many ways are there to choose where the \( i \) dominoes go within the \( n-i \) tiles? There are \( \binom{n-i}{i} \) ways. Since one tiling of an \( n \)-board cannot have two different numbers of dominoes, this collection of tasks is mutually exclusive. Thus, the number of tilings on an \( n \)-board is \( \sum_{i=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n-i}{i} \).

Equating the two ways we counted this number completes the proof.

3 Visual Example

Consider a board of length 5. Recall that there are \( f_5 = 8 \) ways to tile this board. The visualization lists each of these ways on the right side and then highlights them according to the number of dominoes that each tiling contains. As an example, note that there are \( \binom{5-2}{2} = 3 \) ways to tile a 5-board with 2 dominoes.

4 Fibonacci Numbers in Pascal’s Triangle

Because this Fibonacci identity contains a binomial coefficient, it seems intuitive that it could be seen in Pascal’s triangle.\(^1\) Take a minute to study the triangle with this identity in mind. Scroll down slowly and look only at the first triangle. Can you find the Fibonacci sequence?

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\(^1\)For an introduction to the arithmetic triangle (Pascal’s triangle), see the exposition entitled “Pascal’s Identity.”
1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

1 5 10 10 5 1

1 6 15 20 15 6 1

Figure 1: Pascal’s Triangle

By summing the “shallow diagonals” the Fibonacci sequence is revealed [Wei05].

\[
\begin{array}{cccccccc}
1 & & & & & & & \\
1 & 1 & & & & & & \\
1 & 2 & 1 & & & & & \\
1 & 3 & 3 & 1 & & & & \\
1 & 4 & 6 & 4 & 1 & & & \\
1 & 5 & 10 & 10 & 5 & 1 & & \\
1 & 6 & 15 & 20 & 15 & 6 & 1 & \\
\end{array}
\]

Figure 2: The Fibonacci Sequence within Pascal’s Triangle.

To see how this fact relates to the above Fibonacci identity, reconsider the Fibonacci number \( f_5 = 8 \). By Theorem 1, we know that \( f_5 = \sum_{i=0}^{\frac{5}{2}} \binom{5-i}{i} = \binom{5-0}{0} + \binom{5-1}{1} + \binom{5-2}{2} \). In other words, \( 8 = 1 + 4 + 3 \), which is what we notice in the triangle. Because both the Fibonacci sequence and Pascal’s triangle are such unique structures, this relationship is rather surprising.

References


Fibonacci Identity 2
Callie Wurtz

The Motivation for the Fibonacci sequence along with its Preliminary Ideas can be found in the “Fibonacci Overview” file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

1 The Problem Presented

Theorem 1

For $n \geq 0$, \[
\sum_{k=0}^{n} f_k = f_{n+2} - 1.
\]

2 The Solution by Counting

Consider tiling an $(n + 2)$-board with squares and dominoes. We are concerned with how many of these tilings use at least one domino. By our interpretation of Fibonacci numbers, there are $f_{n+2}$ ways to tile a board of length $(n + 2)$. Excluding the “all square” tiling gives $f_{n+2} - 1$ tilings that include at least one domino.

Since we know there must be at least one domino, we can consider the position of the last domino. Let this last domino cover cells $k + 1$ and $k + 2$. (Note that if the board begins with its only domino, then $k = 0$.) This implies that cells $k + 3$ through $n + 2$ must be covered by squares. Therefore, the only cells we are unsure about for $k > 0$ are cells 1 through $k$. Consequently, there are $f_k$ ways to tile these cells. Since the last domino cannot be in two places at once, the Rule of Sum applies. There are thus $f_0 + f_1 + f_2 + \cdots + f_n = \sum_{k=0}^{n} f_k$ ways to tile a board with at least one domino.

Equating the two ways we counted this number completes the proof.

3 Visual Example

Consider a board of length $4 + 2 = 6$. The visualization first lists every way to tile this board. Note that there are $f_6 = 13$ ways to do this. Without the “all-square” tiling, there are $f_{4+2} - 1 = 12$ tilings. Now for the left-hand side, consider the position of the last domino. For example, if the last (and only) domino is the first piece on the board, it covers cells 1 and 2. This implies that the rest of the pieces are squares, and thus, there is only $f_0 = 1$ way that this arrangement can be tiled. The visualization breaks down the remaining summands from the left-hand side and demonstrates that they add to 12.

References

Fibonacci Identity 3

The Motivation for the Fibonacci sequence along with its Preliminary Ideas can be found in the “Fibonacci Overview” file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

1 The Problem Presented

\[ f_{m+n} = f_m f_n + f_{m-1} f_{n-1}. \]

2 The Solution by Counting

Consider tiling a board of length \( m + n \) with squares and dominoes. By our interpretation of the Fibonacci numbers, there are clearly \( f_{m+n} \) ways to tile an \( (m + n) \)-board.

Now consider cell \( m \). We are interested in the breakability of cell \( m \). If cell \( m \) is breakable, then by definition, the tiling can be decomposed into an \( m \)-tiling followed by an \( n \)-tiling. Since there are \( f_m \) ways to tile an \( m \)-board and \( f_n \) ways to tile an \( n \)-board and these tasks are independent of each other, there are \( f_m \cdot f_n \) ways to tile an \( (m + n) \)-board breakable at cell \( m \).

Suppose now that the board is not breakable at cell \( m \). This implies that cells \( m \) and \( m + 1 \) are covered by a domino. Therefore, the tiling must be breakable at cell \( m - 1 \) and there are \( f_{m-1} \) ways to tile cells \( 1 \) through \( m - 1 \). Note that the board is also breakable at cell \( m + 1 \), and there are \( (m+n) - (m+2) + 1 = n - 1 \) cells from \( m + 2 \) through \( m + n \). Those \( n - 1 \) cells can be tiled in \( f_{n-1} \) ways. Thus, there are \( f_{m-1} \cdot f_{n-1} \) ways to tile a board that is unbreakable at cell \( m \). This can be seen in the following picture.

![Diagram of Fibonacci Identity 3](image)

**Figure 1:** Counting tilings of an \((m + n)\)-board based on the breakability at \( m \).

Since the board is either breakable or unbreakable at cell \( m \), there are a total of \( f_m f_n + f_{m-1} f_{n-1} \) ways to tile a board of length \( m + n \).

Equating the two ways we counted this number completes the proof.

\( \square \)
3 Visual Example

Consider a board of length 6, with \( m = 4 \) and \( n = 2 \). Recall that there are \( f_6 \) ways to tile this board. The visualization lists these on the left side of the screen. Now consider the condition of breakability at cell 4. The visualization sorts these tilings into the two cases of the right-hand side. If it is breakable at cell 4, there are \( f_4 \cdot f_2 = 5 \cdot 2 \) tilings. If it is unbreakable at cell 4, there are \( f_3 \cdot f_1 = 3 \cdot 1 \) tilings.

References

Fibonacci Identity 4
Callie Wurtz

The Motivation for the Fibonacci sequence along with its Preliminary Ideas can be found in the “Fibonacci Overview” file. That document also contains a combinatorial interpretation of the Fibonacci numbers that you are advised to read before continuing.

1 The Problem Presented

Theorem 1
For \( n \geq 0 \),
\[
\sum_{k=0}^{n} f_k^2 = f_n f_{n+1}.
\]

2 The Solution by Counting

Picture two domino boards, one of length \( n \) and the other of length \( n + 1 \). We are interested in the number of ways these two boards can be tiled. By our interpretation of Fibonacci numbers, there are \( f_n \) ways to tile an \( n \)-board and \( f_{n+1} \) ways to tile an \( (n + 1) \)-board. Since tiling one board is independent of tiling the other, there are \( f_n \cdot f_{n+1} \) ways to tile both boards.

Now place the \( (n + 1) \)-board directly above the \( n \)-board so that the first cells are aligned as in the following figure.

```
|   |   | ... |
1  2
```
```
|   | ... |   |
1  2
```

Figure 1: There are \( f_n f_{n-1} \) ways to tile these boards.

We are interested in the position of the last fault. Suppose the position of the last fault is at cell \( k \), and note that \( 0 \leq k \leq n \). In other words, the last and only fault may be at cell 0 or the last fault could be anywhere up to the end of the shorter board. Consider tiling the boards by first tiling to the left of the fault (the head) and then tiling to the right of the fault (the tail).

Because the last fault is at cell \( k \), the head of each board has length \( k \). (Recall that there is 1 way to tile a board of length 0.) There are therefore \( f_k \) ways to tile the head of the \( (n + 1) \)-board and also \( f_k \) ways to tile the head of the \( n \)-board. Thus, there are \( f_k^2 \) ways to tile both boards through cell \( k \).

Now consider tiling the tails of both boards. The tail of the \( (n + 1) \)-board consists of cells \( k + 1 \) through \( n + 1 \). The tail of the \( n \)-board is composed of cells \( k + 1 \) through \( n \) for \( k < n \). (If \( k = n \), and the tail of the \( n \)-board is length 0.) Therefore, one of the tails has an even number of cells and the other must have an odd number of cells. Because there can be no faults in the tails, there is only way to tile them. The even-length tail must be tiled with all dominoes and the odd-length tail must begin with a square and then be tiled with all dominoes. Convince yourself that this is the only possibility.
Since tiling cells 1 through \( k \) is independent of tiling after cell \( k \), the number of ways to accomplish both of these tasks is found by multiplying. There are \( f_k^2 \) ways to tile the heads of these two boards and only 1 way to tile the tails, so there are \( f_k^2 \cdot 1 \) tilings of both boards with a fault at cell \( k \). This means that any tiling of such a pair of boards is completely determined by the tiling of their heads. We have done a lot of counting, but we are not quite done. Recall that \( 0 \leq k \leq n \). Since there cannot be two “last faults” for the boards, one position of \( k \) mutually excludes the rest. Therefore, there is a total of \( \sum_{k=0}^{n} f_k^2 \) ways to tile one board of length \( n \) and another board of length \( n+1 \).

Equating the two ways we counted this number completes the proof.

\[ \square \]

### 3 Visual Example

In order to keep the example small, this visualization uses a board of length 3 above a board of length 2. Note that there are \( f_3 f_2 = 3 \cdot 2 = 6 \) ways to tile these two boards. All of these ways are listed on the right side of the screen. Then, each pair of tilings is highlighted according to the location of its last fault.

As an example, consider a pair that has a fault at cell 2. The heads of these boards have length 2, and there are only 2 ways to tile a 2-board. In other words, \( f_2 = 2 \). Consequently, there are \( f_2^2 = 2^2 = 4 \) tilings for a pair of boards with a fault at cell 2.

### References