THE MATHEMATICAL ASSOCIATION OF AMERICA
AMERICAN MATHEMATICS COMPETITIONS

23rd Annual

AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

SOLUTIONS PAMPHLET

Tuesday, March 8, 2005

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution. *Remember that reproduction of these solutions is prohibited by copyright.*

Correspondence about the problems and solutions for this AIME and orders for any of the publications listed below should be addressed to:

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1. (Answer: 942)

   Let r be the radius of each of the six congruent circles, and let A and B be the centers of two adjacent circles. Join the centers of adjacent circles to form a regular hexagon with side $2r$. Let O be the center of $\mathcal{C}$. Draw the radii of $\mathcal{C}$ that contain A and B. Triangle ABO is equilateral, so $OA = OB = 2r$. Because each of the two radii contains the point where the smaller circle is tangent to $\mathcal{C}$, the radius of $\mathcal{C}$ is $3r$, and $K = \pi\left((3r)^2 - 6r^2\right) = 3\pi r^2$. The radius of $\mathcal{C}$ is 30, so $r = 10$, $K = 300\pi$, and $\lfloor K \rfloor = 942$.

2. (Answer: 012)

   Suppose that the nth term of the sequence S_k is 2005. Then $1+(n-1)k = 2005$ so $k(n-1) = 2004 = 2^2 \cdot 3 \cdot 167$. The ordered pairs $(k, n-1)$ of positive integers that satisfy the last equation are $(1, 2004)$, $(2, 1002)$, $(3, 668)$, $(4, 501)$, $(6, 334)$, $(12, 167)$, $(167, 12)$, $(334, 6)$, $(501, 4)$, $(668, 3)$, $(1002, 2)$, and $(2004, 1)$. Thus the requested number of values is 12. Note that the number of divisors of $2^2 \cdot 3 \cdot 167$ can also be found to be $(2+1)(1+1)(1+1) = 12$ by using the formula for the number of divisors.

3. (Answer: 109)

   There are two types of integers n that have three proper divisors. If $n = pq$, where p and q are distinct primes, then the three proper divisors of n are 1, p, and q; and if $n = p^3$, where p is a prime, then the three proper divisors of n are 1, p, and p^2. Because there are 15 prime numbers less than 50, there are $\binom{15}{2} = 105$ integers of the first type. There are 4 integers of the second type because 2, 3, 5, and 7 are the only primes with squares less than 50. Thus there are $105 + 4 = 109$ integers that meet the given conditions.

4. (Answer: 294)

   Let the square formation have s rows and s columns, and let the rectangular formation have x columns and $(x+7)$ rows. Then $x(x+7) = s^2 + 5$, so $x^2 + 7x - (s^2+5) = 0$. Because x is a positive integer, $x = \left(-7 + \sqrt{4s^2 + 69}\right)/2$, and there must be a positive integer k for which $k^2 = 4s^2 + 69$. Then $69 = k^2 - 4s^2 = (k+2s)(k-2s)$. Therefore $(k+2s, k-2s) = (69, 1)$ or $(23, 3)$. Thus $(k, s) = (35, 17)$ or $(13, 5)$, and the maximum number of members this band can have is $17^2 + 5 = 294$.

5. (Answer: 630)

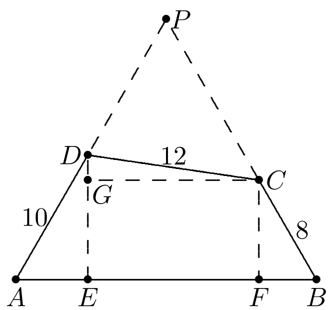
   First consider the orientation of the coins. Label each coin U or D depending upon whether it is face up or face down, respectively. Then for each arrangement of the coins, there is a corresponding string consisting of a total of eight U's and D's that is formed by listing each coin's label starting from the bottom of the stack. An arrangement in which no two adjacent coins are face to face corresponds to such a string that does not contain UD. Thus the first U in the string must have no D's after it. The first U may appear in any of eight positions or not at all, for a total of nine allowable strings. For each of these nine strings, there are $\binom{8}{4}$ ways to pick the positions for the four gold coins, and the positions of the silver coins are then determined. Thus there are $9 \cdot \binom{8}{4} = 630$ arrangements that satisfy Robert's rules of order.

6. (Answer: 045)

The given equation is equivalent to $x^4 - 4x^3 + 6x^2 - 4x + 1 = 2006$, that is, $(x-1)^4 = 2006$. Thus $(x-1)^2 = \pm\sqrt{2006}$, and $x - 1 = \pm\sqrt[4]{2006}$ or $\pm i\sqrt[4]{2006}$. Therefore the four solutions to the given equation are $1 \pm \sqrt[4]{2006}$ and $1 \pm i\sqrt[4]{2006}$. Then $P = (1 + i\sqrt[4]{2006})(1 - i\sqrt[4]{2006}) = 1 + \sqrt{2006}$, so $\lfloor P \rfloor = 45$.

7. (Answer: 150)

Draw lines containing D and C that are perpendicular to $\overline{AB}$ at E and F, respectively. Then $AE = 5$, $DE = 5\sqrt{3}$, $BF = 4$, and $CF = 4\sqrt{3}$. Now draw a line containing C that is perpendicular to $\overline{DE}$ at G. Because $EFCG$ is a rectangle, $GE = CF = 4\sqrt{3}$, so $DG = DE - GE = \sqrt{3}$. Apply the Pythagorean Theorem to $\triangle DGC$ to find that $\sqrt{141} = GC = EF$. Then $AB = AE + EF + FB = 9 + \sqrt{141}$, and $p + q = 150$.



OR

Let P be the intersection of $\overrightarrow{AD}$ and $\overrightarrow{BC}$, and let $AD = a$, $AB = b$, $BC = c$, $CD = d$, $DP = x$, and $PC = y$. Then $\triangle ABP$ is equilateral, and $x + a = y + c = b$. Apply the Law of Cosines to $\triangle DCP$ to obtain $x^2 + y^2 - xy = d^2$, and then substitute to get $(b-a)^2 + (b-c)^2 - (b-a)(b-c) = d^2$. Expand and simplify to get

$$a^2 + b^2 + c^2 = d^2 + ab + bc + ac.$$

For the given quadrilateral, this yields $10^2 + b^2 + 8^2 = 12^2 + 10b + 8b + 80$, and then $b^2 - 18b - 60 = 0$, whose positive solution is $9 + \sqrt{141}$. Thus $p + q = 150$.

8. (Answer: 113)

   Let $y = 2^{111x}$. The given equation is equivalent to $(1/4)y^3 + 4y = 2y^2 + 1$, which can be simplified to $y^3 - 8y^2 + 16y - 4 = 0$. Since the roots of the given equation are real, the roots of the last equation must be positive. Let the roots of the given equation be x_1, x_2, and x_3, and let the roots of the equation in y be y_1, y_2, and y_3. Then $x_1 + x_2 + x_3 = (1/111)(\log_2 y_1 + \log_2 y_2 + \log_2 y_3) = (1/111)\log_2(y_1y_2y_3) = (1/111)\log_2 4 = 2/111$, and $m + n = 113$.

   Note: It can be verified that $y^3 - 8y^2 + 16y - 4 = 0$ has three positive roots by sketching a graph.

9. (Answer: 074)

   A cube can be oriented in 24 ways because each of the six faces can be on top and each of the top face's four edges can be at the front. There are eight corner cubes in the large cube. For the corner cubes, six orientations will expose three orange faces. This is because there are two sets of three orange faces that can be exposed. For each such set, each of the three orange faces can appear in a given position, and the positions of the other two are then determined. Thus the probability that all corner cubes expose three orange faces is $(6/24)^8 = (1/4)^8$. For cubes at the center of an edge, there are 10 orientations that expose two orange faces. This is because there are five sets of two orange faces that share an edge, and each such set can appear in two orientations. The probability that all 12 of these edge cubes expose two orange faces is $(10/24)^{12} = (5/12)^{12}$. A cube that is in the center of a face can have any of the four orange faces outward in four orientations, and thus there is a probability of $(16/24)^6 = (2/3)^6$ that each center cube exposes an orange face. Thus the probability that the entire surface of the larger cube is orange is

   $$\left(\frac{1}{4}\right)^8 \cdot \left(\frac{5}{12}\right)^{12} \cdot \left(\frac{2}{3}\right)^6 = \frac{5^{12}}{2^{34} \cdot 3^{18}},$$

   and $a + b + c + p + q + r = 12 + 34 + 18 + 5 + 2 + 3 = 74$.

   OR

   The large cube contains eight corner unit cubes, twelve unit cubes at the center of an edge, and six unit cubes at the center of a face. All visible faces of a unit cube are orange if and only if the shared edge of its two unpainted faces, except perhaps for an endpoint, is in the interior of the large cube. The number of edges interior to the large cube is three for a corner cube, five for a cube at the center of an edge, and eight for a cube at the center of a face. Thus the probability that the entire surface of the large cube is orange is

   $$\left(\frac{3}{12}\right)^8 \cdot \left(\frac{5}{12}\right)^{12} \cdot \left(\frac{8}{12}\right)^6 = \frac{5^{12}}{2^{34} \cdot 3^{18}},$$

   and, as above, $a + b + c + p + q + r = 74$.

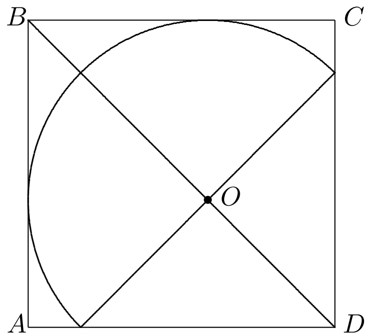
10. (Answer: 047)

Let l be the line containing the median to side $\overline{BC}$. Then l must contain the midpoint of $\overline{BC}$, which is $((12+23)/2, (19+20)/2) = (35/2, 39/2)$. Since l has the form $y = -5x + b$, substitute to find that $b = 107$. Thus the coordinates of A are $(p, -5p + 107)$. Now compute p using the fact that the area of the triangle with coordinates $(0,0)$, (x_1, y_1), and (x_2, y_2) is the absolute value of $(1/2)\begin{vmatrix} x_1 & y_1 \\ x_2 & y_2 \end{vmatrix}$. To use this formula, translate the point $(12, 19)$ to the origin, and, to preserve area, translate points A and C to $A' = (p-12, -5p+107-19) = (p-12, -5p+88)$ and $C' = (23-12, 20-19) = (11, 1)$, respectively. Apply the above formula to obtain $(1/2)|(p-12)\cdot 1 - (-5p+88)\cdot 11| = 70$, which yields $|56p - 980| = 140$. Thus $p = 15$ or $p = 20$, and the corresponding values of q are 32 and 7, respectively. The largest possible value of $p + q$ is 47.

OR

Let M be the midpoint of $\overline{BC}$. The coordinates of M are $(35/2, 39/2)$. An equation of line AM is $y = -5x+107$, so the coordinates of A can be represented as $(p, -5p + 107)$. Line BC has equation $x = 11y - 197$ or, equivalently, $x - 11y + 197 = 0$, so the distance from A to line BC is $\dfrac{|p - 11(-5p + 107) + 197|}{\sqrt{1^2 + 11^2}} = \dfrac{|56p - 980|}{\sqrt{122}}$. The length of $\overline{BC}$ is $\sqrt{1^2 + 11^2}$, so $70 = [\triangle ABC] = \dfrac{1}{2}\sqrt{122} \cdot \dfrac{|56p - 980|}{\sqrt{122}}$. Solve to obtain $p = 20$ or $p = 15$, so $p + q = p - 5p + 107 = -4p + 107$. Thus $p + q = 27$ or 47, and the maximum value of $p + q$ is 47.

11. (Answer: 544)

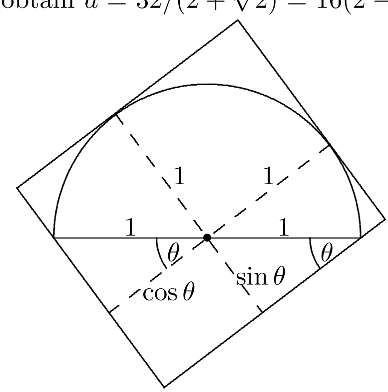


Any semicircle that is contained in a square can be translated to yield a semicircle that is inside the square and tangent to two adjacent sides of the square. Name the square $ABCD$, and, without loss of generality, consider semicircles tangent to $\overline{AB}$ and $\overline{BC}$. The center O of any such semicircle is equidistant from $\overline{AB}$ and $\overline{BC}$ and therefore must lie on diagonal $\overline{BD}$. Let $\mathcal{S}$ be the circle determined by the semicircle. The placement of O that yields the largest semicircle is the point at which the intersection of $\mathcal{S}$ with the square region is a semicircle. This is because if O were placed closer to B, the radius of $\mathcal{S}$ would be smaller; and if O were placed farther from B, the intersection of $\mathcal{S}$ and the square region would be an arc of less than 180°, and so no semicircle centered at O would fit in the square. Because $\overline{BD}$ is a symmetry line for the desired semicircle and the square, $\overline{BD}$ is the perpendicular bisector of the diameter joining the endpoints of the semicircle.

Let the radius of the largest semicircle be r. Then the distance from O to $\overline{AD}$ is $r/\sqrt{2}$. The sum of the distances from O to $\overline{AD}$ and $\overline{BC}$ is 8, so the diameter of the largest semicircle that fits in the square is $2r = \dfrac{2\cdot 8}{1+\frac{1}{\sqrt{2}}} = 16(2-\sqrt{2}) = 32-\sqrt{512}$. Thus $m+n = 32+512 = 544$.

OR

Consider a related problem: find the least possible side-length of a square that contains a semicircle for which the diameter is fixed. Let the orientation of the square with respect to the semicircle be such that the sides of the square and the diameter of the semicircle determine angles of θ and $(\pi/2-\theta)$, with $0 \le \theta \le \pi/2$. Without loss of generality, assume the radius of the semicircle is 1. The opposite sides of the square are parallel, and, in general, if a pair of parallel lines touch a semicircle, then one will be tangent to its arc and one will contain an endpoint of the diameter. The diagram shows two perpendicular pairs of parallel lines with all four lines touching the semicircle. The distance between the lines in each pair is as small as possible because each line of the pair touches the semicircle. Thus the greater of these two distances is the minimal side-length of a square in this orientation that contains the semicircle. Because the distances between the pairs of parallel lines are $1+\cos\theta$ and $1+\sin\theta$, the minimal side-length of a square in this orientation that contains the semicircle is $\max\{1+\cos\theta, 1+\sin\theta\}$. For θ between 0 and $\pi/2$, this length is minimum when $\theta = \pi/4$, so the minimum length for any orientation of the square is $1+\sqrt{2}/2 = (2+\sqrt{2})/2$. To find the maximum value of d, solve the proportion $\dfrac{d}{2} = \dfrac{8}{(2+\sqrt{2})/2}$ to obtain $d = 32/(2+\sqrt{2}) = 16(2-\sqrt{2}) = 32-\sqrt{512}$. Thus $m+m = 544$.



12. (Answer: 025)

If d is a divisor of n, then so is $\frac{n}{d}$. Thus the number of divisors of n must be even unless, for some d, $d = \frac{n}{d}$, that is, $n = d^2$. Hence $\tau(n)$ is odd if and only if n is a square. Therefore, as n increases, $S(n)$ changes parity only when n is a square. Thus $S(n)$ is odd for $1^2 \le n \le 2^2 - 1$, even for $2^2 \le n \le 3^2 - 1$, odd for $3^2 \le n \le 4^2 - 1$, and so on. Consequently

$$\begin{aligned} a &= (2^2 - 1 - 1^2 + 1) + (4^2 - 1 - 3^2 + 1) + (6^2 - 1 - 5^2 + 1) + \cdots + (44^2 - 1 - 43^2 + 1) \\ &= (2^2 - 1^2) + (4^2 - 3^2) + (6^2 - 5^2) + \cdots + (44^2 - 43^2) \\ &= (2+1)(2-1) + (4+3)(4-3) + (6+5)(6-5) + \cdots + (44+43)(44-43) \\ &= 1 + 2 + 3 + \cdots + 44 = 44 \cdot 45/2 = 990. \end{aligned}$$

Then $b = 2005 - 990 = 1015$, so $|a - b| = 25$.

13. (Answer: 083)

Let $P(a, b)$ be the number of permissible paths from $(0, 0)$ to (a, b), and define $P(0, 0) = 1$. If $a = 0$ or $b = 0$, then $P(a, b) = 1$. If $a > 0$ and $b > 0$, then the particle can reach (a, b) from any of the points $(a-1, b)$, $(a-1, b-1)$, $(a, b-1)$. Also, if the particle is at $(a-1, b)$ and next moves to (a, b), then the particle must have entered the row containing these points on a diagonal path, not a vertical one, because otherwise one of the moves to the right towards (a, b) would make a right angle. Thus the number of ways to travel to $(a-1, b)$ in such a way that the path continuing to (a, b) is permissible is

$$P(0, b-1) + P(1, b-1) + \cdots + P(a-2, b-1).$$

A similar statement holds for paths the particle can take to $(a, b-1)$ that result in a permissible path to (a, b). Thus,

$$P(a, b) = \left(\sum_{i=0}^{a-2} P(i, b-1)\right) + P(a-1, b-1) + \left(\sum_{j=0}^{b-2} P(a-1, j)\right).$$

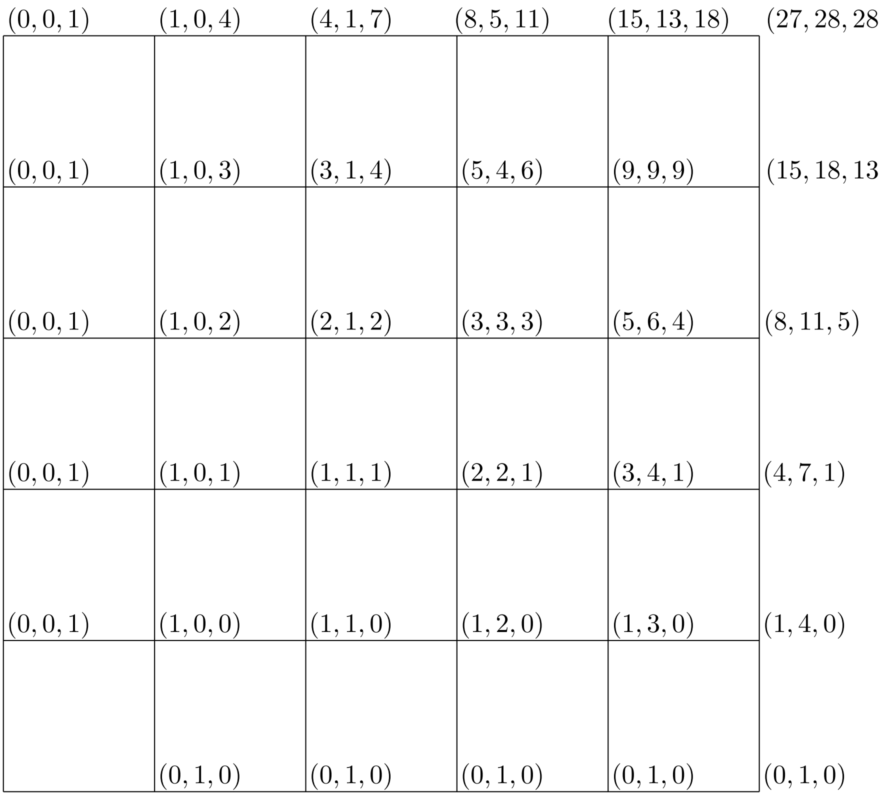
This is simply the sum of the number of permissible paths from the origin to points on the top or right side of the rectangle with vertices $(0, 0)$, $(a-1, 0)$, $(a-1, b-1)$, $(0, b-1)$. With this realization, calculate the number of permissible paths to each lattice point as shown in the grid below, to find that there are 83 permissible paths.

1	5	12	24	46	83
1	4	8	15	27	46
1	3	5	9	15	24
1	2	3	5	8	12
1	1	2	3	4	5
1	1	1	1	1	1

OR

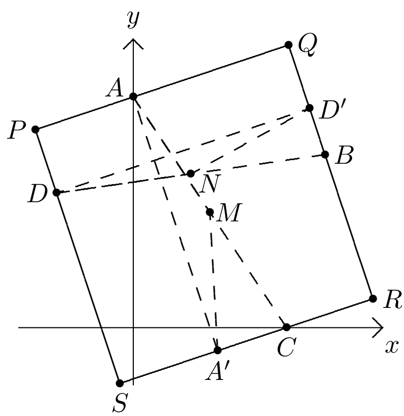
Label each vertex with an ordered triple whose first component represents the number of paths that end at that vertex with a diagonal step, whose second component represents the number of paths that end at that vertex with a step to the right, and whose third component represents the number of paths that end at that vertex with a step up. Begin by labeling the vertices with the triples

$(0, 0, 1)$, $(0, 1, 0)$, and $(1, 0, 0)$ as shown. For the other vertices, the first component at a vertex is the sum of the three components at the vertex diagonally below it and to the left, the second component at a vertex is the sum of the first two components at the vertex directly to its left, and the third component at a vertex is the sum of the first and third components at the vertex directly below it. Use these relationships to complete the labeling of the grid. The requested number of paths is the sum of the components in the upper right vertex, that is, $27 + 28 + 28 = 83$.



14. (Answer: 936)

Because $\overline{AC}$ and $\overline{BD}$ intersect, A and C must be on opposite sides of the square, as must B and D. Name the vertices of the square P, Q, R, and S so that A is on $\overline{PQ}$, B is on $\overline{QR}$, C is on $\overline{RS}$, and D is on $\overline{SP}$. Let h and v represent the horizontal and vertical change, respectively, from P to Q. Then A', the projection of A onto $\overline{RS}$, has coordinates $(0+v, 12-h)$. Let M be the midpoint of $\overline{AC}$. Then M has coordinates $(4,6)$ and $MA = MA'$, so $4^2+6^2 = (v-4)^2+(h-6)^2$. Similarly, D', the projection of D onto $\overline{QR}$, has coordinates $(-4+h, 7+v)$, N, the midpoint of $\overline{BD}$, has coordinates $(3,8)$, and $ND = ND'$, so $7^2+1^2 = (h-7)^2+(v-1)^2$. The two equations imply that $12h+8v = h^2+v^2 = 14h+2v$, and so $h = 3v$. Then $12h+8v = h^2+v^2$ yields $36v+8v = (3v)^2+v^2$, so $v = 44/10$. Thus $K = h^2+v^2 = 10v^2 = 10(44^2/10^2) = 1936/10$, so $10K = 1936$, and the requested remainder is 936.



OR

Let m be the slope of the side of the square containing B. The line containing this side has equation $y-9 = m(x-10)$ or $mx-y+(9-10m) = 0$. Similarly, the line containing the side containing C has equation $y = (-1/m)(x-8)$ or $x+my-8 = 0$. Because the distance from D to the first line is equal to the distance from A to the second line,

$$\frac{|-4m-7+9-10m|}{\sqrt{m^2+1}} = \frac{|12m-8|}{\sqrt{m^2+1}}.$$

Solve to obtain $m = 5/13$ or $m = -3$. For the square obtained with the first slope, some of the points are on extended sides of the square. This is because A and C are on opposite sides of the line with slope $5/13$ that contains B. Thus $m = -3$. Then $K = \dfrac{(12m-8)^2}{m^2+1} = 44^2/10$, so $10K = 1936$, and the requested remainder is 936.

Query: For four arbitrary points A, B, C, D in the plane, what are the necessary and sufficient conditions that a unique square $\mathcal{S}$ exists?

15. (Answer: 038)

Let M be the midpoint of $\overline{AB}$, and let S and N be the points where median $\overline{CM}$ meets the incircle, with S between C and N. Let $\overline{AC}$ and $\overline{AB}$ touch the incircle at R and T, respectively. Assume, without loss of generality, that T is between A and M. Then $AR = AT$. Use the Power-of-a-Point Theorem to conclude that

$$MT^2 = MN \cdot MS \quad \text{and} \quad CR^2 = CS \cdot CN.$$

Because $CS = SN = MN$, conclude that $CR = MT$, and

$$AC = AR + CR = AT + MT = AM = \frac{1}{2}AB = 10.$$

Let $s = (1/2)(AB + BC + CA)$. Then $AT = s - BC$, and

$$MT = MA - AT = \frac{1}{2}AB - s + BC = \frac{BC - AC}{2} = \frac{BC - 10}{2}.$$

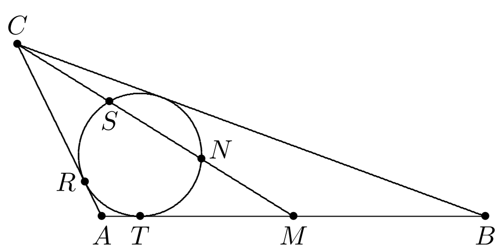
But $MT^2 = MN \cdot MS = (2/9)CM^2$, so $\dfrac{BC - 10}{2} = CM \cdot \dfrac{\sqrt{2}}{3}$. Hence

$$CM = \frac{3}{2\sqrt{2}} \cdot (BC - 10).$$

Apply the Law of Cosines to triangles AMC and ABC to obtain

$$\frac{10^2 + 10^2 - CM^2}{2 \cdot 10 \cdot 10} = \cos A = \frac{10^2 + 20^2 - BC^2}{2 \cdot 10 \cdot 20}.$$

Then $BC^2 = 100 + 2 \cdot CM^2$, so $BC^2 = 100 + (9/4)(BC - 10)^2$. The solutions of this equation are 26 and 10, but $BC > AB - AC = 10$. It follows that $BC = 26$, and then that $CM = 12\sqrt{2}$. The length of the altitude from A in isosceles $\triangle AMC$ is therefore $2\sqrt{7}$. Thus $[ABC] = 2[AMC] = 24\sqrt{14}$, and $m + n = 38$.



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