The Mathematical Association of America
American Mathematics Competitions

25th Annual *(Alternate)*

# AMERICAN INVITATIONAL MATHEMATICS EXAMINATION (AIME)

# SOLUTIONS PAMPHLET

Wednesday, March 28, 2007

This Solutions Pamphlet gives at least one solution for each problem on this year's AIME and shows that all the problems can be solved using precalculus mathematics. When more than one solution for a problem is provided, this is done to illustrate a significant contrast in methods, e.g., algebraic vs geometric, computational vs. conceptual, elementary vs. advanced. The solutions are by no means the only ones possible, nor are they necessarily superior to others the reader may devise.

We hope that teachers inform their students about these solutions, both as illustrations of the kinds of ingenuity needed to solve nonroutine problems and as examples of good mathematical exposition. Routine calculations and obvious reasons for proceeding in a certain way are often omitted. This gives greater emphasis to the essential ideas behind each solution.

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1. (Answer: 372)

   If a sequence contains no more than one 0, there are $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$ sequences formed from the characters A, I, M, E, 2, 0, and 7. If a sequence contains two 0's, the 0's can be placed in $\binom{5}{2} = 10$ ways, the remaining characters can be chosen in $\binom{6}{3} = 20$ ways, and those remaining characters can be arranged in $3! = 6$ ways, for a total of $10 \cdot 20 \cdot 6 = 1200$ sequences. Thus $N = 2520 + 1200 = 3720$, and $\frac{N}{10} = 372$.

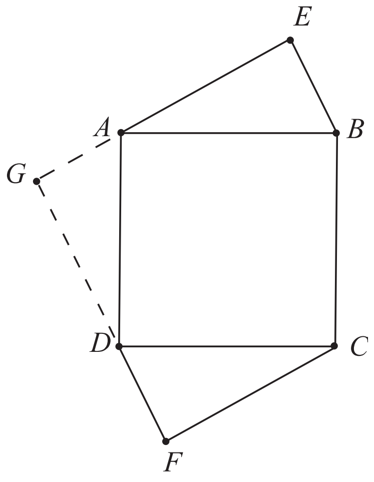
2. (Answer: 200)

   For any such ordered triple $(a, b, c)$, because $a$ is a factor of $b + c$, $a$ is also a factor of 100. Thus $a$ is an element of $\{1, 2, 4, 5, 10, 20, 25\}$, and $\dfrac{b}{a}$ and $\dfrac{c}{a}$ are positive integers for which $\dfrac{b}{a} + \dfrac{c}{a} = \dfrac{100}{a} - 1$ (Note that if $a = 50$ or 100, then at least one of $b$ and $c$ is zero). Because $\dfrac{b}{a}$ and $\dfrac{c}{a}$ are positive integers, there are (for each choice of $a$) $\dfrac{100}{a} - 2$ pairs $\dfrac{b}{a}$ and $\dfrac{c}{a}$. Thus there are

   $$\frac{100}{1} + \frac{100}{2} + \frac{100}{4} + \frac{100}{5} + \frac{100}{10} + \frac{100}{20} + \frac{100}{25} - 2 \cdot 7 = 214 - 14 = 200 \text{ such triples.}$$

3. (Answer: 578)

   Extend $\overline{AE}$ past $A$ and $\overline{DF}$ past $D$ to meet at $G$. Note that $\angle ADG = 90^\circ - \angle CDF = \angle DCF = \angle BAE$ and $\angle DAG = 90^\circ - \angle BAE = \angle ABE$. Thus $\triangle AGD \cong \triangle BEA$. Therefore $EG = FG = 17$, and because $\angle EGF$ is a right angle, $EF^2 = 2 \cdot 17^2 = 578$.



4. (Answer: 450)

The fact that 60 workers produce 240 widgets and 300 whoosits in two hours implies that 100 workers produce 400 widgets and 500 whoosits in two hours, or 200 widgets and 250 whoosits in one hour. Let $a$ be the time required for a worker to produce a widget, and let $b$ be the time required for a worker to produce a whoosit. Then $300a + 200b = 200a + 250b$, which is equivalent to $b = 2a$. In three hours, 50 workers produce 300 widgets and 375 whoosits, so $150a + mb = 300a + 375b$ and $150a + 2ma = 300a + 750a$. Solving the last equation yields $m = 450$.

5. (Answer: 888)

The graph passes through the points $(0, 9), (24\frac{7}{9}, 8), (49\frac{5}{9}, 7), \ldots, (198\frac{2}{9}, 1), (223, 0)$, where each decrease of 1 unit in $y$ results in an increase of $24\frac{7}{9}$ units in $x$. Therefore the region can be divided into rectangles with dimensions $8 \times 24$, $7 \times 25$, $6 \times 25$, $5 \times 25$, $4 \times 24$, $3 \times 25$, $2 \times 25$, and $1 \times 25$, for a total of 888 squares.

**OR**

Consider the rectangle with vertices $(0, 0)$, $(0, 9)$, $(223, 0)$, and $(223, 9)$. There are 2007 1 by 1 squares within this rectangle. The diagonal from $(0, 0)$ to $(223, 9)$ crosses exactly one of these squares between $x = n$ and $x = n+1$ for most of the 223 possible values of $n$. There are exactly 8 values of $n$ for which the diagonal crosses one of the horizontal lines $y = m (1 \leq m \leq 8)$, and for these values the diagonal crosses two squares. The diagonal never passes through any corners, because 9 and 223 are relatively prime and $9 \cdot 223 = 2007$. Thus, out of the 2007 squares, $223 + 8$ of them are crossed by the diagonal, leaving 1776 squares untouched. Half of these, or 888 of them, lie below the diagonal.

6. (Answer: 640)

Let $a_1 a_2 a_3 \cdots a_k$ be the decimal representation of a parity-monotonic integer. It is not difficult to check that for each fixed $a_{i+1}$, there are four choices for $a_i$; for example, if $a_{i+1} = 8$ or 9, then $a_i \in \{1, 3, 5, 7\}$; if $a_{i+1} = 4$ or 5, then $a_i \in \{1, 3, 6, 8\}$, and so on. There are 10 choices for the digit $a_k$, and 4 choices for each of the remaining digits. Hence there are $4^{k-1} \cdot 10$ $k$-digit parity-monotonic integers, and the number of four-digit parity-monotonic integers is $4^3 \cdot 10 = 640$.

7. (Answer: 553)

Because $k \leq \sqrt[3]{n_i} < k+1$, it follows that $k^3 \leq n_i < (k+1)^3 = k^3 + 3k^2 + 3k + 1$. Because $k$ is a divisor of $n_i$, there are $3k + 4$ possible values for $n_i$, namely $k^3, k^3 + k, \ldots, k^3 + 3k^2 + 3k$. Hence $3k + 4 = 70$ and $k = 22$. The desired maximum is $\dfrac{k^3 + 3k^2 + 3k}{k} = k^2 + 3k + 3 = 553$.

8. (Answer: 896)

Let $h$ be the number of 4 unit line segments and $v$ be the number of 5 unit line segments. Then $4h + 5v = 2007$. Each pair of adjacent 4 unit line segments and each pair of adjacent 5 unit line segments determine one basic rectangle. Thus the number of basic rectangles determined is $B = (h-1)(v-1)$. To simplify the work, make the substitutions $x = h-1$ and $y = v-1$. The problem is now to maximize $B = xy$ subject to $4x + 5y = 1998$, where $x$, $y$ are integers. Solve the second equation for $y$ to obtain

$$y = \frac{1998}{5} - \frac{4}{5}x,$$

and substitute into $B = xy$ to obtain

$$B = x\left(\frac{1998}{5} - \frac{4}{5}x\right).$$

The graph of this equation is a parabola with $x$ intercepts 0 and 999/2. The vertex of the parabola is halfway between the intercepts, at $x = 999/4$. This is the point at which $B$ assumes its maximum. However, this corresponds to a nonintegral value of $x$ (and hence $h$). From $4x + 5y = 1998$ both $x$ and $y$ are integers if and only if $x \equiv 2 \pmod 5$. The nearest such integer to $999/4 = 249.75$ is $x = 252$. Then $y = 198$, and this gives the maximal value for $B$ for which both $x$ and $y$ are integers. This maximal value for $B$ is $252 \cdot 198 = 49896$, and the requested remainder is 896.

9. (Answer: 259)

Let $G$ and $H$ be the points where the inscribed circle of triangle $BEF$ is tangent to $\overline{BE}$ and $\overline{BF}$, respectively. Let $x = EP$, $y = BG$, and $z = FH$. Then by equal tangents, $EG = x$, $BH = y$, and $FP = z$. Note that $y + z = BF = BC - CF = 364$, and $x + y = \sqrt{63^2 + 84^2} = \sqrt{21^2(3^2 + 4^2)} = 105$. Also note that $\triangle BEF \cong \triangle DFE$, so $FQ = x$. Thus $PQ = FP - FQ = z - x = (y+z) - (x+y) = 364 - 105 = 259$.

**OR**

Let $x$, $y$, and $z$ be defined as in the first solution, and let $O$ be the foot of the perpendicular from $E$ to $\overline{BF}$. Applying the Pythagorean Theorem to triangle $EOF$ yields $EF = \sqrt{EO^2 + FO^2} = \sqrt{63^2 + (448 - 2 \cdot 84)^2} = \sqrt{7^2(9^2 + 40^2)} = 7 \cdot 41 = 287$. Thus $z + x = EF = 287$, and $x + y = 105$ and $y + z = 364$, as shown in the first solution. Adding these three equations together and dividing by 2 yields $x + y + z = 378$. Thus $x = 378 - 364 = 14$, $y = 378 - 287 = 91$, and $z = 378 - 105 = 273$. Therefore $PQ = z - x = 273 - 14 = 259$.

10. (Answer: 710)

There are $2^6$ subsets of $S$, and for $0 \le k \le 6$, there are $\binom{6}{k}$ subsets with $k$ elements, so the probability that $A$ has $k$ elements is $\binom{6}{k}/2^6$. If $A$ has $k$ elements, there are $2^k$ subsets of $S$ contained in $A$ and $2^{6-k}$ subsets contained in $S-A$. One of these, the empty set, is contained in both $A$ and $S-A$, so the probability that $B$ is contained in either $A$ or $S-A$ is $\dfrac{2^k+2^{6-k}-1}{2^6}$. The requested probability is therefore

$$\sum_{k=0}^{6} \frac{\binom{6}{k}}{2^6} \cdot \frac{2^k+2^{6-k}-1}{2^6} = \frac{1}{2^{12}}\left(\sum_{k=0}^{6} 2^k\binom{6}{k} + \sum_{k=0}^{6} 2^{6-k}\binom{6}{k} - \sum_{k=0}^{6}\binom{6}{k}\right)$$

$$= \frac{1}{2^{12}}\left(2\sum_{k=0}^{6} 2^k\binom{6}{k} - \sum_{k=0}^{6}\binom{6}{k}\right) = \frac{1}{2^{12}}\left(2\cdot 3^6 - 2^6\right) = \frac{3^6-2^5}{2^{11}} = \frac{697}{2^{11}}.$$

Thus $m+n+r = 697+2+11 = 710$.

**OR**

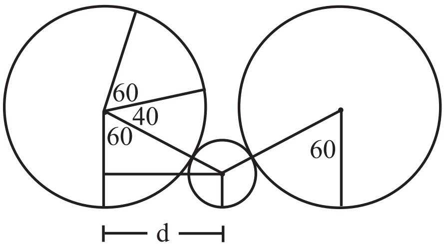
Let $S = \{1,2,3,4,5,6\}$. With $S_1 = A$ and $S_2 = B$, let $M$ be the $6\times 2$ matrix in which $m_{ij} = 1$ if $i \in S_j$ and 0 otherwise. There are $2^{12}$ such matrices. Observe that $B \subseteq A$ precisely when each row of $M$ is 11, 10, or 00. There are $3^6$ such matrices. Similarly, $B \subseteq S-A$ precisely when each row of $M$ is 01, 00, or 10, and again there are $3^6$ such matrices. The intersection of these two types of matrices are those in which each row is 00 or 10, and there are $2^6$ such matrices. The requested probability is therefore

$$\frac{3^6+3^6-2^6}{2^{12}} = \frac{2\cdot 3^6-2^6}{2^{12}} = \frac{3^6-2^5}{2^{11}} = \frac{697}{2^{11}}, \text{ as before.}$$

11. (Answer: 179)

Let the larger tube roll until it is tangent to the smaller tube as shown in the diagram. At that point, the centers of the tubes are a horizontal distance $d$ apart, where $d$ is one leg of a right triangle with hypotenuse $72+24 = 96$ and other leg $72-24 = 48$. It follows that the triangle is a 30 - 60 - 90 triangle with a $30^\circ$ angle at the center of the smaller tube, $d$ is equal to $48\sqrt{3}$, and the larger tube rests on a point of its circumference $60^\circ$ from the point of its circumference where it is tangent to the smaller tube.

When the larger tube finishes rolling over the smaller tube, it is tangent to the smaller tube on the other side of the smaller tube. Its center has moved a horizontal distance of $2d = 96\sqrt{3}$. It has rolled over an arc of $180^\circ - 2(30^\circ) = 120^\circ$ of the smaller tube, and thus it has rolled over an arc of one-third of $120^\circ$, or $40^\circ$ of the larger tube. The point of its circumference where the larger tube rests after rolling over the smaller tube is $60° + 40° + 60° = 160°$ from the point where it rested before rolling over the smaller tube. Thus the larger tube has rolled over an arc of $160°$ while moving horizontally a distance of $96\sqrt{3}$. When the larger tube completes one revolution, it has rolled horizontally by rolling through $360° - 160° = 200°$ of arc and moving a distance of $96\sqrt{3}$. Therefore the total horizontal distance covered is $\frac{200}{360} \cdot 72 \cdot 2\pi + 96\sqrt{3} = 80\pi + 96\sqrt{3}$. Thus $a + b + c = 80 + 96 + 3 = 179$.




12. (Answer: 091)

The sequence is geometric, so there exist numbers $a$ and $r$ such that $x_n = ar^n$. It follows that

$$308 = \sum_{n=0}^{7} \log_3(x_n) = \sum_{n=0}^{7} \log_3(ar^n) = \sum_{n=0}^{7} [\log_3(a) + n\log_3(r)] =$$

$$8\log_3(a) + \left(\sum_{n=0}^{7} n\right)\log_3(r) = 8\log_3(a) + 28\log_3(r).$$

Thus $2\log_3(a) + 7\log_3(r) = 77$. Furthermore,

$$\log_3\left(\sum_{n=0}^{7} x_n\right) = \log_3\left(\sum_{n=0}^{7} ar^n\right) = \log_3\left(a \cdot \frac{r^8 - 1}{r - 1}\right) = \log_3\left(ar^7 \cdot \frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}}\right) =$$

$$\log_3(a) + 7\log_3(r) + \log_3\left(\frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}}\right),$$

which is between 56 and 57.

Because the terms are all integral powers of 3, it follows that $a$ and $r$ must be powers of 3. Also, the sequence is increasing, so $r$ is at least 3. Therefore

$$1 = \frac{1 - \frac{1}{r}}{1 - \frac{1}{r}} < \frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}} < \frac{1}{1 - \frac{1}{3}} = \frac{3}{2} < 3, \text{ and } 0 < \log_3\left(\frac{1 - \frac{1}{r^8}}{1 - \frac{1}{r}}\right) < 1.$$

Also note that since $a$ and $r$ are powers of 3, $\log_3(a) + 7\log_3(r)$ is an integer and therefore must equal 56. Thus $\log_3(a) + 7\log_3(r) = 56$. The two equations $\log_3(a) + 7\log_3(r) = 56$ and $2\log_3(a) + 7\log_3(r) = 77$ have the solution $\log_3(a) = 21$ and $\log_3(r) = 5$.

It follows that $\log_3(x_{14}) = \log_3(ar^{14}) = \log_3(a) + 14\log_3(r) = 21 + 14 \cdot 5 = 91$.

13. (Answer: 640)

Number the rows from bottom to top, with the bottom row numbered 0 and the top row numbered 10. Let the entries in row 0, from left to right, be $x_0, x_1, \ldots, x_{10}$, where each $x_k$ is 0 or 1. In row $k$ (the $k$th row from the bottom) label the squares from left to right by $0, 1, \ldots, 10-k$. It can then be shown by induction that the number in row $k$ and square $j$, $0 \le j \le 10-k$, is

$$\binom{k}{0}x_{j+0} + \binom{k}{1}x_{j+1} + \cdots + \binom{k}{k}x_{j+k} = \sum_{i=0}^{k}\binom{k}{i}x_{j+i}.$$

Thus the entry in the top square (in row 10) is

$$\sum_{i=0}^{10}\binom{10}{i}x_i.$$

It is easy to check that $\binom{10}{k}$ is a multiple of 3 for $2 \le k \le 8$. Thus

$$\sum_{i=0}^{10}\binom{10}{i}x_i \equiv x_0 + \binom{10}{1}x_1 + \binom{10}{9}x_9 + x_{10} \equiv x_0 + x_1 + x_9 + x_{10} \pmod 3.$$

For this last expression to be a multiple of 3, either $x_0 = x_1 = x_9 = x_{10} = 0$ or three of these four numbers are 1 and the fourth is 0. Thus there are five choices of $x_0, x_1, x_9, x_{10}$ that make the sum a multiple of 3. Furthermore, each of $x_2, x_3, x_4, \ldots, x_8$ can be either 0 or 1, so these 7 values can be assigned in $2^7$ ways. Thus there are $5 \cdot 2^7 = 640$ initial distributions that result in the number in the top square being a multiple of 3.

14. (Answer: 676)

If the leading term of $f(x)$ is $ax^m$, then the leading term of $f(x)f(2x^2) = ax^m \cdot a(2x^2)^m = 2^m a^2 x^{3m}$, and the leading term of $f(2x^3 + x) = 2^m a x^{3m}$. Hence $2^m a^2 = 2^m a$, and $a = 1$. Because $f(0) = 1$, the product of all the roots of $f(x)$ is $\pm 1$. If $f(\lambda) = 0$, then $f(2\lambda^3 + \lambda) = 0$. Assume that there exists a root $\lambda$ with $|\lambda| \neq 1$. Then there must be such a root $\lambda_1$ with $|\lambda_1| > 1$. Then $|2\lambda^3 + \lambda| \ge 2|\lambda|^3 - |\lambda| > 2|\lambda| - |\lambda| = |\lambda|$. But then $f(x)$ would have infinitely many roots, given by $\lambda_{k+1} = 2\lambda_k^3 + \lambda_k$, for $k \ge 1$. Therefore $|\lambda| = 1$ for all of the roots of the polynomial. Thus $\lambda\overline{\lambda} = 1$, and $(2\lambda^3 + \lambda)\overline{(2\lambda^3 + \lambda)} = 1$. Solving these equations simultaneously for $\lambda = a + bi$ yields $a = 0$, $b^2 = 1$, and so $\lambda^2 = -1$. Because the polynomial has real coefficients, the polynomial must have the form $f(x) = (1 + x^2)^n$ for some integer $n \ge 1$. The condition $f(2) + f(3) = 125$ implies $n = 2$, giving $f(5) = 676$.

15. (Answer: 389)

Let $O_A$, $O_B$, and $O_C$ be the centers of $\omega_A$, $\omega_B$, and $\omega_C$, respectively. Then $\overline{O_AO_B} \parallel \overline{AB}$, $\overline{O_BO_C} \parallel \overline{BC}$, and $\overline{O_CO_A} \parallel \overline{CA}$. Also, the lines $AO_A$, $BO_B$, and $CO_C$ are concurrent at $I$, the incenter of triangle $ABC$, and therefore there is a dilation $\mathcal{D}$ centered at $I$ that sends triangle $O_AO_BO_C$ to triangle $ABC$. Let $R$ and $r$ be the circumradius and inradius of triangle $ABC$, respectively, and let $R_1$ and $r_1$ be the circumradius and inradius of triangle $O_AO_BO_C$, respectively. Then $\dfrac{R}{r} = \dfrac{R_1}{r_1}$. By Heron's formula,

$$\begin{aligned}[ABC] &= \frac{\sqrt{(13+14+15)(13+14-15)(14+15-13)(15+13-14)}}{4} \\ &= \frac{r(13+14+15)}{2} = \frac{13 \cdot 14 \cdot 15}{4R},\end{aligned}$$

implying that $r = 4$ and $R = \frac{65}{8}$. Let $x$ be the radius of $\omega$. Because $I$ is the center of $\mathcal{D}$, $r_1 = r - x$. Let $S$ be the center of $\omega$. Then $S$ is equidistant from $O_A$, $O_B$, and $O_C$, that is, $S$ is the circumcenter of triangle $O_AO_BO_C$. Thus $R_1 = SO_A = 2x$. Therefore

$$\frac{2x}{4-x} = \frac{2x}{r-x} = \frac{R_1}{r_1} = \frac{R}{r} = \frac{65}{32}.$$

Solving the last equation gives $x = \dfrac{260}{129}$, and $m + n = 389$.

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