Equilibrium Dividend Policy

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Abstract

A firm's optimal dividend policy is examined. This study differs from previous research in that: first, while most of the previous studies use discrete-time model or descriptive methods, such as regression analysis and event study, this study uses continuous-time model; second, whereas most of the past studies focus on the behavior of individual firms, this examines a general market equilibrium condition to develop an equilibrium valuation model; third, unlike other major studies, we introduce the concept of an opportunity cost as one of the variables affecting dividend policy. Under this framework and some assumptions, we succeed in deriving an optimal equilibrium level of dividend in the aggregate market. By comparing our results in this study to those of previous studies using comparative static analysis, we find that M&M (Miller & Modigliani) theories can be applied to the real world only under very unrealistic assumptions. In addition, we show the mechanism of how a firm's debt policy, the cost of capital, and the corporate tax rate affect the dividend policy of a firm. This is the first theoretical approach to incorporate aggregate and continuous-time framework into the dividend policy, and the empirical test of our model must be made to prove the validity of the model in the future.

I. Introduction

As Brealey & Myers (1991) point out, the dividend policy of a firm is one of the most controversial subjects in the modern finance theory. They argue that Many people believe dividends are good, others believe they are bad, and still others believe they are irrelevant. If pressed, we stand somewhere in the middle, but we cannot be dogmatic about it (p.922). Miller & Modigliani (1961) showed that, when investors can create any payout pattern they want by selling and purchasing shares, the expected return required to hold these shares will be invariant to the way in which firms 'package' gross-dividend payments and new issues of stock. Since neither the firms expected future net cash flows nor its discount rate is affected by the choice of dividend policy per se, its current market value cannot be changed by a change in that policy. Thus, a dividend policy is irrelevant. Although, under Miller & Modigliani proposition, there are no priori
reasons for firms to follow any systematic dividend policy, there are also no penalties if they choose to do so. It is important to note that the key to the Miller & Modigliani argument is that the investment decisions of a firm are completely independent of dividend policy. The firm can pay any level of dividends it wishes without affecting investment decisions. If dividends plus desired investment outlays use more cash flow than is provided from operations, the firm should seek external financing (e.g., equity). Thus, the desire to maintain a level of dividends should never affect the investment decision. But, the assumptions under which they derive their conclusion are not so realistic, and they obviously do not hold precisely. Firms and investors do pay income taxes, firms do incur floatation costs, investors do incur transaction costs, and both taxes and transaction costs could cause the cost of the capital of a firm to be affected by dividend policy. Further, managers often have better information than outside investors. Therefore, their conclusions on dividend irrelevancy could not be valid under the real world conditions.

In some respects, the most critical assumption inherent in Miller & Modigliani theory is that the dividend policy does not affect investors' required rate of return on equity. This issue has been hotly debated in academic circles. Gordon (1959, 1962) and Lintner (1962), on the one hand, argued that the required rate of return increases as the dividend payout is reduced, because investors can be more sure of receiving dividend payments than the income from capital gains which are expected to result from retained earnings. They say, in effect, the investors value a dollar of expected dividends more highly than a dollar of expected capital gains because the dividend yield component is less risky than the component in the total expected return.\(^1\) On the other hand, Miller, Modigliani and Brennan (1971) call the Gordon & Lintner argument The Bird-in-the-Hand Fallacy because, in their view, many investors are going to reinvest their dividends in the same or similar firms anyway, and, in any event, the riskiness of the firms' cash flows to investors in the long run is determined only by the riskiness of its cash flows from operating assets, and not by its dividend payout policy. Litzenberger & Ramaswami (1979) propose another theory, based on tax effects\(^2\). Since dividends are effectively taxed at higher rates than capital gains, investors should require higher rates of return on stocks with high dividend yields. Thus, according to their theory, a firm should pay a low (or zero) dividend in order to maximize its value. On the other hand, Black & Scholes (1974), Miller & Scholes (1978, 1982), Hess (1983), Eades, Hess and Kim (1984) present the analysis and evidence suggesting that, as an empirical manner, tax effects per se do not appear to affect the cost of capital.

Besides these three major theories describing the dividend policy of a firm, some other researchers propose some different paradigms. Easterbrook (1984) proposes an agency-theory explanation for dividends. Although some of these analyses might provide reasons to believe that investors are not indifferent between cash dividends and capital gains, the empirical evidence to date is still inconclusive for rejecting the Miller & Modigliani proposition.

As we discussed briefly, most of the previous studies (with few exceptions, such as Shiller (1981), Marsh & Merton (1987), Hakansson (1982) on both normative and positive models of dividends) focused on the micro behavior of individual firms. Even though the foregoing studies appear to support Miller & Modigliani proposition from the viewpoint of the micro behavior of individual firms, they do not necessarily rule out the possibility that there may exist an aggregate equilibrium supply of dividends. This point basically motivates this study. Thus, this study is different from previous studies in that it uses an aggregate equilibrium model similar to Marsh & Merton (1987). In addition, in this study, continuous-time stochastic model is utilized to describe the behavior of dividends and other key variables affecting a dividend policy. From the market equilibrium condition, the optimal dividend policy for the individual firms can be derived.

\(^1\) This argument is sometimes called 'Bird-in-the-Hand' theory.
\(^2\) This theory is called 'Tax Differential Theory.'
The model specified here has a similar form to real option models, which are widely used in the valuation of real assets.\(^3\)

II. The Market Setting

1. Assumptions

The economy is constrained by the following set of assumptions.

(A1) Capital market is perfectly competitive. Since there exists sufficiently large number of firms issuing equity in the capital market and, actually, the entry and exit to the capital market is almost free, this assumption seems reasonable in the real world. Grenadier (1995) uses the same assumption to derive competitive equilibrium lease rate in the market. (A2) The investment decision of a firm is independent of the dividend policy. This assumption is directly from the results of previous studies, such as Miller & Modigliani (1961), Fama (1974), and Miller & Scholes (1978). Since the goal of this study is not to examine the characteristics of the dividend policy of an individual firm, but to derive aggregate equilibrium solution in the market as a whole, individual firms investment behavior is assumed, and the aggregate solution will be focused in the efficient manner. If this assumption does not hold, dividend policy might be a function of investment decision, leading to a more complicated model. Since firms can issue equity and borrow debt for the fund of investment freely in the capital market by the assumption (A1), this assumption does make sense. (A3) Stock prices follow geometric Brownian motion (hereafter, GBM). Needless to say, this stochastic specification is widely used without debate in the financial literature. This assumption is very important to derive the behavior of aggregate dividends in the market. The functional equation to represent the stochastic dynamics (GBM) is as follows:

\[
dS = \mu_S dt + \sigma_S dZ_S
\]  

where \(S\) = instantaneous stock price,\(^4\),

\(\mu_S\) = instantaneous conditional expected change in \(S\),

\(t\) = time dimension,

\(\sigma_S\) = instantaneous conditional standard deviation of \(S\),

\(dZ_S\) = the increment of a standard Wiener process in \(S\).

(A4) Payment of dividends incurs an 'opportunity cost'. This opportunity cost is due to the fact that the fund distributed as dividends would be invested in other projects with, at least, riskless rate of return, if it were not used as dividends.\(^5\) In this study, since the cash flow from any arbitrary investment is

\(^3\) It is well-known fact that the real-option valuation model is one of the hottest areas in the current financial economics. In this sense, Dixit (1989) and Grenadier (1995) models are very helpful to understand the framework of this study.

\(^4\) This is a market index, such as S&P 500 or DJIA.

\(^5\) Eades, Hess, and Kim (1994) discuss this issue very well, even though they do not use the term 'opportunity cost' explicitly. They argue in pp. 1630-1632 that 'we also expect interest rates on cash-equivalent securities to affect the level of dividend capturing. Corporations usually invest excess cash in short-term and low-risk securities...The lost tax premium of debt, or higher before-tax rate of return, is an opportunity cost of dividend capturing...'
random walk, the behavior of an opportunity cost is assumed to follow GBM as follows:

\[ dC = \mu_c dt + \sigma_c dZ_c \]  

(2)

where \( C \) = instantaneous opportunity cost, 
\( \mu_c \) = instantaneous conditional expected change in \( C \), 
\( \sigma_c \) = instantaneous conditional standard deviation of \( C \), 
\( dZ_c \) = the increment of a standard Wiener process in \( C \).

In addition, it is reasonable to think that the cost be a function of dividend and opportunity cost of capital: namely, \( C = f(D, r) \), where \( D \) is a dividend and \( r \) is the opportunity cost of capital. We can consider \( r \) as a riskless rate, such as T-bill rate as we can see in the Eades, Hess, & Kim (1994), because corporations usually invest excess cash in short-term and low-risk (almost riskless) securities. They argue that corporate investment in financial securities is typically a temporary use of funds that have a more permanent home (e.g., payment of dividends or investment in real assets)(p.1631). \( (A5) \) The capital structure for an average firm\(^6\) is composed of only debt and common equity. This assumption is necessary to find equilibrium cost of capital in the market.

2. Aggregate Demand for and Supply of Dividends in the Market

Generally speaking, the value of a firm is a function of many variables, such as cash flows, cost of capital, capital structure, dividend, tax rates, and others. However, Since, if we introduce all those variables into a model, the model might be very complex and closed-form solution can hardly be obtained, this study focuses on the 'marginal effect\(^7\)' of dividend on the firm value. Since a stock price is a function of an opportunity cost described in the assumption (A4) and dividend (i.e., \( S=f_1(C, D) \), where \( D \) is the dollar amount of dividend payment), a firm value can be expressed as a function of cost and dividend as follows:

\[ V = f_2(S) = f_2[f_1(C, D)] = f_2*f_1(C, D), \text{ if } f_1 \text{ and } f_2 \text{ are one-to-one functions.} \]

But, as we can see in the many introductory textbooks in finance, it is natural to assume that \( f_1 \) and \( f_2 \) be one-to-one functions, indicating having inverse functions. This fact means that \( S \) uniquely determines the value of \( f_1 \) and, as a result, the value of \( f_1 \) also uniquely decides the value of \( f_2 \). For the convenience, the functional dependency of a firm value on \( C \) and \( D \) can be defined as \( V(C, D) \). However, to distinguish between a group of firms that do not pay dividends currently and a group of firms that pay dividends currently, two definitions in the firm value\(^8\) will be used as follows:

\[ V(C, D) = \text{the value of a firm in the group which is waiting for the optimal moment to pay a dividend} \]

\[ W(C, D) = \text{the value of a firm in the group which is currently paying dividends} \]  

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6) Even though firms can use different kinds of assets, such as preferred stocks, warrants, and others, for the necessary supply of funds, we assume that most of the firms (so, an average firm) have the capital structure of common stocks and debt.

7) Here, marginal effect means that only the change in dividend policy is considered holding other policy variables like a capital structure and an investment strategy constant.

8) This notation is similar to the those used in Dixit (1989) (i.e., \( V_0 \) and \( V_1 \)) and in Grenadier (1995) (i.e., \( V(P,C) \) and \( W(P,C) \)).
Before describing the dynamics of aggregate dividends, it is useful to explain the behavior of an individual firms’ dividend policy. Since the price of any asset is determined by the demand and supply mechanism in the market economy, it is natural that the demand and supply be defined accurately to derive equilibrium price.

*First*, in this study, the demand for dividends can be equivalently defined as the demand for common stock by assumption (A9). According to previous studies, it is a very common fact that the dividend announcement affects stock price, even though the underlying reason for the price change is somewhat controversial (i.e., some argue signaling or information effect and others tax effect) according to Litzenberger & Ramaswamy (1982) and Hess (1982). Thus, it is reasonable to posit that a dividend itself has some effect on the stock price. In addition, as Litzenberger & Ramaswamy (1982) and other researchers argue, dividends are non-linearly associated with stock prices. Thus, in this study, the following inverse demand function is used to describe the demand function for the dividend:

\[ S(t) = Q(t)^{-a} \ast D(t) \]  \hspace{1cm} (4)

where \( S(t) \) is the stock price at time \( t \); \( Q(t) \) is the aggregate supply of equity at time \( t \) in the market; \( D(t) \) is the aggregate supply of dividend at time \( t \) in the market; \( a \) is assumed a constant elasticity\(^9\) of the demand for an equity. In the equation (4), we can think that dividends have multiplicative (i.e., non-linear) demand shock on stock prices. *Second*, the supply of dividends depends on competitive firms that may pay dividends at any time at the expense of 'opportunity costs'. Since while dividend-paying firms can make additional capital gains by paying dividends, they also should pay opportunity costs, the net cash flows at each time will be the difference between additional capital gains and opportunity costs.

Therefore, the interaction of competitive, value maximizing supplies of and derived demand for dividends results in an equilibrium determination of optimal aggregate dividend level in the market. Before we derive the value of a firm, it is necessary to mention the reason that the continuous-time, not discrete, model can be used in the dividend study. If we focus on the individual firm, it is truly evident that the dividend payment is discrete event occurring at specific dividend-payment dates (e.g., since most firms pay dividend each quarter, dividends occur only four times a year). But, if we consider all firms in the market, it is also natural that we use continuous occurrence of dividends around the year. Thus, as we can use a continuous dividend-yield model which is frequently adopted in the pricing of options (e.g., options in stock index) with dividend payment\(^{10}\), we can use a continuous-time model for the aggregate dividend, because aggregate equity produce almost continuous flows of dividend payments.

### III. Model of Aggregate-Dividend Dynamics

Under given assumptions and the demand and supply mechanism described above, the value of a firm can be derived by applying a stochastic calculus. From equation (4), the inverse demand function can be rewritten as follows:

\[ D = S \ast Q^a. \]  \hspace{1cm} (5)\(^{11}\)

\(^9\) In this model, price elasticity of demand is assumed to be constant. This means that demand for dividends is constant regardless of the level of market index. This seems reasonable, because those who want dividends buy some equity without worrying about the level of market index. Also, see the reasons mentioned in Grenadier (1995).

\(^{10}\) In the Black-Scholes European option pricing equation with dividend payment, dividend yield is considered by multiplying \( e^{-rt} \), where \( y \) is a dividend yield), to the asset price in the equilibrium setting.

\(^{11}\) For the notational convenience, \( t \) (time dimension) will be dropped hereafter.
By applying Ito's Lemma to equation (5), we can derive that

$$dD = \mu_s dt + \sigma_s dZ_s + (\alpha D/Q)dQ. \quad (6)$$

In the traditional option-pricing literature, contingent claims are priced through arbitrage arguments. However, such an approach requires assumptions about the free tradability (or liquidity) of the underlying asset. The dividend itself is not freely tradable, even though equity with dividend has liquidity, and much transaction cost is necessary to buy dividend through equity. In addition, short sale is impossible for dividend. Thus, such arbitrage arguments are not applicable to the dividend case. Clearly, in the absence of a freely traded underlying security (e.g., typically, the value of a piece of land on which a call option is written), an equilibrium approach may be used, even though an appropriate equilibrium model must be chosen. In this study, continuous-time equilibrium model similar to Dixit (1989) is applied.

Consider the instantaneous return on \(W(C, D)\) over a region in which a dividend policy is unchanging. By Ito's Lemma, the instantaneous change in \(W\) is

$$dW = \left[ \frac{1}{2} \sigma_C^2 W_{CC} + \rho \sigma_C \sigma_s C D W_{CD} + \frac{1}{2} \sigma_S^2 W_{DD} + \mu_C C W_C + \mu_S D W_D \right] dt + \sigma_C C W_{dZ} + \sigma_S D W_{dZ} \quad (7)$$

where \(\rho\) is the correlation coefficient between \(dZ_C\) and \(dZ_S\). In addition to the capital gains earned on the asset, \(dW\), the asset also yields additional cash flows due to dividend payment of \(E\left(\frac{dS}{S}\right)D\) and an opportunity cost of \(E\left(\frac{dC}{C}\right)D\), indicating that the additional net capital gain due to dividend payment is \(E\left[\frac{dS}{S} - \frac{dC}{C}\right]D\), where \(E\) is an expectation operator. But, from equations (1) and (2),

$$E[dS/S] = E[\mu_s dt + \sigma_s dZ_s] = \mu_s dt,$$
$$E[dC/C] = E[\mu_C dt + \sigma_C dZ_c] = \mu_C dt. \quad \text{Thus,}$$
$$E[dS/S - dC/C]D = (\mu_s - \mu_C)D dt. \quad (8)$$

Therefore, the total expected return on \(W\) per unit time is, from equations (7) and (8),

$$= \left[ \frac{1}{2} \sigma_C^2 W_{CC} + \rho \sigma_C \sigma_s C D W_{CD} + \frac{1}{2} \sigma_S^2 W_{DD} + \mu_C C W_C + \mu_s D W_D \right] dt + (\mu_s - \mu_C)D dt$$

Simplifying above equation yields the following equilibrium partial differential equation (PDE):

$$0 = \frac{1}{2} \sigma_C^2 W_{CC} + \rho \sigma_C \sigma_s C D W_{CD} + \frac{1}{2} \sigma_S^2 W_{DD} + \mu_C C W_C + \mu_s D W_D - kwW + (\mu_s - \mu_C)D \quad (9)$$

12) Full derivation is in the [APPENDIX-1].
13) Full derivation is in the [APPENDIX-2].
14) Since the expected total capital gain earned from the investment in equity is \(E[dS/S](N*S)\), where \(N\) is total number of stock outstanding, and the capital gain which we can get by investing the amount of \(D\) in the stocks with same expected return is \(E[dS/S]*D\).
15) \(kw\) is the cost of capital for the firm which has capital structure in assumption (A5). In the equilibrium, it is determined in the market as follows: \(kw = k_d(1\!-\!r)B/(B+E) + k_rE/(B+E)\), where \(k_d\) and \(k_r\) are the cost of debt and the cost of equity, respectively, \(r\) is a corporate tax rate, and \(B\) and \(E\) denote the total amount of debt and equity, respectively. We can apply the CAPM to derive the equilibrium rates of \(k_d\) and \(k_r\).
With the same reasoning as \( W(C, D) \), PDE for the \( V(C, D) \) can be derived as follows:

\[
0 = \dfrac{\partial}{\partial c} c^2 V_{CC} + \rho \sigma c \sigma C \sigma CD_{CD} + \dfrac{\partial}{\partial s} s^2 D_{DD} + \mu_c C V_C + \mu_s D V_D - k_W V
\]  

(10)\(^{16}\)

To solve PDEs (9) and (10), we should know additional boundary conditions. Since the values of two kinds of firms, \( V(C, D) \) and \( W(C, D) \), are connected by individual firm behavior and equilibrium conditions, it is necessary to consider individual firm behavior. A firm currently not paying dividends will decide to pay dividend and opportunity cost, \( C(t) \), when it is individually optimal. Solving this problem necessitates two types of conditions. I.e., the value matching condition is \( V(C', D') = W(C', D') - C' \), where \( C' \) and \( D' \) denote values of \( C(t) \) and \( D(t) \) which trigger to pay dividends: and smooth-pasting or high-contact conditions are \( V_C(C', D') = W_C(C', D') - 1 \) and \( V_D(C', D') = W_D(C', D') \)\(^{17}\). These conditions guarantee that the non-dividend paying firm chooses to pay dividends at the optimal point, meaning that these optimality conditions determine the trigger values that maximize the value of the firm.

Up to this point, individual optimal behavior has been described and used to derive the PDE for firm values. To find a closed-form solution to the PDEs (9) and (10), a competitive equilibrium can be utilized. In a competitive equilibrium, as we can see in assumption (A1), there is no economic profit, because free entry and exit eat away all excess profits. In other words, if there is an excess profit, this induces immediate new entry, and the profit will disappear right away. Thus, the free-entry condition in competitive equilibrium in assumption (A1) leads the following additional boundary condition:

\[
W(C', D') = C'
\]  

(11)

Since all firms are solving the same optimization problem and will each choose the same optimal policy, this condition ensures that there can be no excess profit accruing to entry to the dividend market. Thus, this condition ensures that \( V(C, D) = 0 \); i.e., the marginal value of a firm\(^{18}\) which does not pay dividend is always zero, meaning that the (additional or marginal) firm value is considered to be zero when the firm does not pay dividend, ceteris paribus. Therefore, combining PDEs (9) & (10) and the boundary conditions results in following PDE in equilibrium:

\[
0 = \dfrac{\partial}{\partial c} c^2 W_{CC} + \rho \sigma c \sigma C \sigmaDW_{CD} + \dfrac{\partial}{\partial s} s^2 D_{DD} + \mu_c C W_C + \mu_s D W_D - k_W W + (\mu_s - \mu_C) D
\]

subject to

\[
W(C', D') = C', \ W_D(C', D') = 0, \ W_C(C', D') = 0, \ W(C, 0) = 0,
\]

(13)

(14)

where conditions (13) are from value-matching, smooth-pasting, and competitive equilibrium conditions and (14) are regularity conditions. \( W(C, 0) = 0 \) has the same meaning as \( V(C, D) = 0 \) as explained in the footnote 18. But, the condition \( W(\infty, D) = [(\mu_s - \mu_C)D]/k_W \) ensures that, for very large opportunity costs, there is almost no new entry to the dividend market. Thus, the value of a firm that is currently paying dividend is equivalent to the perpetuity value of additional capital gain due to dividend payout.

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16) \( k_W \) is equal to the \( k_W \) if a dividend policy is independent of the capital structure of a firm.
17) For details of high-contact conditions, see Merton (1973).
18) The concept of 'marginal value of a firm' is consistent with the concept 'marginal effect' that we introduce in the section II-2. It should be emphasized that we focus only on the additional (marginal) effect of change in dividend on the firm value in this study. That is, \( V(C, D) = 0 \) means that the firm value is equivalent to zero when the firm does not pay dividends, ceteris paribus.
The solution to PDE (12) subject to (13) & (14) is as follows:

\[ W(C, D) = (-\gamma)^{\frac{1}{\beta-1}} \frac{B}{\beta-1} \frac{1}{C} (B-1) \frac{D}{\beta-1} + \gamma D \]

(15)

where \( \beta \equiv \left( t s^2 + \rho c c s + l c^2 - \mu s \right) + (t s^2) \left( (\rho c c s + \mu s - l c^2 + t s^2)^2 - 2 s^2 \right) \mu c - kw \)

\[ \gamma \equiv (\mu s - \mu c) / (\rho c c s - l c^2 + \mu s + \mu c - kw). \]

Similarly, \( V(C, D) \) can be expressed as

\[ V(C, D) = (-\gamma)^{\frac{1}{\beta-1}} \frac{B}{\beta-1} \frac{1}{C} (B-1) \frac{D}{\beta-1} \]

(16)

where \( \beta \equiv \left( t s^2 + \rho c c s + l c^2 - \mu s \right) + (t s^2) \left( (\rho c c s + \mu s - l c^2 + t s^2)^2 - 2 s^2 \right) \mu c - kw \)

\[ \gamma \equiv (\mu s - \mu c) / (\rho c c s - l c^2 + \mu s + \mu c - kw). \]

It is interesting to note that only differences between \( V(C, D) \) and \( W(C, D) \) are \( \gamma D \) term in equation (15) and \( kw \) & \( kv \) terms in \( \gamma \). This might have very important implication for an optimal dividend, similar to many other real options literature, such as Dixit (1989), Pindyck (1991), McDonald & Siegel (1985), and Brennan & Schwartz (1985). That is, since the numerator of \( \gamma \) may represent the risk-adjusted discount rate, \( \gamma D \) is perpetuity value of the additional capital gain made when the firm pays dividends forever. Therefore, the first part in equation (15) can be interpreted as "the value of an option to eliminate dividend". Similarly, the value \( V(C, D) \) in equation (16) can be interpreted as "the value of an option to start paying dividend".

In addition, the optimal dividend can be expressed as a function of \( C^* \) as follows:

\[ D^* = \frac{C^*}{\gamma(\beta-1)} \]

(17)

To find a closed form solution for \( D \), it is necessary to estimate the value of an exogenous variable \( C^* \). Although various functional forms are possible for \( C \), linear relationship, like \( C = A + (1+r)D \) (where \( A \) is constant), is assumed in this study, because extra funds can be invested, at least, in the riskless rates. This means that if we invest extra funds in more risky assets, \( C \) can be increased and \( 'A' \) should be positive. Thus,

\[ D^* = A \left( \frac{B}{\gamma(\beta-1)} \right) \left( \frac{1}{1 + (1+r)} \right) \]

(18)

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19) Full derivation of solution to PDE (12) is in the [APPENDIX-3].

20) If \( t s^2 + \rho c c s + l c^2 < l c^2 \), then \( \beta > 1 \); otherwise, \( \beta < 1 \).

21) If the numerator of is simply \( kw \), it is evident that \( \gamma D \) is perpetuity value of additional capital gains earned from dividend payments. Unfortunately, the numerator is \( (\rho c c s - l c^2 + \mu s + \mu c - kw) \). Since \( c^2 \) is less than \( c \), \( \rho c c s - l c^2 \) should be positive. Thus, \( \mu s + \mu c - kw \) determines the sign of numerator of \( \gamma \). In addition, the numerator of \( \gamma \) reflects the risk in \( C \) and \( S \). So, we define it as a 'risk-adjusted' discount rate.

22) See also [APPENDIX-3].
Now that we express \( W(C, D) \) and \( D^* \) as exogenous state variables (\( \mu_C, \mu_S, \sigma_C, \sigma_S, \rho, \gamma, B \) (debt), \( Q, S, \) and \( A \)), the remaining thing to do is to estimate those variables. It should be done in the future research with empirical test of the model derived in this study.

**IV. Comparative Statics Analysis**

To compare this study with other previous studies, comparative statics analysis is useful, because we can see the effect of changes in the key variables on \( D^* \).

*First*, to investigate the effect of change in \( D^* \) on \( W(C, D) \), elasticity can be used as follows:

\[
\frac{\partial W(C, D)}{\partial D} = (-)^g \beta \left( \frac{B}{\beta - 1} \right)^{\beta - 1} \left( \frac{1}{\beta - 1} \right)^{C - 1/\beta} \beta^{\beta - 1} + \gamma. 
\]

(19)

In equation (19), the sign of \( \partial W(C, D)/\partial D \) is dependent on the values \( \beta \) and \( \gamma \), i.e., the state variables \( \mu_C, \mu_S, \sigma_C, \sigma_S, \rho, k_w \). If \( \frac{1}{4} \sigma_S^2 + \rho \sigma_C \sigma_S + \mu_S = \frac{1}{4} \sigma_C^2 \) and \( \mu_C = \mu_S \), then \( \partial W(C, D)/\partial D = 0 \), supporting the Miller & Modigliani proposition (i.e., irrelevancy of a dividend on the firm value). Particularly, since the condition of \( \mu_C = \mu_S \) means that additional expected capital gain is equal to the expected opportunity cost of capital, this implies \( V(C, D) = W(C, D) \). That is, a dividend is irrelevant. If \( \mu_C \neq \mu_S \), a dividend policy affects a firm value in the equilibrium. Therefore, the appropriate estimation of exogenous state variables is critical to determine the relevancy of a dividend policy.

*Second*, to see how a corporate tax affects a dividend policy, \( \partial D/\partial \tau \) should be investigated as follows:

\[
\frac{\partial D}{\partial \tau} = \frac{\partial D}{\partial \beta} \frac{\partial \beta}{\partial \gamma} \frac{\partial \gamma}{\partial k_w} \frac{\partial k_w}{\partial \tau} 
\]

by a chain rule, the sign of \( \partial D/\partial \gamma \) is determined by the signs of \( \partial D/\partial \beta (\gamma) \), \( \partial \beta (\gamma)/\partial k_w \), and \( \partial k_w/\partial \tau \). But, from equations, \( k_w = k_w (1 - \theta) + k_e \left( \frac{B}{B + E} \right)^{23} \), we can find that \( \partial k_w/\partial \tau < 0 \), and, from equations for \( \beta \) and \( \gamma \), it can be shown that \( \partial \beta/\partial k_w < 0 \) and \( \partial \gamma/\partial k_w < 0 \). In addition, from equation (18), if \( \frac{1}{4} \sigma_S^2 + \rho \sigma_C \sigma_S + \mu_S < \frac{1}{4} \sigma_C^2 \) and \( \mu_C \mu_S \), then \( \partial D/\partial \beta < 0 \) and \( \partial D/\partial \gamma < 0 \). As a result, we can find the following relationship:

\[
\partial D/\partial \gamma > 0, \text{ if } \frac{1}{4} \sigma_S^2 + \rho \sigma_C \sigma_S + \mu_S < \frac{1}{4} \sigma_C^2 \text{ and } \mu_C \mu_S, \text{ and} \\
\partial D/\partial \tau > 0, \text{ if } \frac{1}{4} \sigma_S^2 + \rho \sigma_C \sigma_S + \mu_S \geq \frac{1}{4} \sigma_C^2 \text{ and } \mu_C \geq \mu_S.
\]

The first relationship implies that, under a given condition (actually, this condition is more realistic than the latter), if tax rate is increased, then dividend payment triggering non-dividend-paying firms to enter the dividend market should be increased. This means that the higher corporate tax rate, the more dividends should be paid out. This fact seems reasonable, because firms had better pay out dividends as much as possible to increase debt to maximize tax shields if corporate tax rate is high.

*Third*, similarly, we can analyze the effect of a debt amount (\( B \)) on the dividend policy. Using the same

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23 \( k_d \) and \( k_e \) are a cost of capital and a cost of equity, respectively, whose equilibrium rates are determined in the market by an equilibrium model such as CAPM: \( B \) and \( E \) are total debt and equity for the dividend paying firms, respectively, in the market.
method as the above, we can easily show that

$$\frac{\partial D}{\partial B} > 0, \text{ if } \frac{1}{2}\sigma_s^2 + \rho\sigma_c\sigma_s + \mu_s < \frac{1}{2}\sigma_c^2 \text{ and } \mu_c < \mu_s, \text{ and } \mu_c < \mu_s,$$

$$\frac{\partial D}{\partial B} < 0, \text{ if } \frac{1}{2}\sigma_s^2 + \rho\sigma_c\sigma_s + \mu_s > \frac{1}{2}\sigma_c^2 \text{ and } \mu_c > \mu_s.$$

This is an expected result in the above analysis, because the higher tax rate increases both B and D. The fact that B and D have a positive relationship gives bondholders an extremely important implication. That is, if a firm increases a dividend payout, it signals also a bad news to the bondholders, because the probability that the firm may increase the debt amount is high. This fact is very consistent with the common sense that a dividend payment induces an agency problem, because bondholders are concerned about the increase in dividend payment to shareholders at the expense of them.

Fourth, to compare this model with the analysis of Eades, Hess, & Kim (1994), the impact of riskless rate on the dividend capturing is investigated by using that $\frac{\partial D}{\partial r}>0$. This is easily derived from the equation (18). According to Eades, Hess, and Kim (1994), T-bill rates are negatively related with dividend capturing (p.1632), contrary to this study. They use the relationship between business cycle and dividend capturing to explain their result. But, this study uses ‘opportunity cost’ concept as follows: if $r$ goes up, the opportunity cost will also go up, indicating that the triggering point of D should be increased. The reason for the difference between two studies might be partly because their model focuses on the individual firm behavior, while this model is aggregate model, partly because of some misspecification of the model.

Finally, the effect of a cost of capital (WACC: $k_w$) on D is investigated. As already analyzed in the 'second' above,

$$\frac{\partial D}{\partial k_w} < 0, \text{ if } \frac{1}{2}\sigma_s^2 + \rho\sigma_c\sigma_s + \mu_s < \frac{1}{2}\sigma_c^2 \text{ and } \mu_c < \mu_s,$$

$$\frac{\partial D}{\partial k_w} > 0, \text{ if } \frac{1}{2}\sigma_s^2 + \rho\sigma_c\sigma_s + \mu_s > \frac{1}{2}\sigma_c^2 \text{ and } \mu_c > \mu_s.$$

This result has two important implications: first, a dividend policy is independent of a cost of capital only if $\frac{1}{2}\sigma_s^2 + \rho\sigma_c\sigma_s + \mu_s = \frac{1}{2}\sigma_c^2$ and $\mu_c = \mu_s$, which are not realistic conditions, and, otherwise, it is not independent; second, a cost of capital is negatively related with a dividend, which can be explained by the fact that the increase in the $k_w$ requires higher cost of equity and debt, indicating the decrease in the dividend due to the decrease in equity.

V. Concluding Remarks

In this paper, it was shown that a continuous-time and aggregate model can be applied in the study of a dividend policy. Combining an individual firm behavior and an aggregate market behavior yields the equilibrium-optimal dividend policy in the perfectly competitive market setting. The major results of this study are as follows: dividend irrelevance theory can be valid only when very specific and unrealistic
conditions are met: a corporate tax rate, a debt policy, a cost of capital can affect the dividend policy under specific conditions.

Even though this approach produces some important implications for the dividend policy, some problems should be solved in the future research. Since the assumption (A4) is so strong, it is more desirable to relax it to yield more realistic results. Since the optimal dividend policy is critically dependent on the exogenous variables, the appropriate estimation methods should be developed for the variables. Empirical research must be done to make sure the model is valid to be applied in the real world.

[Appendices]

[Appendix-1] Proof of equation (6)

\[
dD = DsdS + DqdQ + t(DsdSS^2 + 2DsdSdQ + DqdQ^2)
\]

= DsdS + DqdQ (since Dss = sdQ = DQQ = 0)

= Q^2(\mu_sSdt + \sigma_sSdZ_s) + a SQ^2 dt (by assumption A3)

= \mu_s(SQ^2)dt + \sigma_s(SQ^2)dZ_s + (a / Q(SQ^2)dt

= \mu_sSdt + \sigma_sSdZ_s + (a D/Q)dQ (from equation (5)). Q.E.D.

[Appendix-2] Proof of equation (7)

\[
dW = W_{cc}C + W_{dd}D + t(W_{cc}S^2 + 2W_{cc}SD + W_{dd}D^2)
\]

= W_{cc}[\mu_cCdt + \sigma_cCdZ_c] + W_{dd}[\mu_sDdt + \sigma_sDdZ_s + (a D/Q)dQ] + t(W_{cc}S^2)dt

= \mu_cCdt + \sigma_cC\sigma_sSdW_{CD} + \mu_sDdt + \sigma_sDdZ_s + (a D/Q)dQ + tW_{dd}S^2 dt

= \mu_cCdt + \mu_sDdt + \sigma_cSdW_{CD} + \sigma_sDdZ_s + (a D/Q)dQ + tW_{dd}S^2 dt

= \mu_cCdt + \mu_sDdt + \sigma_cSdW_{CD} + \sigma_sDdZ_s. Q.E.D.

[Appendix-3] Proof of equations (15) and (17)

To solve PDE (12) subject to boundary conditions (13) and (14), new variables are defined as follows for the convenience:

\[
X = \frac{D}{C} \text{ and } Y = \frac{1}{C} W(C, D)
\]  

(a1)

Since \( W = CY, \ WC = Y, \)

\[
W_{cc} = \frac{\partial Y}{\partial C} \frac{\partial X}{\partial C} = Y_x(\frac{D}{C^2}), \text{ (by a chain rule)}
\]

\[
W_{CD} = \frac{\partial Y}{\partial D} = C \frac{\partial Y}{\partial X} \frac{\partial X}{\partial D} = Y_x(\frac{1}{C})
\]

\[
W_{dd} = C \frac{\partial Y}{\partial D} = C \frac{\partial Y}{\partial X} \frac{\partial X}{\partial D} = Y_x C(\frac{1}{C}) = Y_x
\]

Then, equation (12) becomes

\[
0 = 4 \sigma_s^2(1/C)Y_{xx} + \rho \sigma_c \sigma_s C(1/C)Y_x + \frac{C^2}{2}(-D/C^2) + \mu_sDY_x + \mu_cCY - kwCY + (\mu_s - \mu_c)D
\]
Dividing each side by \( C \) yields
\[
0 = s^2(\frac{D}{C})^2Y_{XX} + (\rho \sigma_c \sigma_s + \mu_s - s^2)(D/C)Y_X + (\mu_c - kw)Y + (\mu_s - \mu_c)(D/C).
\] (a2)

Substituting equation (a1) into equation (a2) yields the following differential equation,
\[
0 = s^2X^2Y_{XX} + (\rho \sigma_c \sigma_s - s^2 + \mu_s)XY_X + (\mu_c - kw)Y + (\mu_s - \mu_c)X
\] (a3)
subject to
\[
Y(0) = 0, \quad Y(X^*) = (1/C^*)W(C^*,D^*) = (1/C^*)C^* = 1, \quad Y_X(X^*) = 0,
\] (a4)
where \( X^* = D^*/C^* \).
The PDE (a3) subject to boundary conditions (a4) can be transformed into the following simplified form,
\[
aX^2Y_{XX} + bXY_X + cY = Xd + e
\] (a5)
where \( a = s^2, \quad b = \rho \sigma_c \sigma_s - s^2 + \mu_s, \quad c = \mu_c - kw, \quad d = \mu_s - \mu_c, \quad e = 0. \)
Since we know the solution to PDE (a5)\(^{(24)}\), we can obtain the following solution to PDE (a3) as follows:
\[
Y(X) = A_1 X^\beta + \frac{X(\mu_s - \mu_c)}{\rho \sigma_c \sigma_s - \frac{1}{2} \sigma_c^2 + \mu_s + \mu_c - kw}
\]
\[
\beta = \left( \frac{1}{\sigma_s^2} \right)^{\frac{1}{2}} \frac{1}{2} \frac{\sigma_s^\gamma - \rho \sigma_c \sigma_s + \frac{1}{2} \sigma_c^2 - \mu_s + [(\rho \sigma_c \sigma_s - \frac{1}{2} \sigma_c^2 + \mu_s - \frac{1}{2} \sigma_s^2)^2 - 2 \sigma_s^2(\mu_c - kw)]^{\frac{1}{2}}}{\frac{1}{2} \sigma_c^2}
\]
By applying boundary conditions (a4), we can find
\[
X^* = \frac{\beta}{\lambda(\beta - 1)} \quad \text{and} \quad A_1 = (\gamma / \beta)^\delta \left( \frac{1}{\beta - 1} \right)^{\gamma
\gamma - \gamma}
\]
where \( \beta \) and \( \gamma \) are defined as equation (15).
As a result, we can derive
\[
Y(X) = (\gamma / \beta)^\delta[(\beta - 1)]^{1 - \beta}X^\beta + \gamma X, \quad \text{and}
\]
\[
X^* = \beta / (\gamma(\beta - 1)) = D^*/C^* \quad \text{(proof of equation (17))}.
\]
But, from equation (a1), since \( W(C,D) = CY(X) \),
\[
W(C,D) = (\gamma / \beta)^\delta \left( \frac{1}{\beta - 1} \right)^{\gamma - \gamma}C^{1 - \beta}D^\beta + \gamma D. \quad Q.E.D.
\]

References


