Rotations in $\mathbb{E}^3$ and Möbius Transform

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A direct bijection of rotations in Euclidean 3-space $\mathbb{E}^3$ and Möbius transform(fractional transform) in $\mathbb{C}$ will be constructed.

1 Quaternions

1.1 Definition of Quaternions

Define the space $\mathbb{H}$ as follows.

$$\mathbb{H} \equiv \{q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k} \mid a, b, c, d \in \mathbb{R}\}$$

A quaternion q is an element of $\mathbb{H}$. Multiplication and addition of two quaternions are also defined on $\mathbb{H}$.

$$\begin{aligned} q &= q_0 + q_1\mathbf{i} + q_2\mathbf{j} + q_3\mathbf{k} \\ u &= u_0 + u_1\mathbf{i} + u_2\mathbf{j} + u_3\mathbf{k} \\ q + u &= (q_0 + u_0) + (q_1 + u_1)\mathbf{i} + (q_2 + u_2)\mathbf{j} + (q_3 + u_3)\mathbf{k} \\ qu &= (q_0u_0 - q_1u_1 - q_2u_2 - q_3u_3) \\ &+(q_0u_1 + q_1u_0 + q_2u_3 - q_3u_2)\mathbf{i} \\ &+(q_0u_2 + q_2u_0 + q_3u_1 - q_1u_3)\mathbf{j} \\ &+(q_0u_3 + q_3u_0 + q_1u_2 - q_2u_1)\mathbf{k} \end{aligned}$$

One can infer that quaternions are a kind of generalised complex numbers. Unlike ordinary numbers(which are fields), commutative law does not hold for multiplication of quaternions. For example, $\mathbf{ij} = -\mathbf{ji} = \mathbf{k}$. By adopting the convention to interpret $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ as unit vectors of x, y, and z axis of Euclidean 3 space $\mathbb{E}^3$, one can express multiplication of quaternions through vector algebra.

$$\begin{aligned} q &= (q_0, \vec{q}) \\ u &= (u_0, \vec{u}) \\ qu &= (q_0u_0 - \vec{q}\cdot\vec{u},\ q_0\vec{u} + u_0\vec{q} + \vec{q}\times\vec{u}) \end{aligned}$$

Given a quaternion q, one can defined the conjugate quaternion $\bar{q}$ as well.

$$\begin{aligned} q &= (q_0, \vec{q}) \\ \bar{q} &= (q_0, -\vec{q}) = -\tfrac{1}{2}(q + \mathbf{i}q\mathbf{i} + \mathbf{j}q\mathbf{j} + \mathbf{k}q\mathbf{k}) \end{aligned}$$

The norm of a quaternion q, $|q|$, is defined in an analoguous way to complex numbers.

$$|q| = \sqrt{q\bar{q}} = \sqrt{\bar{q}q} = \sqrt{q_0^2 + \vec{q}\cdot\vec{q}}$$

From now on, the unit imaginary numbers **i**, **j**, and **k** will be written as i, j, and k since there is no risk of confusion.

1.2 Rotations in $\mathbb{E}^3$ and Quaternions

Let's call unit quaternions as quaternions with unit norm. Rotation of a vector q in $\mathbb{E}^3$ can be written with unit quaternions.

$$q = (0,\ \vec{q})$$
$$u = (\cos(\theta/2), \sin(\theta/2)\ \hat{n}),\ |\hat{n}|^2 = 1$$
$$uq\bar{u} = (0,\ \hat{n}(\hat{n}\cdot\vec{q}) + \cos\theta[\vec{q} - \hat{n}(\hat{n}\cdot\vec{q})] + \sin\theta\ \{\hat{n}\times[\vec{q} - \hat{n}(\hat{n}\cdot\vec{q})]\})$$

Inspecting the result, one concludeds that the vector $\vec{q}$ was rotated by angle θ with $\hat{n}$ as the axis of rotation. This means that an element of SO(3), the group of rotations in $\mathbb{E}^3$ preserving the origin, can be matched to each unit quaternion. The unit quaternions u and $-u$ describes the same rotation, so this correspondence is 2-to-1.

On the other hand, one can match unit quaternions to 2×2 matrices by the following rule. The i appearing on the right side is just a normal unit imaginary number $i = \sqrt{-1}$ of complex numbers, and $\vec{\sigma}$ refers to the Pauli matrices σ_1, σ_2, and σ_3 in vector form.

$$q = (q_0,\ \vec{q}) \leftrightarrow U = q_0\mathbb{I}_2 + (-i)\vec{q}\cdot\vec{\sigma}$$

This matching rule preserves the algebraic structure of unit quaternions, so is an isomorphism. As unit quaternions q are matched the corresponding 2×2 matrix U also exhibits the following feature.

$$\det U = 1,\ U^\dagger = U^{-1}$$

It is also possible to show that the matched matrices have determinant 1, and all unitary 2×2 matrices with determinant 1 can be expressed in such a way. Thus, the group(group multiplication given by usual multiplication rule) of unit quaternions and SU(2) are isomorphic, having the topological structure of 3-dimensional sphere S^3. The group SO(3) is obtained by identifying the antipodal points of unit quaternion space S^3, so SO(3) has the topological structure of RP^3. Nature classifies particles into two groups by their ability to discern the two; fermions that do and bosons that don't.

1.3 Spliting Quaternions

The (field of) complex numbers $\mathbb{C}$ can be identified with the subset of quaternions $\mathbb{H}(\mathbb{C} = \{z = x + yi \mid x, y \in \mathbb{R}\})$, and usual multiplication rule including commutative law applies to $\mathbb{C}$. Since most people are accostomed to multiplication rule of complex numbers, let's divide a quaternion q into two complex number parts.

$$
\begin{aligned}
q &= a + bi + cj + dk \\
&= (a + bi) + (cj + dk) \\
&= (a + bi) + (c + di)j
\end{aligned}
$$

$$
\begin{aligned}
\therefore q &= \alpha + \beta j \\
\alpha &= a + bi \\
\beta &= c + di
\end{aligned}
$$

The unit quaternion j defies to go away. Complex numbers and j interact in the following ways.

$$
\begin{gathered}
z \in \mathbb{C} \\
zj = j\bar{z}, \ jz = \bar{z}j \\
jj = -1 \\
jzj = -\bar{z}
\end{gathered}
$$

$\bar{z}$ refers to complex conjugation$(z = x + yi \leftrightarrow \bar{z} = x - yi)$. When j and a complex number z is written in series, interchanging their places conjugates z.

When a quaternion q is split, square of its norm is the sum of square norm of its constituents.

$$
\begin{gathered}
q = \alpha + \beta j, \ \alpha, \beta \in \mathbb{C} \\
|q|^2 = q\bar{q} = \bar{q}q = |\alpha|^2 + |\beta|^2
\end{gathered}
$$

Thus, the complex numbered parts of a unit quaternion u satisfies the following condition.

$$
\begin{gathered}
u = \alpha + \beta j, \ \alpha, \beta \in \mathbb{C} \\
|u|^2 = |\alpha|^2 + |\beta|^2 = 1
\end{gathered}
$$

2 Stereographic Projection by Quaternions

The definition of stereographic projection will not be given. If in need of review, consult Wikipedia: http://en.wikipedia.org/wiki/Stereographic_projection

2.1 Mapping $\mathbb{C}$ to a Plane in $\mathbb{E}^3$

The unit quaternions induces rotations on pure quaternions, the quaternions without real component. To induce rotations by unit quaternions, one needs to map complex numbers to pure quaternions. An obviously simple way to do this is to multiply complex numbers by j.

$$
z = x + yi \in \mathbb{C} \to z' = zj = xj + yk \in \mathbb{H}
$$

2.2 Mapping a Plane to a Sphere

We now project the plane onto a sphere. Thinking of the space of pure quaternions as $\mathbb{E}^3$, one can use the vector point $q_0 = -i$ to project the plane $\mathbb{C}j \equiv \{z' = xj + yk \mid x, y \in \mathbb{R}\}$ onto a sphere $S^2 \equiv \left\{q = ai + bj + ck \;\middle|\; a, b, c \in \mathbb{R}, a^2 + b^2 + c^2 = 1\right\}$. This is elementary vector algebra;

find nonzero α such that

$$q = q_0 + \alpha(z' - q_0), q\bar{q} = 1$$

After some calculations, it can be found that

$$\alpha = \frac{2}{1+|z|^2}$$

$$\therefore q = \frac{1}{1+|z|^2}\left[2z' + (1-|z|^2)i\right] = \frac{1}{1+|z|^2}\left[2zj + (1-|z|^2)i\right] \equiv P(z)$$

2.3 Mapping a Sphere to a Plane

The inverse process of mapping the sphere to the plane is hard to calculate from the algebraic expression of mapping the plane to the sphere. The way of working around this is to realise that j and k components of q are multiplied by the inverse of 1 plus i component. Expressed in equations,

$$q = ai + bj + ck \to z' = \frac{bj + ck}{1+a}$$

To get the i component algebraically, one can utilise the relation

$$q = ai + bj + ck \leftrightarrow iqi = -ai + bj + ck$$

Cranking through calculations, one gets the algebraic expression for mapping the sphere to the plane.

$$q = ai + bj + ck, iqi = -ai + bj + ck$$
$$q + iqi = 2(bj + ck)$$
$$-(iq + qi) = 2a$$

$$\therefore z' = zj = \frac{q + iqi}{2 - (qi + iq)}$$

As z is the variable we will be using, we multiply both sides with $-j$.

$$z = P^{-1}(q) \equiv \frac{(q + iqi)(-j)}{2 - (qi + iq)}$$

One can show that $P^{-1}(P(z)) = z$ by direct calculation, thereby confirming that the expression obtained is indeed the inverse projection from the sphere onto the plane.

2.4 Adding up

The quantity we wish to calculate is the Möbius transform corresponding to the rotation induced by a unit quaternion v. Using the defined functions $P : \mathbb{C} \to S^2$ and $P^{-1} : S^2\backslash\{-i\} \to \mathbb{C}$ the Möbius transformation $w = w(z)$ can be expressed as

$$w = P^{-1}(vP(z)\bar{v})$$

Split v into two complex numbers by $v = \alpha + \beta j, |\alpha|^2 + |\beta|^2 = 1$. After some tedius calculations, one can show that

$$w = \frac{\alpha\beta i(z - i\beta/\alpha)(\bar{z} - i\alpha/\beta)}{|\beta|^2(z + i\bar{\alpha}/\bar{\beta})(\bar{z} - i\alpha/\beta)}$$

Tidying up, the Möbius transformation corresponding to the rotation induced by $v = \alpha + \beta j$ on S^2 is

$$w = \frac{\alpha z - i\beta}{-i\bar{\beta}z + \bar{\alpha}}$$

or the transformation corresponding to the matrix

$$M = \begin{pmatrix} \alpha & -i\beta \\ -i\bar{\beta} & \bar{\alpha} \end{pmatrix}$$