MATHEMATICAL ASSOCIATION OF AMERICA

Solutions Pamphlet

American Mathematics Competitions

62nd Annual

AMC 12 B

American Mathematics Contest 12 B

Wednesday, February 23, 2011

This Pamphlet gives at least one solution for each problem on this year's contest and shows that all problems can be solved without the use of a calculator. When more than one solution is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic *vs* geometric, computational *vs* conceptual, elementary *vs* advanced. These solutions are by no means the only ones possible, nor are they superior to others the reader may devise.

We hope that teachers will inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. *However, the publication, reproduction or communication of the problems or solutions of the AMC 12 during the period when students are eligible to participate seriously jeopardizes the integrity of the results. Dissemination via copier, telephone, e-mail, World Wide Web or media of any type during this period is a violation of the competition rules.*

Correspondence about the problems/solutions for this AMC 12 and orders for any publications should be addressed to:

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The problems and solutions for this AMC 12 were prepared by the MAA's Committee on the AMC 10 and AMC 12 under the direction of AMC 12 Subcommittee Chair:

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1. **Answer (C):** The given expression is equal to

$$\frac{12}{9}-\frac{9}{12}=\frac{4}{3}-\frac{3}{4}=\frac{16-9}{12}=\frac{7}{12}.$$

2. **Answer (E):** The sum of her first 5 test scores is 385, yielding an average of 77. To raise her average to 80, her 6^{th} test score must be the difference between $6\cdot 80=480$ and 385, which is 95.

3. **Answer (C):** Bernardo has paid $B-A$ dollars more than LeRoy. If LeRoy gives Bernardo half of that difference, $\frac{B-A}{2}$, then each will have paid the same amount.

4. **Answer (E):** Because $161=23\cdot 7$, the only two digit factor of 161 is 23. The correct product must have been $32\cdot 7=224$.

5. **Answer (A):**

Because N is divisible by 3, 4, and 5, the prime factorization of N must contain one 3, two 2s, and one 5. Furthermore $2^2\cdot 3\cdot 5=60$ is divisible by every integer less than 7. Therefore the numbers with this property are precisely the positive multiples of 60. The second smallest positive multiple of 60 is 120, and the sum of its digits is 3.

6. **Answer (C):** Let O be the center of the circle, and let the degree measures of the minor and major arcs be $2x$ and $3x$, respectively. Because $2x+3x=360^\circ$, it follows that $x=72^\circ$ and $\angle BOC=2x=144^\circ$. In quadrilateral $ABOC$, the segments AB and AC are tangent to the circle, thus $\angle ABO=\angle ACO=90^\circ$ and $\angle BAC=360^\circ-(144^\circ+90^\circ+90^\circ)=36^\circ$.

7. **Answer (B):** Because $x\le 99$ and $\frac{1}{2}(x+y)=60$, it follows that $y=120-x\ge 120-99=21$. Thus the maximum value of $\frac{x}{y}$ is $\frac{99}{21}=\frac{33}{7}$.

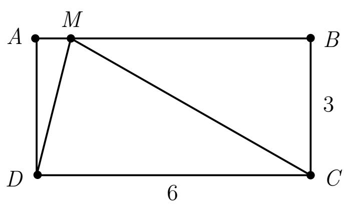
8. **Answer (A):** The only parts of the track that are longer walking on the outside edge rather than the inside edge are the two semicircular portions. If the radius of the inner semicircle is r, then the difference in the lengths of the two paths is $2\pi(r+6)-2\pi r=12\pi$. Let x be her speed in meters per second. Then $36x=12\pi$, and $x=\frac{\pi}{3}$.

9. **Answer (D):** Consider all ordered pairs (a, b) with each of the numbers a and b in the closed interval $[-20, 10]$. These pairs fill a 30×30 square in the coordinate plane, with an area of 900 square units. Ordered pairs in the first and third quadrants have the desired property, namely $a \cdot b > 0$. The areas of the portions of the 30×30 square in the first and third quadrants are $10^2 = 100$ and $20^2 = 400$, respectively. Therefore the probability of a positive product is $\frac{100+400}{900} = \frac{5}{9}$.

OR

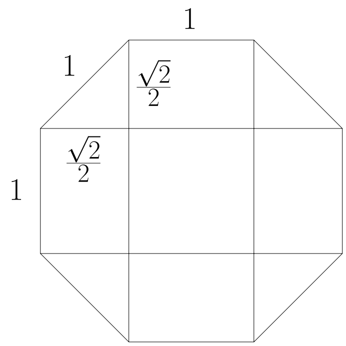
Each of the numbers is positive with probability $\frac{1}{3}$ and negative with probability $\frac{2}{3}$. Their product is positive if and only if both numbers are positive or both are negative, so the requested probability is $\left(\frac{1}{3}\right)^2 + \left(\frac{2}{3}\right)^2 = \frac{5}{9}$.

10. **Answer (E):** Sides $\overline{AB}$ and $\overline{CD}$ are parallel, so $\angle CDM = \angle AMD$. Because $\angle AMD = \angle CMD$, it follows that $\triangle CMD$ is isosceles and $CD = CM = 6$. Therefore $\triangle MCB$ is a $30-60-90°$ right triangle with $\angle BMC = 30°$. Finally, $2 \cdot \angle AMD + 30° = \angle AMD + \angle CMD + 30° = 180°$, so $\angle AMD = 75°$.



11. **Answer (B):** Because $AB = 1$, the smallest number of jumps is at least 2. The perpendicular bisector of $\overline{AB}$ is the line with equation $x = \frac{1}{2}$, which has no points with integer coordinates, so 2 jumps are not possible. A sequence of 3 jumps is possible; one such sequence is $(0, 0)$ to $(3, 4)$ to $(6, 0)$ to $(1, 0)$.

12. **Answer (A):** Assume the octagon's edge is 1. Then the corner triangles have hypotenuse 1 and thus legs $\frac{\sqrt{2}}{2}$ and area $\frac{1}{4}$ each; the four rectangles are 1 by $\frac{\sqrt{2}}{2}$ and have area $\frac{\sqrt{2}}{2}$ each, and the center square has area 1. The total area is $4 \cdot \frac{1}{4} + 4 \cdot \frac{\sqrt{2}}{2} + 1 = 2 + 2\sqrt{2}$. The probability that the dart hits the center square is $\frac{1}{2+2\sqrt{2}} = \frac{\sqrt{2}-1}{2}$.



13. **Answer (B):** The largest pairwise difference is 9, so $w - z = 9$. Let n be either x or y. Because n is between w and z,

$$9 = w - z = (w - n) + (n - z).$$

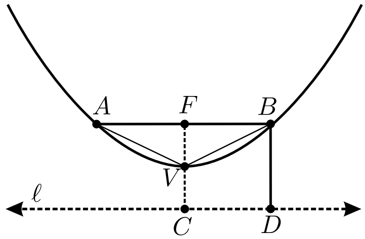
Therefore the positive differences $w - n$ and $n - z$ must sum to 9. The given pairwise differences that sum to 9 are $3 + 6$ and $4 + 5$. The remaining pairwise difference must be $x - y = 1$.

The second largest pairwise difference is 6, so either $w - y = 6$ or $x - z = 6$. In the first case the set of four numbers may be expressed as $\{w, w-5, w-6, w-9\}$. Hence $4w - 20 = 44$, so $w = 16$. In the second case $w - x = 3$, and the four numbers may be expressed as $\{w, w-3, w-4, w-9\}$. Therefore $4w - 16 = 44$, so $w = 15$. The sum of the possible values for w is $16 + 15 = 31$.

Note: The possible sets of four numbers are $\{16, 11, 10, 7\}$ and $\{15, 12, 11, 6\}$.

14. **Answer (D):** Let ℓ be the directrix of the parabola, and let C and D be the projections of F and B onto ℓ, respectively. For any point in the parabola, its distance to F and to ℓ are the same. Because V and B are on the parabola, it follows that $p = FV = VC$ and $2p = FC = BD = FB$. By the Pythagorean Theorem, $VB = \sqrt{FV^2 + FB^2} = \sqrt{5}p$, and thus $\cos(\angle FVB) = \frac{FV}{VB} = \frac{p}{\sqrt{5}p} = \frac{\sqrt{5}}{5}$. Because $\angle AVB = 2(\angle FVB)$, it follows that

$$\cos(\angle AVB) = 2\cos^2(\angle FVB) - 1 = 2\left(\frac{\sqrt{5}}{5}\right)^2 - 1 = \frac{2}{5} - 1 = -\frac{3}{5}.$$



OR

Establish as in the first solution that $FV = p$, $FB = 2p$, and $VB = \sqrt{5}p$. Then $AB = 2 \cdot FB = 4p$, and by the Law of Cosines applied to $\triangle ABV$,

$$\cos \angle AVB = \frac{VA^2 + VB^2 - AB^2}{2(VA)(VB)} = \frac{5p^2 + 5p^2 - 16p^2}{2(5p^2)} = -\frac{3}{5}.$$

Note: The segment AB is called the *latus rectum.*

15. **Answer (D):** Factoring results in the following product of primes:

$$\begin{aligned} 2^{24} - 1 &= (2^{12} - 1)(2^{12} + 1) = (2^6 - 1)(2^6 + 1)(2^4 + 1)(2^8 - 2^4 + 1) \\ &= 63 \cdot 65 \cdot 17 \cdot 241 = 3 \cdot 3 \cdot 5 \cdot 7 \cdot 13 \cdot 17 \cdot 241. \end{aligned}$$

The two-digit integers that can be formed from these prime factors are:

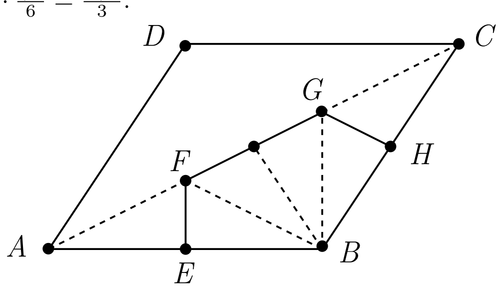
$$\begin{gathered} 17, \quad 3 \cdot 17 = 51, \quad 5 \cdot 17 = 85, \\ 13, \quad 3 \cdot 13 = 39, \quad 5 \cdot 13 = 65, \quad 7 \cdot 13 = 91, \\ 3 \cdot 7 = 21, \quad 5 \cdot 7 = 35, \quad 3 \cdot 3 \cdot 7 = 63, \\ 3 \cdot 5 = 15, \quad \text{and} \quad 3 \cdot 3 \cdot 5 = 45. \end{gathered}$$

Thus there are 12 positive two-digit factors.

16. **Answer (C):** Let E and H be the midpoints of $\overline{AB}$ and $\overline{BC}$, respectively. The line drawn perpendicular to $\overline{AB}$ through E divides the rhombus into two regions: points that are closer to vertex A than B, and points that are closer to vertex B than A. Let F be the intersection of this line with diagonal $\overline{AC}$. Similarly, let point G be the intersection of the diagonal $\overline{AC}$ with the perpendicular to $\overline{BC}$ drawn from H. Then the desired region R is the pentagon $BEFGH$.

Note that $\triangle AFE$ is a $30-60-90^\circ$ triangle with $AE = 1$. Hence the area of $\triangle AFE$ is $\frac{1}{2} \cdot 1 \cdot \frac{1}{\sqrt{3}} = \frac{\sqrt{3}}{6}$. Both $\triangle BFE$ and $\triangle BGH$ are congruent to $\triangle AFE$, so they have the same areas. Also $\angle FBG = 120^\circ - \angle FBE - \angle GBH =$

$60°$, so $\triangle FBG$ is an equilateral triangle. In fact, the altitude from B to $\overline{FG}$ divides $\triangle FBG$ into two triangles, each congruent to $\triangle AFE$. Hence the area of $BEFGH$ is $4 \cdot \frac{\sqrt{3}}{6} = \frac{2\sqrt{3}}{3}$.



17. **Answer (B):** Note that

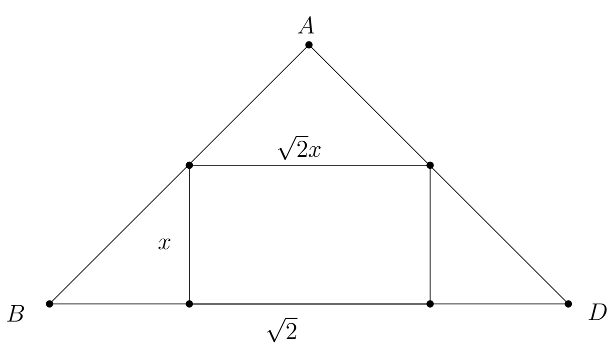
$$h_1(x) = \log_{10}\left(\frac{10^{10x}}{10}\right) = \log_{10}\left(10^{10x-1}\right) = 10x - 1.$$

Therefore $h_2(x) = 10^2x-(1+10)$, $h_3(x) = 10^3x-(1+10+10^2)$, and in general,

$$h_n(x) = 10^n x - \sum_{k=0}^{n-1} 10^k.$$

Hence $h_n(1)$ is an n-digit integer whose units digit is 9 and whose other digits are all 8's. The sum of the digits of $h_{2011}(1)$ is $8 \cdot 2010 + 9 = 16{,}089$.

18. **Answer (A):** Let A be the apex of the pyramid, and let the base be the square $BCDE$. Then $AB = AD = 1$ and $BD = \sqrt{2}$, so $\triangle BAD$ is an isosceles right triangle. Let the cube have edge length x. The intersection of the cube with the plane of $\triangle BAD$ is a rectangle with height x and width $\sqrt{2}x$. It follows that $\sqrt{2} = BD = 2x + \sqrt{2}x$, from which $x = \sqrt{2} - 1$.



Hence the cube has volume

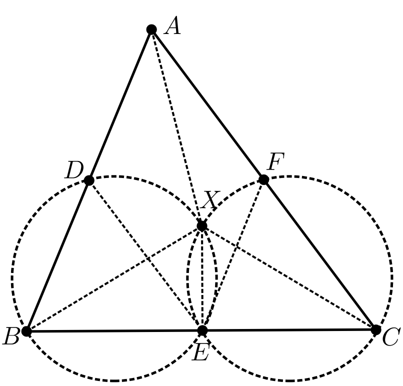
$$(\sqrt{2}-1)^3 = (\sqrt{2})^3 - 3(\sqrt{2})^2 + 3\sqrt{2} - 1 = 5\sqrt{2} - 7.$$

OR

Let A be the apex of the pyramid, let O be the center of the base, let P be the midpoint of one base edge, and let the cube intersect $\overline{AP}$ at Q. Let a coordinate plane intersect the pyramid so that O is the origin, A on the positive y-axis, and $P = \left(\frac{1}{2}, 0\right)$. Segment AP is an altitude of a lateral side of the pyramid, so $AP = \frac{\sqrt{3}}{2}$, and it follows that $A = \left(0, \frac{\sqrt{2}}{2}\right)$. Thus the equation of line AP is $y = \frac{\sqrt{2}}{2} - \sqrt{2}x$. If the side length of the cube is s, then $Q = (\frac{s}{2}, s)$, so $s = \frac{\sqrt{2}}{2} - \sqrt{2} \cdot \frac{s}{2}$. Solving gives $s = \sqrt{2} - 1$, and the result follows that in the first solution.

19. **Answer (B):** For $0 < x \le 100$, the nearest lattice point directly above the line $y = \frac{1}{2}x + 2$ is $\left(x, \frac{1}{2}x + 3\right)$ if x is even and $\left(x, \frac{1}{2}x + \frac{5}{2}\right)$ if x is odd. The slope of the line that contains this point and $(0, 2)$ is $\frac{1}{2} + \frac{1}{x}$ if x is even and $\frac{1}{2} + \frac{1}{2x}$ if x is odd. The minimum value of the slope is $\frac{51}{100}$ if x is even and $\frac{50}{99}$ if x is odd. Therefore the line $y = mx + 2$ contains no lattice point with $0 < x \le 100$ for $\frac{1}{2} < m < \frac{50}{99}$.

20. **Answer (C):** Because $\overline{DE}$ is parallel to $\overline{AC}$ and $\overline{EF}$ is parallel to $\overline{AB}$ it follows that $\angle BDE = \angle BAC = \angle EFC$. By the Inscribed Angle Theorem, $\angle BDE = \angle BXE$ and $\angle EFC = \angle EXC$. Therefore $\angle BXE = \angle EXC$. Furthermore $BE = EC$, so by the Angle Bisector Theorem $XB = XC$. Note that $\angle BXC = 2\angle BXE = 2\angle BDE = 2\angle BAC$, and by the Inscribed Angle Theorem, it follows that X is the circumcenter of $\triangle ABC$, so $XA = XB = XC = R$ the circumradius of $\triangle ABC$.



Let $a = BC$, $b = AC$, and $c = AB$. The area of $\triangle ABC$ equals $\frac{1}{4R}(abc)$, and by Heron's Formula it also equals $\sqrt{s(s-a)(s-b)(s-c)}$, where $s = \frac{1}{2}(a+b+c)$. Thus

$$R = \frac{abc}{4\sqrt{s\,(s-a)\,(s-b)\,(s-c)}} = \frac{13 \cdot 14 \cdot 15}{4\sqrt{21 \cdot 8 \cdot 7 \cdot 6}} = \frac{65}{8},$$

and $XA + XB + XC = 3R = \frac{195}{8}$.

21. **Answer (D):** Let the arithmetic and geometric means of x and y be $10a + b$ and $10b + a$, respectively. Then

$$\frac{x+y}{2} = 10a + b \Rightarrow (x+y)^2 = 400a^2 + 80ab + 4b^2$$

and

$$\sqrt{xy} = 10b + a \Rightarrow xy = 100b^2 + 20ab + a^2,$$

so

$$(x-y)^2 = (x+y)^2 - 4xy = 396(a^2 - b^2) = 11 \cdot 6^2 \cdot (a+b)(a-b)$$

Because x and y are distinct, a and b are distinct digits, and the last expression is a perfect square if and only if $a + b = 11$ and $a - b$ is a perfect square. The cases $a - b = 1$, 4, and 9 give solutions $(a, b) = (6, 5)$, $(7.5, 3.5)$, and $(10, 1)$, respectively. Because a and b are digits only the first solution is valid. Thus $(x-y)^2 = 11 \cdot 6^2 \cdot 11 = 66^2$ and $|x - y| = 66$. Note that the given conditions are satisfied if $\{x, y\} = \{32, 98\}$.

22. **Answer (D):** Let $T_n = \triangle ABC$. Suppose $a = BC$, $b = AC$, and $c = AB$. Because $\overline{BD}$ and $\overline{BE}$ are both tangent to the incircle of $\triangle ABC$, it follows that $BD = BE$. Similarly, $AD = AF$ and $CE = CF$. Then

$$\begin{aligned} 2BE &= BE + BD = BE + CE + BD + AD - (AF + CF) \\ &= a + c - b, \end{aligned}$$

that is, $BE = \frac{1}{2}(a + c - b)$. Similarly $AD = \frac{1}{2}(b + c - a)$ and $CF = \frac{1}{2}(a + b - c)$. In the given $\triangle ABC$, suppose that $AB = x+1$, $BC = x-1$, and $AC = x$. Using the formulas for BE, AD, and CF derived before, it must be true that

$$\begin{aligned} BE &= \frac{1}{2}\left((x-1) + (x+1) - x\right) = \frac{1}{2}x, \\ AD &= \frac{1}{2}\left(x + (x+1) - (x-1)\right) = \frac{1}{2}x + 1, \text{ and} \\ CF &= \frac{1}{2}\left((x-1) + x - (x+1)\right) = \frac{1}{2}x - 1. \end{aligned}$$

Hence both (BC, CA, AB) and (CF, BE, AD) are of the form $(y-1, y, y+1)$. This is independent of the values of a, b, and c, so it holds for all T_n. Furthermore, adding the formulas for BE, AD, and CF shows that the perimeter of

T_{n+1} equals $\frac{1}{2}(a+b+c)$, and consequently the perimeter of the last triangle T_N in the sequence is

$$\frac{1}{2^{N-1}}(2011+2012+2013) = \frac{1509}{2^{N-3}}.$$

The last member T_N of the sequence will fail to define a successor if for the first time the new lengths fail the Triangle Inequality, that is, if

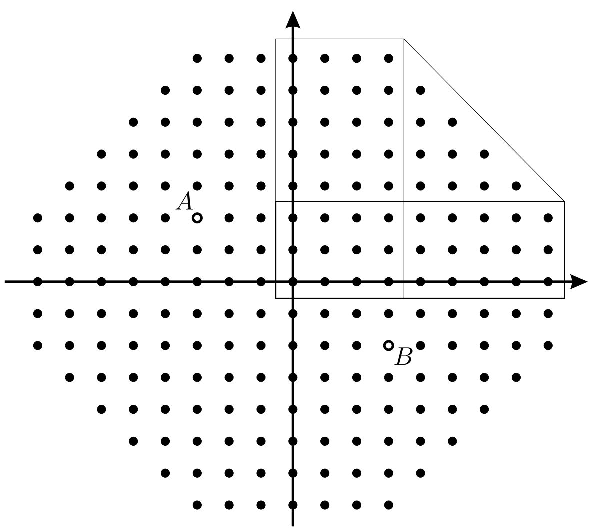
$$-1+\frac{2012}{2^N}+\frac{2012}{2^N} \le 1+\frac{2012}{2^N}.$$

Equivalently, $2012 \le 2^{N+1}$ which happens for the first time when $N = 10$. Thus the required perimeter of T_N is $\frac{1509}{2^7} = \frac{1509}{128}$.

23. **Answer (C):** Let $X = (x, y)$. The distance traveled by the bug from A to X is at least $|x+3|+|y-2|$. Similarly, the distance traveled by the bug from X to B is at least $|x-3|+|y+2|$. It follows that X belongs to a path from A to B traveled by the bug if and only if

$$d = |x-3|+|x+3|+|y-2|+|y+2| \le 20.$$

The expression for d is invariant if x is replaced by $-x$ or y is replaced by $-y$. By symmetry, it is enough to count the number of points X with $x \ge 0$ and $y \ge 0$, multiply by 4, and subtract the points that were overcounted, that is those in the x-axis or in the y-axis. Consider four cases:



Case 1. $0 \le x \le 3$ and $0 \le y \le 2$. In this case $|x-3|+|x+3| = 6$ and $|y-2|+|y+2| = 4$. Thus $d = 10 < 20$ and there are $4 \cdot 3 = 12$ points X in this case. This includes the origin and 5 other points for which $xy = 0$.

Case 2. $0 \le x \le 3$ and $y \ge 3$. In this case $|x-3|+|x+3| = 6$ and $|y-2|+|y+2| = 2y$. Thus $d = 6 + 2y \le 20$ if and only if $y \le 7$. There are $4 \cdot 5 = 20$ points X in this case. This includes 5 points for which $xy = 0$.

Case 3. $x \ge 4$ and $0 \le y \le 2$. In this case $|x-3| + |x+3| = 2x$ and $|y-2| + |y+2| = 4$. Thus $d = 4 + 2x \le 20$ if and only if $x \le 8$. There are $5 \cdot 3 = 15$ points X in this case. This includes 5 points for which $xy = 0$.

Case 4. $x \ge 4$ and $y \ge 3$. In this case $|x-3|+|x+3| = 2x$ and $|y-2|+|y+2| = 2y$. Thus $d = 2x + 2y \le 20$ if and only if $x + y \le 10$. The number of points X in this case is equal to

$$\sum_{x=4}^{7}\sum_{y=3}^{10-x} 1 = \sum_{x=4}^{7}(10 - x - 2) = \sum_{x=4}^{7}(8 - x) = 4 + 3 + 2 + 1 = 10,$$

and there are no points with $xy = 0$.

By symmetry the required total is $4(12 + 20 + 15 + 10) - 2(5 + 5 + 5) - 3 = 4 \cdot 57 - 2 \cdot 15 - 3 = 195$.

24. **Answer (B):** Factoring or using the quadratic formula with z^4 as the variable yields $P(z) = (z^4 - 1)(z^4 + (4\sqrt{3} + 7))$. Moreover, $4\sqrt{3} + 7 = (\sqrt{3} + 2)^2$ and $2(\sqrt{3}+2) = 2\sqrt{3}+4 = (\sqrt{3}+1)^2$; thus $4\sqrt{3}+7 = (\frac{1}{2}(\sqrt{6}+\sqrt{2}))^4$. If $w = \frac{1}{2}(\sqrt{3}+1)$, then the eight zeros of $P(z)$ are $1, -1, i, -i, w(1 + i), w(-1 + i), w(-1 - i)$, and $w(1 - i)$.

The distances from 1 to the other zeros are

$$|1 - (-1)| = 2, |1 \pm i| = \sqrt{2}, |1 - w(1 \pm i)| = \sqrt{(1 - w)^2 + w^2} = \sqrt{2}, \text{ and}$$
$$|1 - w(-1 \pm i)| = \sqrt{(1 + w)^2 + w^2} = \sqrt{2\sqrt{3} + 4} = \sqrt{3} + 1.$$

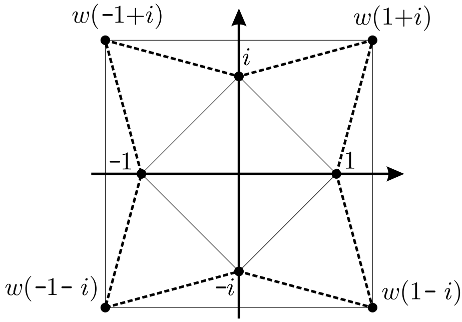
Similarly, the distances from $w(1 + i)$ to the other zeros are

$$|w(1 + i) - w(1 - i)| = |w(1 + i) - w(-1 + i)| = 2w = \sqrt{3} + 1,$$
$$|w(1 + i) - w(-1 - i)| = 2\sqrt{2}w = \sqrt{6} + \sqrt{2},$$

and by symmetry,

$$|w(1 + i) - 1| = |w(1 + i) - i| = \sqrt{2}, \text{ and}$$
$$|w(1 + i) + 1| = |w(1 + i) + i| = \sqrt{3} + 1.$$

Because the set of zeros is 4-fold symmetric with respect to the origin, it follows that every line segment joining two of the zeros has length at least $\sqrt{2}$. This shows that any polygon with vertices at the zeros has perimeter at least $8\sqrt{2}$. Finally, note that the polygon with consecutive vertices $1, w(1+i)$, i, $w(-1+i)$, -1, $w(-1 - i)$, $-i$, and $w(1 - i)$ has perimeter $8\sqrt{2}$.



25. **Answer (D):** Let

$$100 = qk + r, \text{ with } q, r \in \mathbb{Z} \text{ and } |r| \leq \frac{k-1}{2}, \text{ and}$$

$$n = q_1 k + r_1, \text{ with } q_1, r_1 \in \mathbb{Z} \text{ and } |r_1| \leq \frac{k-1}{2},$$

so that $\left[\frac{100}{k}\right] = q$ and $\left[\frac{n}{k}\right] = q_1$. Note that $\left[\frac{n+mk}{k}\right] = \left[\frac{n}{k}\right] + m$ for every integer m. Thus n satisfies the required identity if and only if $n + mk$ satisfies the identity for all integers m. Thus all members of a residue class mod k either satisfy the required equality or not; moreover, k divides 99! for every $1 \leq k \leq 99$, so every residue class mod k in the interval $1 \leq n \leq 99!$ has the same number of elements. Suppose $r \geq 0$. If $r_1 \geq r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1)\, k + (r - r_1)\,,$$

where $0 \leq r - r_1 \leq \frac{k-1}{2}$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 = \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. Similarly, if $r_1 < r - \frac{k-1}{2}$, then

$$100 - n = (q - q_1 + 1)\, k + (r - r_1 - k)\,,$$

where $-\frac{k-1}{2} \leq r - r_1 - k \leq -1$. Thus $\left[\frac{100-n}{k}\right] = q - q_1 + 1 > \left[\frac{100}{k}\right] - \left[\frac{n}{k}\right]$. It follows that the only residue classes r_1 that satisfy the identity are those in the interval $r - \frac{k-1}{2} \leq r_1 \leq \frac{k-1}{2}$. Thus for $r \geq 0$,

$$P(k) = \frac{1}{k}\left(\frac{k-1}{2} + 1 - \left(r - \frac{k-1}{2}\right)\right) = \frac{k-r}{k} = 1 - \frac{|r|}{k}.$$

Similarly, if $r < 0$ then the identity is satisfied only by the residue classes r_1 in the interval $-\frac{k-1}{2} \leq r_1 \leq r + \frac{k-1}{2}$. Thus for $r < 0$,

$$P(k) = \frac{1}{k}\left(r + \frac{k-1}{2} + 1 - \left(-\frac{k-1}{2}\right)\right) = \frac{k+r}{k} = 1 - \frac{|r|}{k}.$$

To minimize $P(k)$ in the range $1 \leq k \leq 99$, where k is odd, first suppose that $r = \frac{k-1}{2}$. Note that $P(k) = \frac{1}{2} + \frac{1}{2k}$, $100 = qk + \frac{k-1}{2}$, and so $201 = k\,(2q+1)$.

The minimum of $P(k)$ in this case is achieved by the largest possible k under this restriction. Because $201 = 3\cdot 67$, it follows that the largest factor k of 201 in the given range is $k = 67$. In this case $P(67) = \frac{1}{2} + \frac{1}{2\cdot 67} = \frac{34}{67}$. Second, suppose $r = \frac{1-k}{2}$. In this case $P(k) = \frac{1}{2} + \frac{1}{2k}$ and $199 = k(2q-1)$. Because 199 is prime, it follows that $k = 1$ and $P(k) = 1 > \frac{34}{67}$. Finally, if $|r| \le \frac{k-3}{2}$, then

$$\begin{aligned} P(k) &= 1 - \frac{|r|}{k} > 1 - \frac{k-3}{2k} = \frac{1}{2} + \frac{3}{2k} \\ &\ge \frac{1}{2} + \frac{3}{2\cdot 99} > \frac{1}{2} + \frac{1}{2\cdot 67} = \frac{34}{67}. \end{aligned}$$

Therefore the minimum value of $P(k)$ in the required range is $\frac{34}{67}$.

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