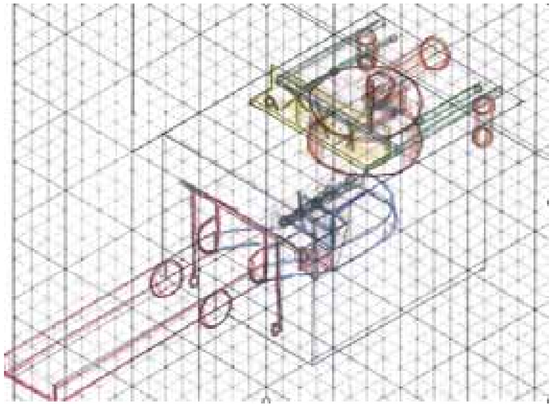
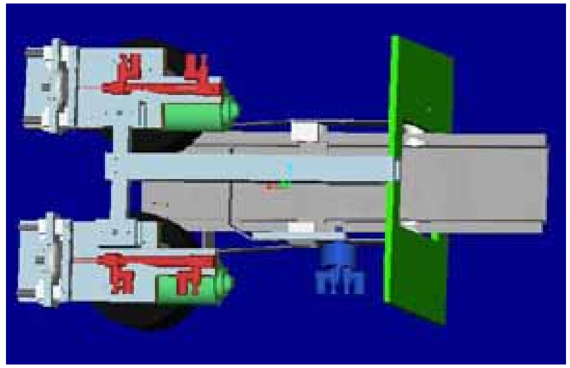
FUNdaMENTALS of Design

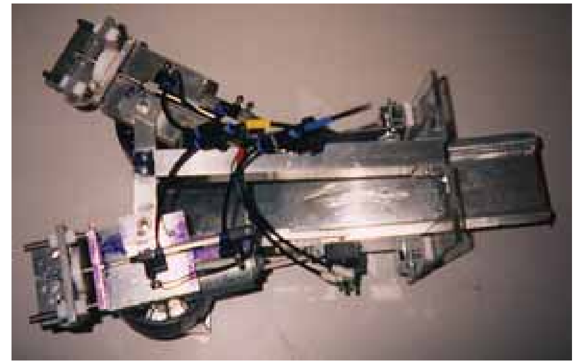
Topic 8

Structures



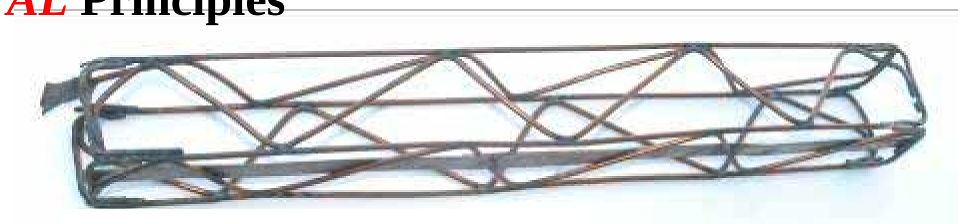
Topic 8
Structures

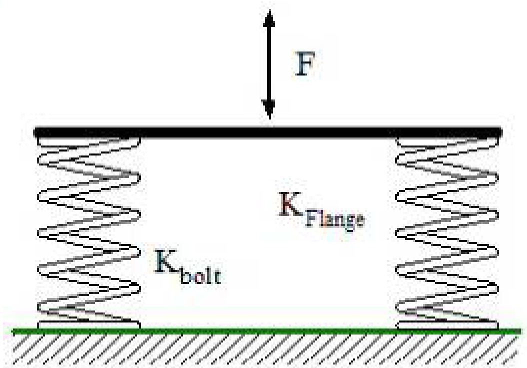


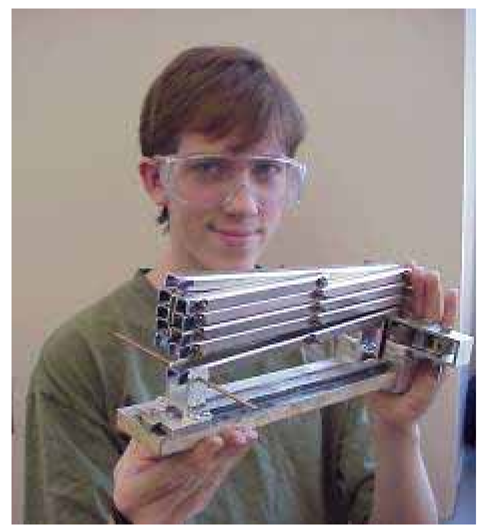


Topics

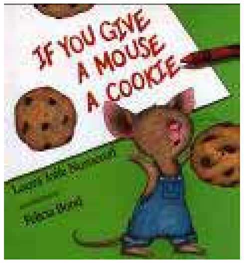
- Beginnings
- ***FUN**da**MENTAL*** **Principles**
- Materials
- Visualization
- Layout
- Stability
- Loadings
- Stiffness
- Strength
- Trusses
- Laminates & Composites







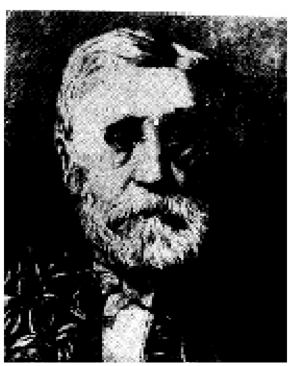
Bryan Ruddy's most amazing Lazy-Tongs structure (2001)



Beginnings

Rony Kubat's most professional 2.007 machine EVER!

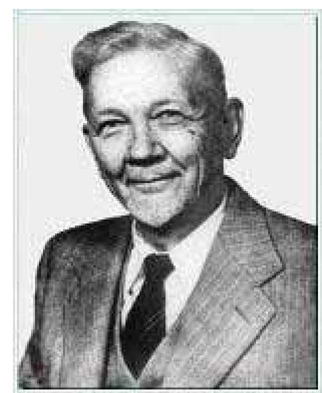
- People have always sought to create ever larger, more complex structures
  - A structure might be able to hold its own weight, but then how much of a load could it carry?
- Bridges represent the greatest structural challenges:
  - Whenever a longer bridge was needed, adding more material also increased the weight…
    - http://www.bizave.com/portland/bridges/Bridge-Gallery1.html
      http://www.discovery.com/stories/technology/buildings/brdg_explore.html#photo
- The great mathematicians of the 18^{th} century set their minds to the task of developing mathematical formulas for predicting the strength of structures
  - See Stephen Timoshenko, *History of the Strength of Materials*
- History repeats itself (Patterns!)
  - Your machine has limited size & weight, yet you want your machine to reach out to the world

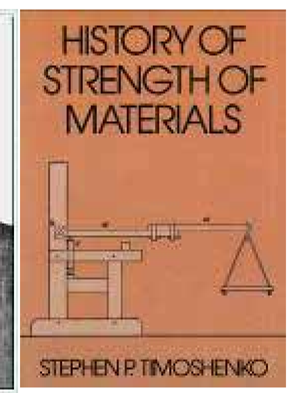


Barré de Saint-Venant
1797-1886

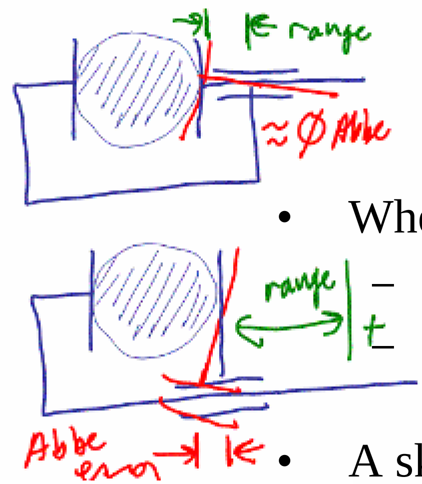
Leonhard Euler
1707-1783

John McBean's longest 2.007 truss ever!

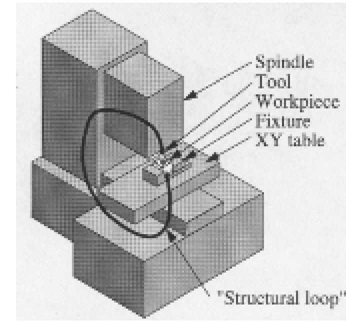




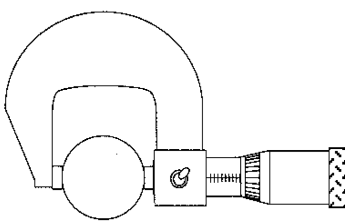
Stephen Prokofyevich Timoshenko
(1878-1972)
http://smitu.cef.spbstu.ru/timoshenko_en.htm

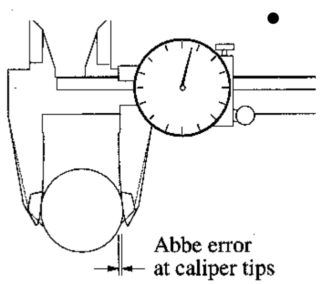


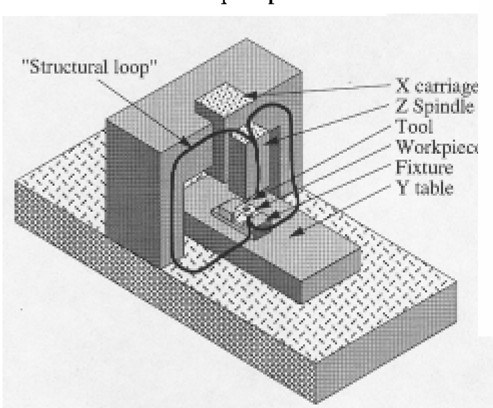
*FUN*da*MENTAL* Principles

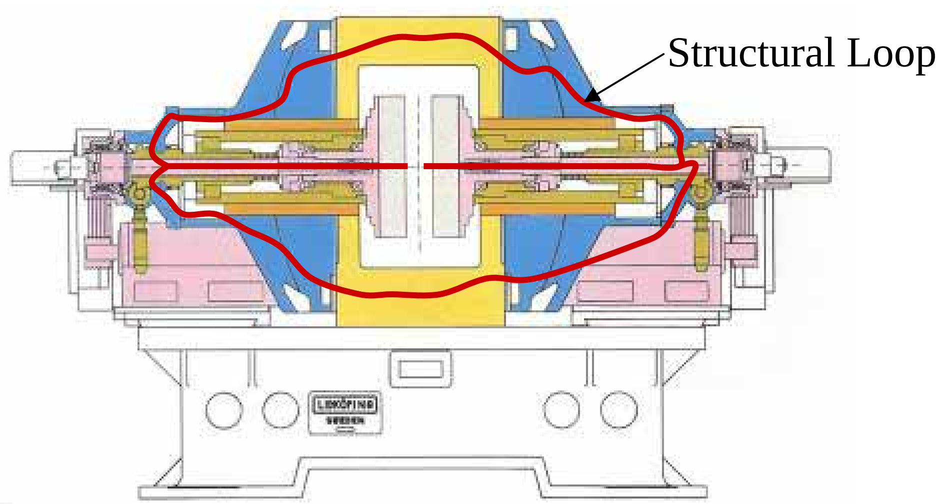


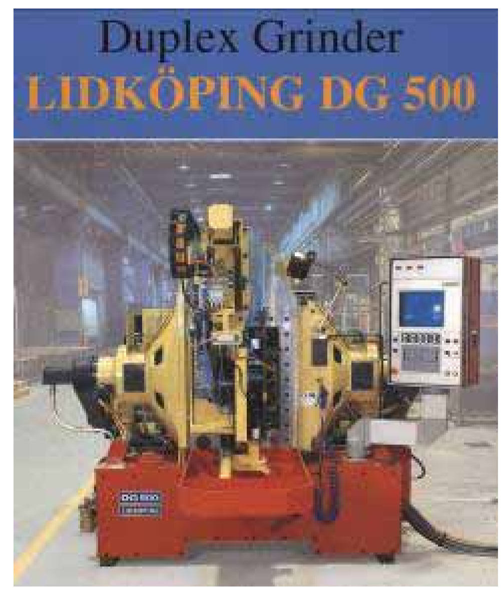
- When the first sketch of the structure is made:
  - Arrows indicating forces and moments should also be sketched
  - The path of how these forces and moments flow from the point of action to the point of reaction, shows the *structural loop*
- A sketch of the structural loop is a great visual design aid
  - A closed structural loop indicates high stability and the likely use of symmetry to achieve a robust design
  - An open structural loop is not bad, it means "proceed carefully"
  - Remember Aesop's fables & "The Oak Tree and the Reeds"
- Example: automobiles to disk drives to semiconductors, exist because of double-disk grinders' ability to create parallel flat surfaces:

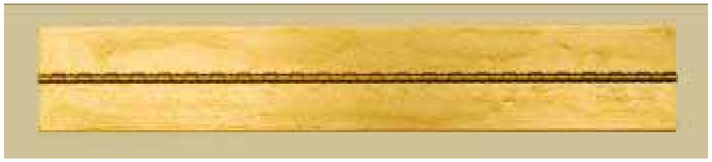








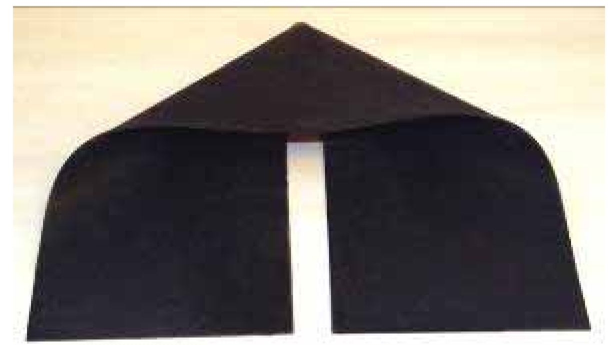




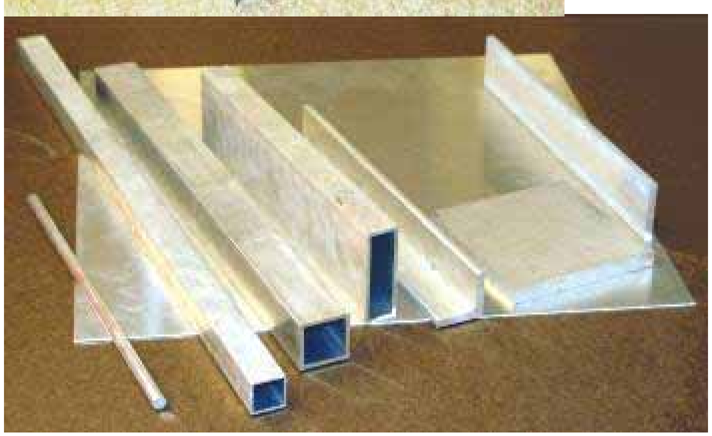
Materials

- Materials make the machine just as sure as any creative design, and are often selected based on strength, stiffness, manufacturability, and wear and corrosion resistance
  - Metals have very high strength-to-weight ratios and are easily machined formed, and joined
  - Woods high directional strength/weight and are easily joined
  - Plastics can have structural and low friction & wear-resistant properties and are easily molded, formed, machined

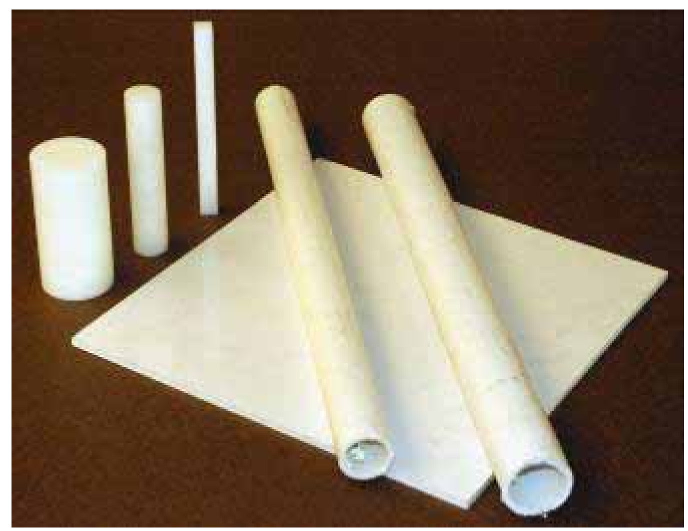
Mild steel shapes in the 2.007 kit



It's a bird! It's a plane! NO, look, it's a sheet of rubber from the 2.007 kit



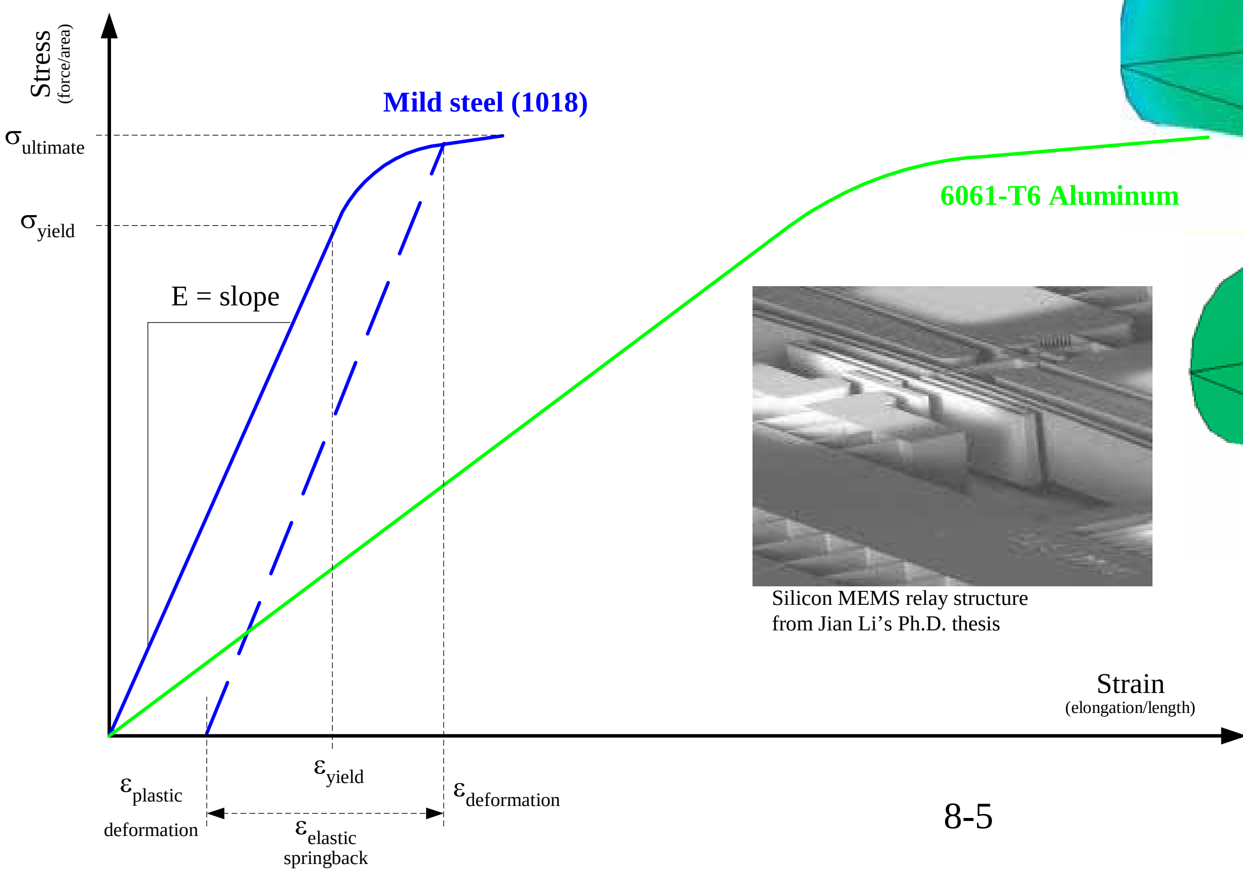
Aluminum shapes in the 2.007 kit



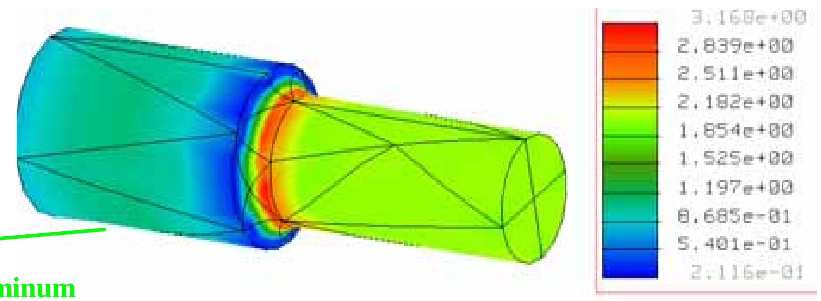
Plastic shapes in the 2.007 kit

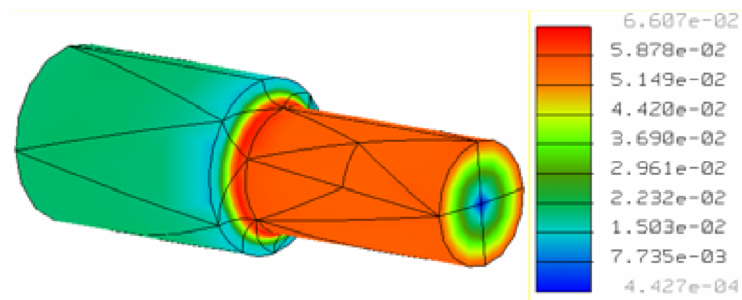
Materials: *Wear, Strength & Stiffness*

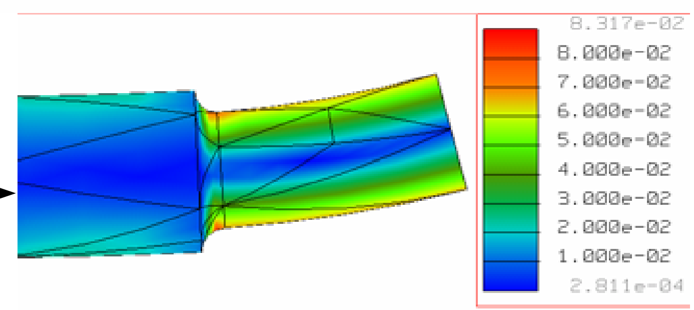
- *Axial*, *torsion*, and *bending* loads can be applied to structures and components
  - An equivalent stress needs to be determined and compared to the material's yield stress
- Yielding in a component's material can mean the component has failed, as will the machine, or it can be used to form a component during manufacturing
  - Elastic instability (buckling) can affect shafts and columns in compression or torsion
  - Know your limits! (See pages 5-20 to 5-23!)

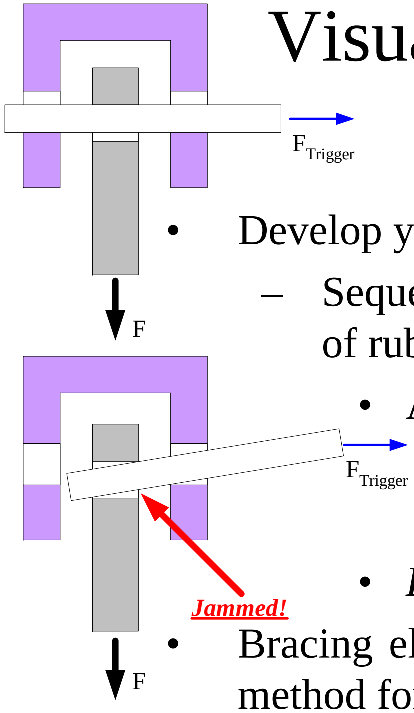


Silicon MEMS relay structure
from Jian Li's Ph.D. thesis





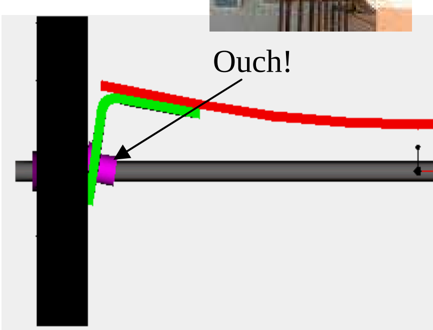


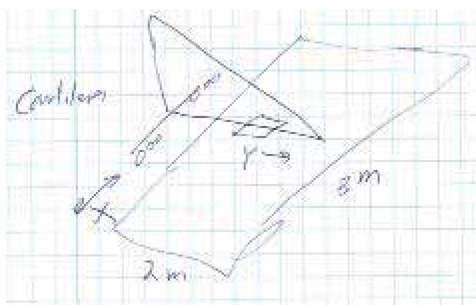


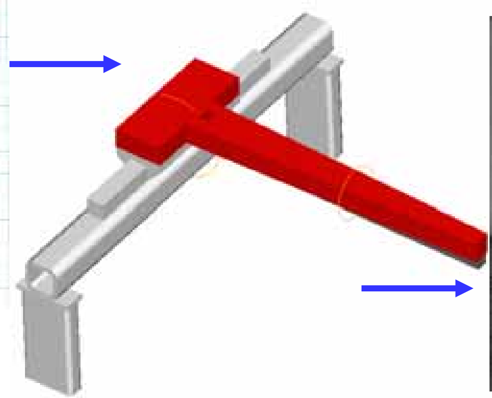
Visualization

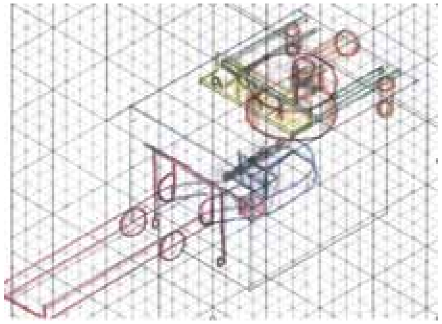
- Develop your ability to imagine a structure deforming as loads are applied
  - Sequentially imagine that each element of the design is a piece of rubber, while other elements are steel
    - Apply forces to the system and see how it deforms
      - Does the deformation cause problems?
      - How can structure be changed to minimize deformations?
    - *Play the movie in your mind*

- Bracing elements with triangles (plate-type gussets or beam-type trusses) is the most efficient method for strengthening a structure!
- Creating CAD or paper and pushpin models is an effective way to visualize a structure
  - Even if you are planning on using finite element analysis, a simple model can help you later determine if the results are meaningful!

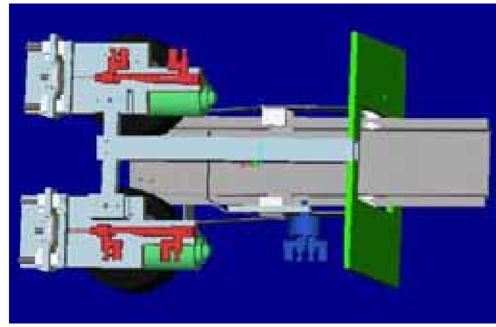


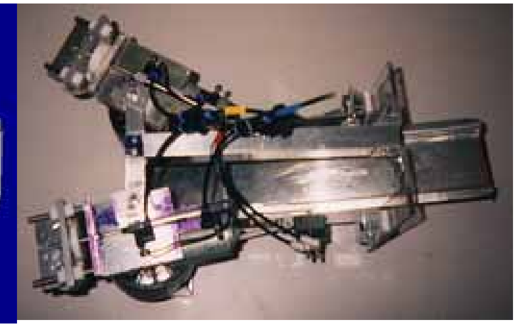




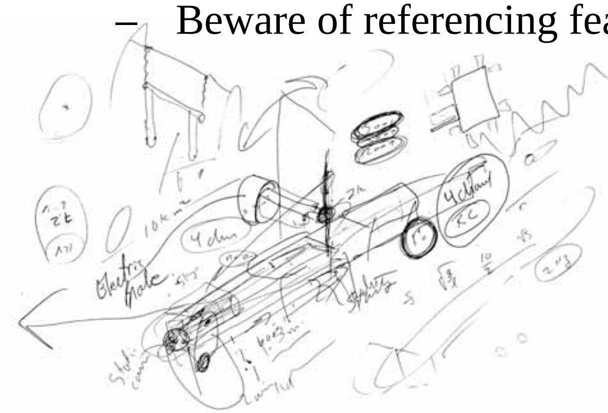


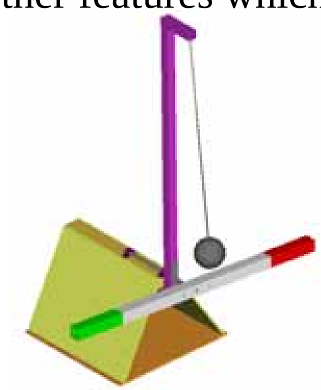
Layout





- *Layout* is where the designer starts to define relative placement of elements and the supporting structure
  - The ***Layout*** is the first embodiment of the ***design intent*** and defines boundaries on the structure
  - *You can create several simple stick-figure layout sketches of different concepts*
  - *Use appropriate analysis (e.g., 1st order error budget) guided by the layouts to help select the best!*
- A *Layout Drawing* is the graphical interpretation of the FRDPARRC table’s Design Parameters
  - As a design progresses from **Strategy, to Concept,** to **Modules**, to **Components**, the *layout* is the first step towards creating the details
- A hand-drawn sketch & notes suffices for an initial layout, & it is a road map for creating a solid model
- A solid model can serve as a layout, as long as one takes care to not add a lot of detail
  - Use datum planes and curves referenced to a global coordinate system
  - Beware of referencing features to other features which may leater be moved or deleted





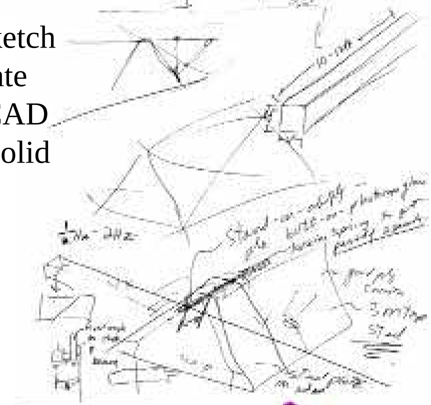
Will Delhagen & Alex Jacobs in
MIT’s 2001 2.007 contest ***Tiltilator!***

Layout: *Sketches*

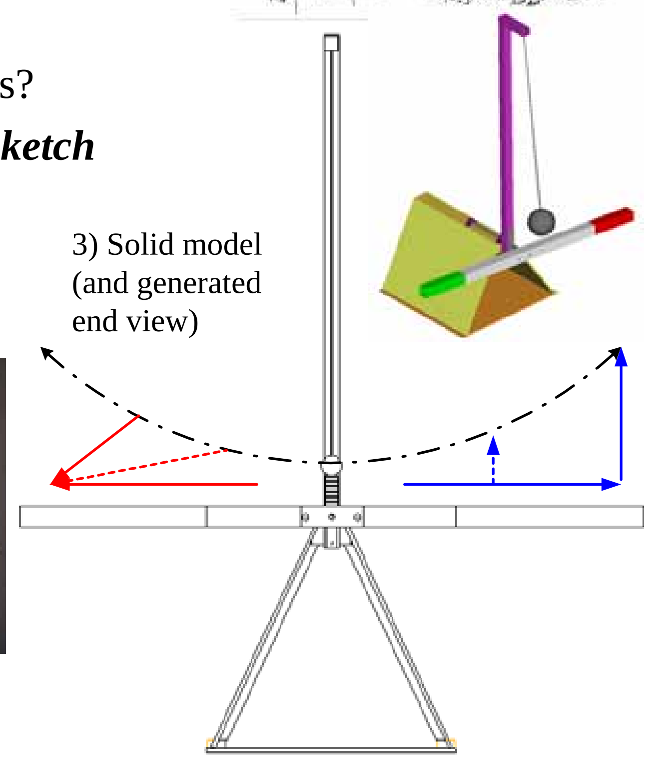
1) Early concept development (used in PREP)

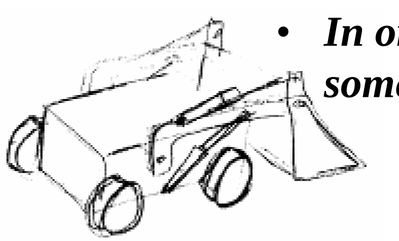
- Use *Motion diagrams* and *stick figures* to help define and select your concept as initial starting points for your layout
  - Design is like a flexible anagram
    - You are allowed not only to rearrange things
      - You are allowed to add or subtract things!
    - Use your knowledge of the FRs and DPs and of fundamental principles to catalyze the creation of the layout
  - E.g., the red-line-strategy machine grabs the pendulum with a flexible arm and then goes to the end of the beam
    - What kinds of structures can enable a machine to do this?
    - ***In order to define the structure, you will also have to sketch some basic ideas for the mechanism***

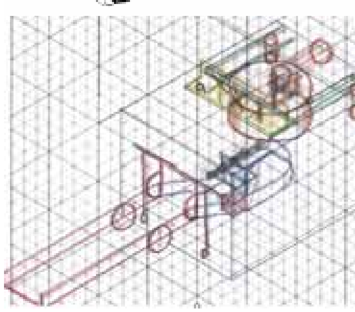
2) Layout sketch (used to create simple 2D CAD drawing or solid model)

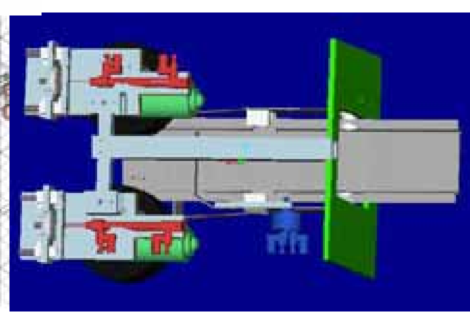


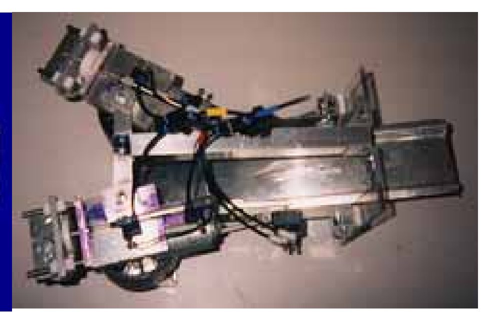
3) Solid model (and generated end view)











Alex Sprunt's Wall-Crawler

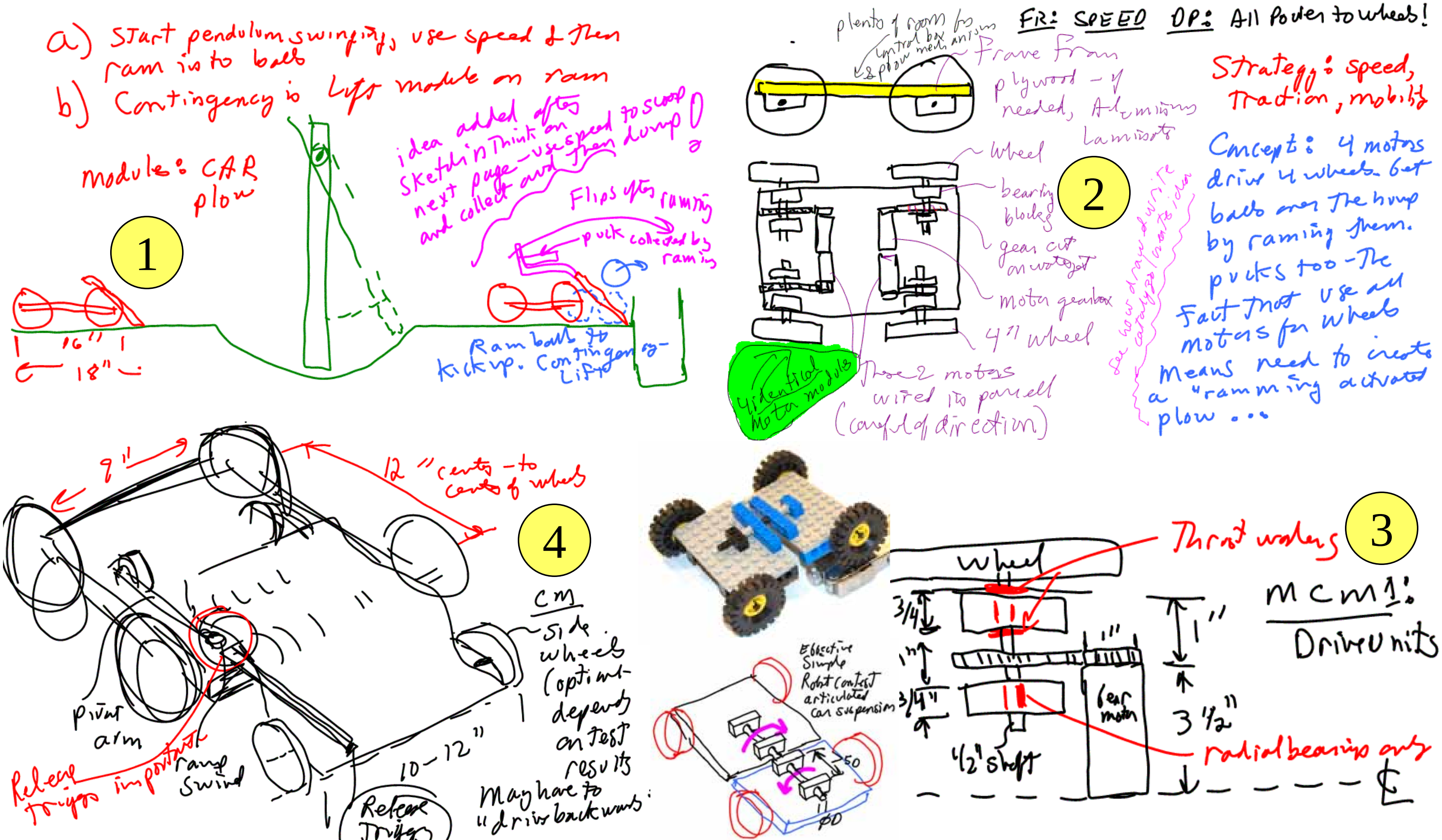
Layout: *FRDPARRC*

- Use a FRDPARRC table to guide creation of initial layout sketches
  - Example: For the *MIT & the Pendulum* contest, create layouts for ***Concepts*** for *Start pendulum swinging and collect balls and pucks* ***Strategy***

Functional Requirements	Possible Design Parameters (**Modules FR's**)	Analysis	References	Risk	Counter-measures
Gather pucks and balls and deposit in goal	1)Pick up and score one at a time 2)**Harvest lots and dump loads**	1)Time/Motion study, Friction/slip, Linkage design 2)Friction, slip, linkage design	Physics text and past contests. Farm equipment websites	1)Not enough time to make multiple trips 2)Gather bin is too large	1)Gather 2 or 3 objects 2)Gather 2 or 3 objects
Actuate pendulum from ground	**1)Vehicle knocks pendulum as it drives by** 2)Fixed-to-ground spinning actuator	**?**	**?**	**?**	**?**
Block opponent	**1)Botherbot** 2)Pendulum clamp **3)Cover goal**	**?**	**?**	**?**	**?**

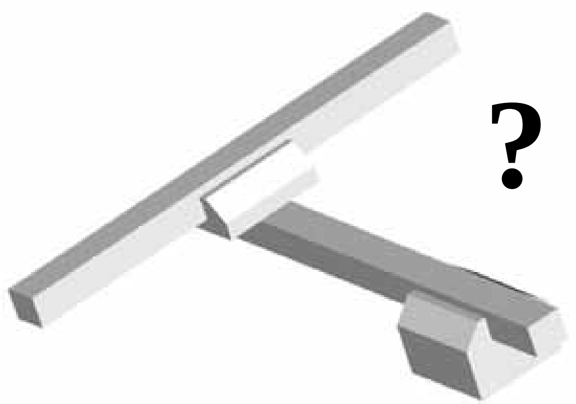
Layout: *"GeekPlow" Example*

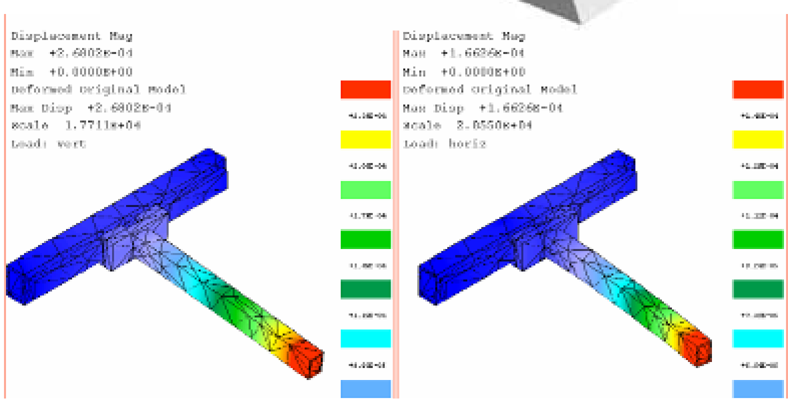
- Appropriate detail for layout sketches and *Peer Review Evaluation Process* (PREP) of a machine (sketched with a Tablet PC) created according to the previous FRDPARRC table:

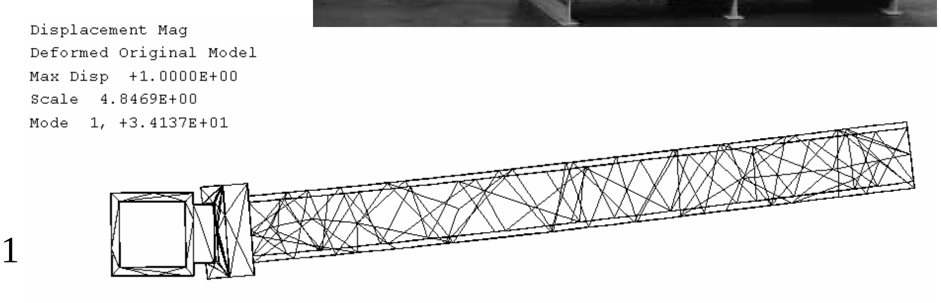


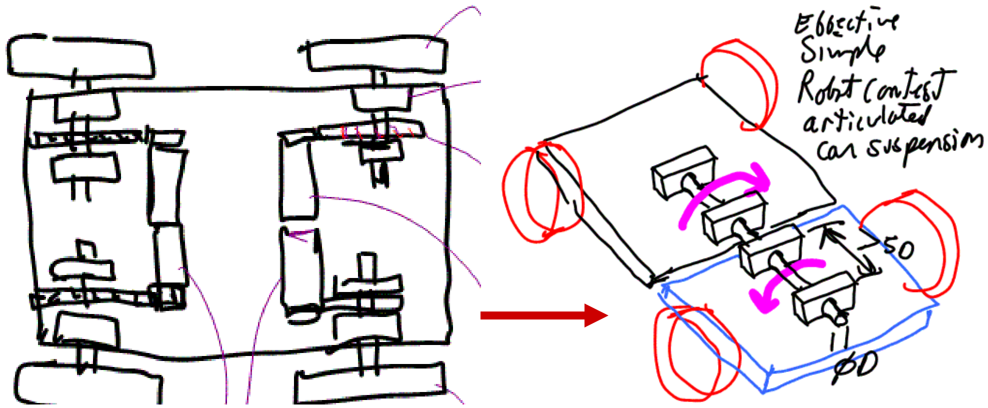
Layout: *Analysis & Bench Level Experiments*

- Since layout involves creating the overall skeleton or supporting structure, it also is a first chance to define the overall structural performance
  - The key is to understand how the *structural loop* behaves
    - Sequentially imagine each component is made of a soft material…visualize…
    - First order calculations are in order
    - If the machine is complex, Finite Element Analysis (FEA) may also be used
    - If analysis is too costly (e.g., time to do), consider a *Bench Level Experiment*
    - Deflections, stresses, and vibration modes can all be initially estimated





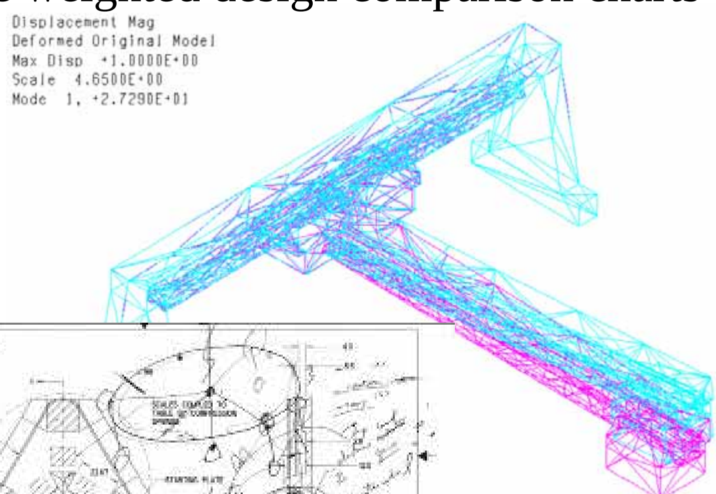


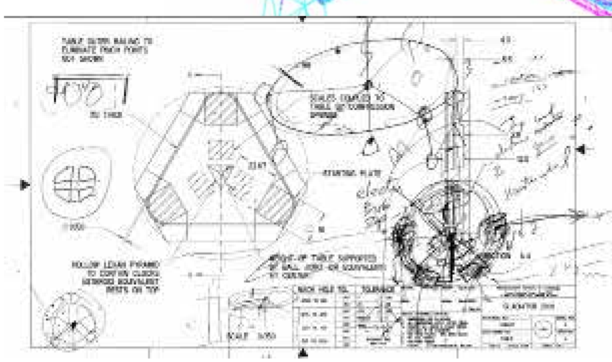


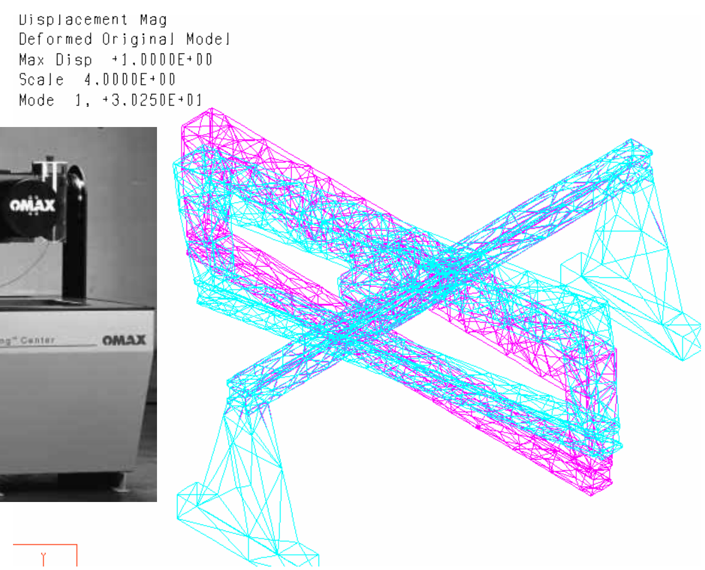
Layout: *Evolution & Comparison*

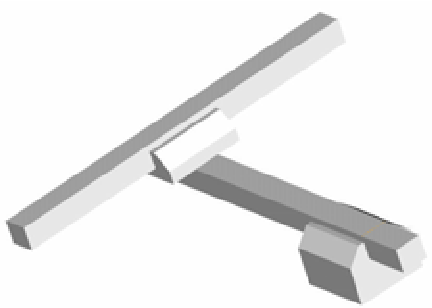
- There are often two or more possible design paths
  - Use analysis, manufacturability, & robustness design reviews
- The overall structure must be defined before module development can commence in earnest
  - Make the design amenable to evolution as detail later emerges
- Use weighted design comparison charts

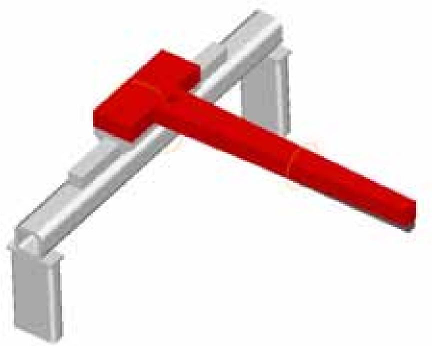
Comparison between one large or two small bearings		
LWH55 (T2 preload)		
Moment (kgf-m)	**42**	
Deflection (rad, minutes)	0.000582	**2**
Moment stiffness (kgf-m/rad)	71,463	
distance to load (m)	**0.12**	
Load (kgf)	**100**	
resulting moment (kgf-m)	12	
resulting deflection (rad)	0.000168	
resulting deflection (microns)	20.2	
equivelent stiffness (N/micron, lbs/inch)	49	286,086
rated moment load capacity (kgf-m)	431	
equivelent load capacity at outside edge (kgf)	3,592	
Lwhd15 (T2 preload)		
Force (kgf)	200	
deflection (microns)	10	
Lateral stiffness	20	
distance between bearing rails	0.2	
stiffness (kgf-m/rad)	800,000	
rated load at edge of table (2 trucks) (kgf)	1900	

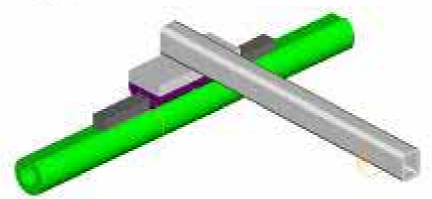






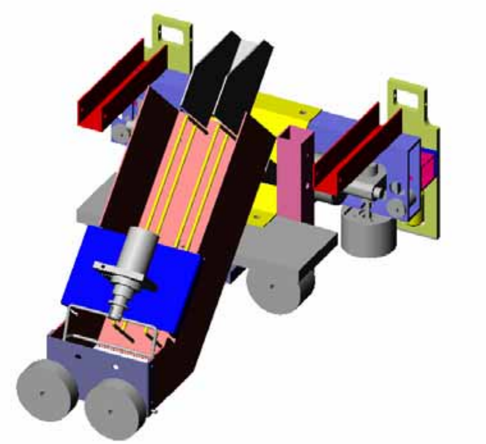


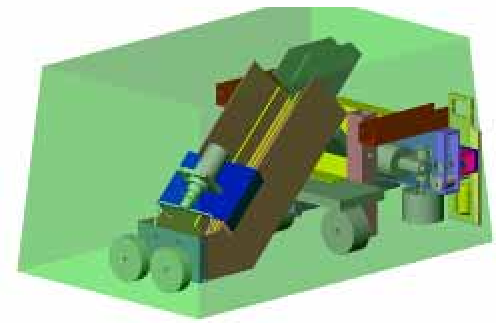




Layout: *Solid Models*

- In order to create an appropriate level-of-detail solid model layout drawing:
  - – Use the FRDPARRC tables from **Strategy, to Concept,** to M**odules**, to **Components** to understand what functions the structure must perform
    - The chicken-and-egg issue is that no detail yet exists, only sketches and spreadsheets
      - – Most machines must have an overall structure, a skeleton or frame, to which modules are attached
        - » Creating the skeleton or frame is the critical first step in catalyzing the generation of detail for all the modules
        - » Details are added as the design progresses

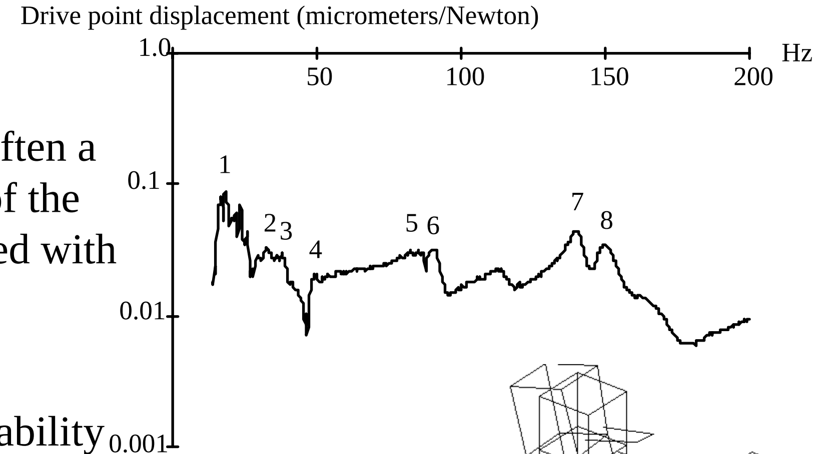




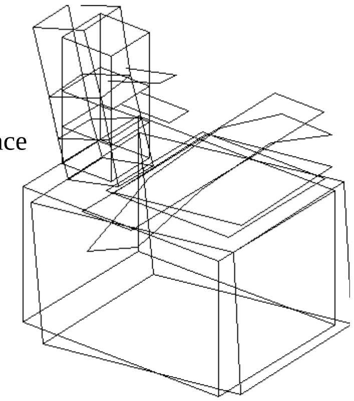
Rony Kubat's solid model and machine. Rony's solid model fit in the solid model size-constraint box as did his real machine in the real box. It performed as modeled and very nearly won the 2.007 *MechEverest* contest as it scored a perfect 50 points EVERY time…

Stability

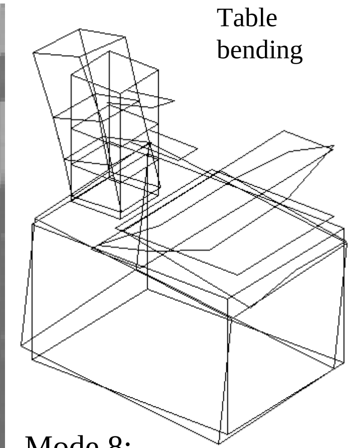
- Static Stability:
  - For robot contest machines, tipping-over stability is often a prime *Functional Requirement* that drives the shape of the overall structure and where the modules will be located with respect to each other
- Dynamic stability and Buckling (see page 5-23!):
  - Are structural resonances excited that can lead to instability and degradation of components or the process?
  - Do axial compression forces cause the component to buckle?
- Positive uses (apply reciprocity!)
  - Pile drivers, ultrasonic machining, triggers…



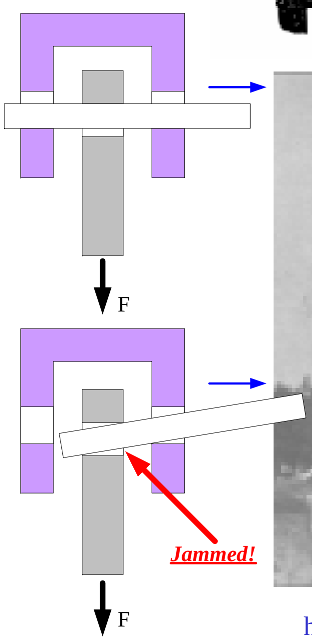
Experimental modal analysis of a small surface grinder (performed by Eric Marsh when he was Prof. Slocum's Ph.D. student)

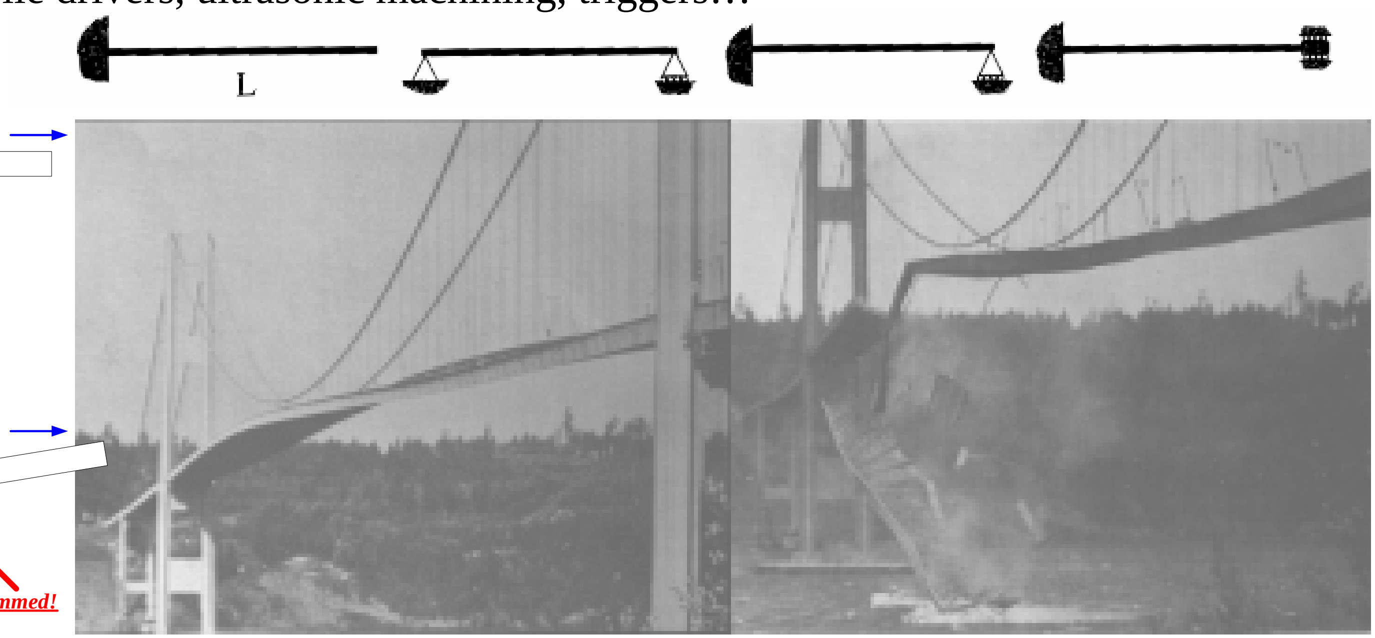


Mode 7: Table bending



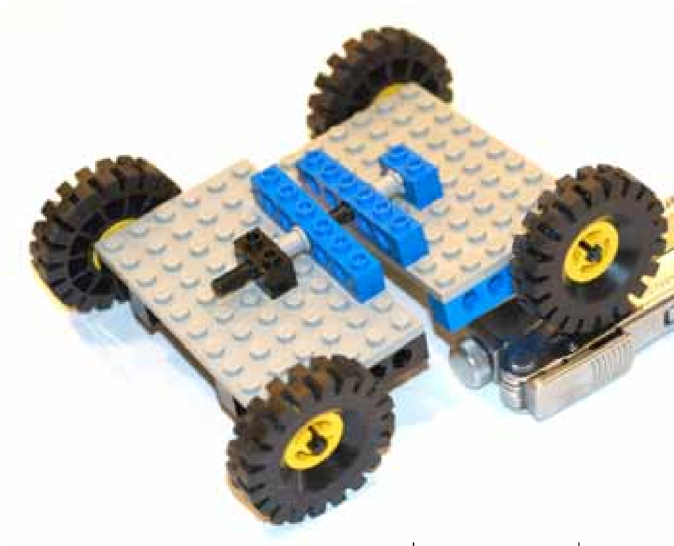
Mode 8: Column bending



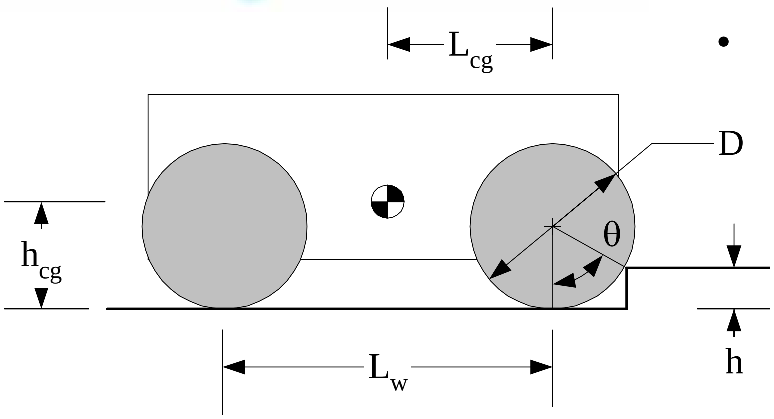


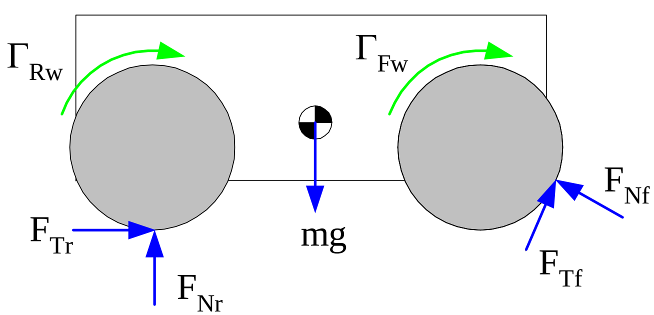
http://www.eng.iastate.edu/explorer/Bridge/collapse.htm

Stability: *Obstacles*



- Two wheel drive vehicles: The rear wheels have to push hard enough to make the front wheels climb the obstacle
- Four wheel drive vehicles: The rear wheels also provide the normal force needed for the front wheels to apply a tractive effort to help climb over the obstacle
- What do the free-body diagrams show about "pushing" verses "pulling" the wheels over a bump?
  - Is it better to try and climb a bump straight-on (both front wheels engage it at the same time) or one wheel at a time?
- Experiment with the spreadsheet *Driving_over_step.xls*



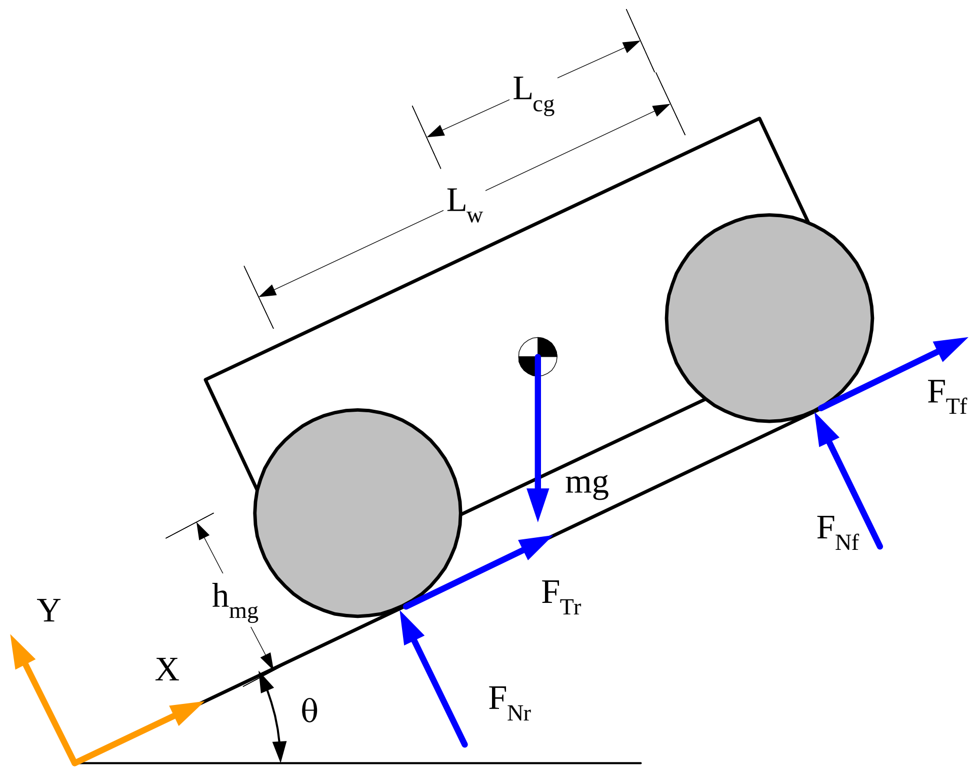


Option 1: Nice and easy slow drive over the step	2 wheels contact	1 wheel contact
Normal force between a rear wheel and the ground, FNr/2 (N)	13.9	13.9
Minimum required coefficient of friction, mumin2, mumin1	0.43	0.24
Normal force between a front wheel and the step FNf/2 (N)	11.2	13.3
Total motor tractive effort, gm (N)	16.0	16
Total motor limited tractive force from from both rear wheels, Frwmax (N)	8.0	8
Total motor limited tractive force from from both front wheels, Ffwmax (N)	8.0	8
Total friction limited tractive effort, gmu (N)	15.1	16.3
Total friction limited tractive force from from both rear wheels before slip, Frwmumax (N)	8.4	8.4
Total friction limited tractive force from from both rear wheels before slip, Ffwmumax (N)	6.7	8.0
Total minimum tractive effort required, gmin (N)	21.7	13.22
Total tractive effort required by both rear wheels, FTr (N)	12.06	6.77
Total tractive effort required by both front wheels, FTf (N)	9.66	6.45
Step Climable?		
Can the machine climb over the step?	no	yes
Option 2: Ramming speed!		
Ideal forward velocity required to get over the step, v (mm/s)	626	
Δ potential energy on top of step, DPE (N-m)	0.98	

Stability: *Slopes & Balance*

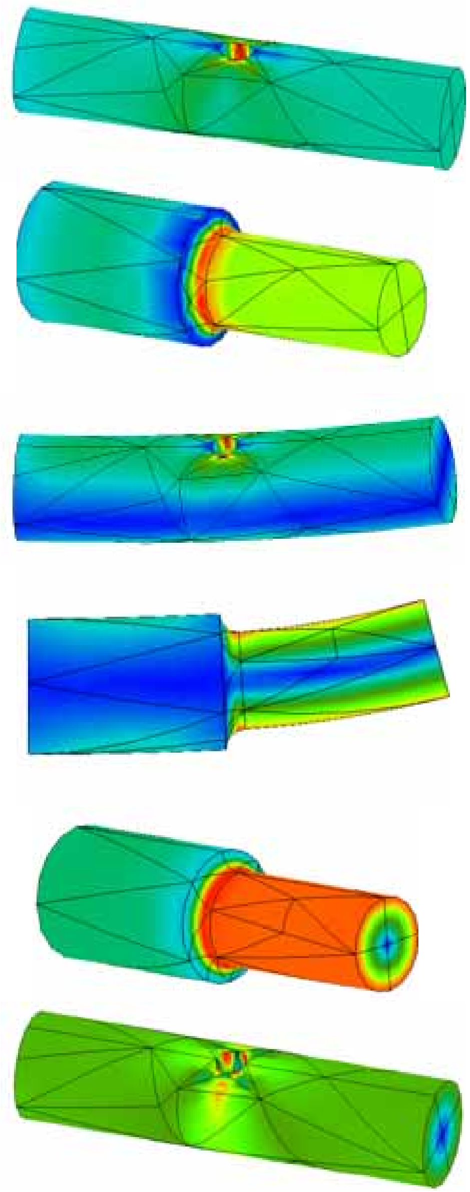
- Can the machine drive up the hill?
  - What is better for climbing hills, FWD or RWD?
  - Do you really need AWD?
- When will the machine tip over?
  - What happens when the force vector due to gravity just passes through the rear wheels ground contact points?

Funny image found on www, photographer not credited, would like to, email slocum@mit.edu

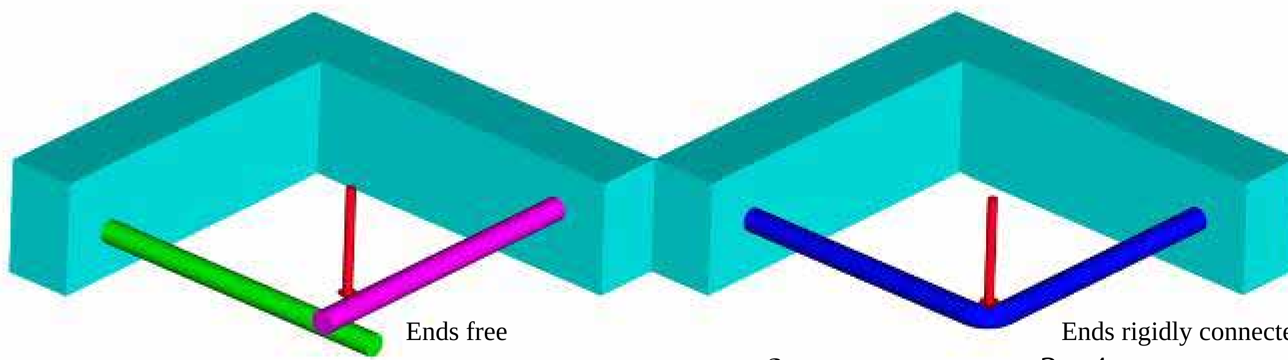


Driving_up_slope.xls		
System	**Values**	
Rear wheels' diameter, Drw (mm)	**100**	
Front wheels' diameter, Dfw (mm)	**100**	
Distance between wheels, Lw (mm)	**250**	
Distance center of front wheel to center of gravity, Lcg (mm)	**125**	
Height of center of mass about plane, hmg (mm)	**50**	
Slope angle, theta (deg, rad)	**20.00**	**0.35**
Machine mass, m (kg)	**4**	
Machine weight, mg (N)	**39.2**	
Maximum drive torque applied to both rear wheels, grw (N-mm)	**400**	
Maximum drive torque applied to both front wheels, gfw (N-mm)	**400**	
Coefficient of friction, mu	**0.5**	
for FWD $\gamma_f = 1$ and $\gamma_r = 0$, for RWD $\gamma_f = 0$ and $\gamma_r = 1$, and for AWD $\gamma_f = 1$ and $\gamma_r = 1$		
Enter 1 for rear wheel drive or AWD, γr	**1**	
Enter 1 for front wheel drive or AWD, γf	**1**	
Nice and easy slow drive up the ramp (no acceleration)		
Normal force between both rear wheels and ground, FNr (N)	**21.1**	
Normal force between both front wheels and step FNf (N)	**15.7**	
Maximum rear wheel tractive force from drive torque, Frwmax (N)	**8**	
Maximum rear wheel force from drive torque, Ffwmax (N)	**8**	
Maximum rear wheel tractive force before slip, Frwmumax (N)	**10.5**	
Maximum front wheel tractive force before slip, Ffwmumax (N)	**7.9**	
Total tractive force generated by both rear wheels, FTr (N)	**8.00**	
Total tractive effort required by both front wheels, FTf (N)	**7.87**	
Total tractive effort of the machine, FTm (N)	**15.87**	
Force from gravity acting along the incline, Fg (N)	**13.41**	
Can the machine climb up the ramp?	**yes**	
Tip-over angle (deg, rad)	**68.2**	**1.19**
Required minimum friction coefficients:		
Front wheel drive only required coefficient of friction	**0.85**	
Rear wheel drive only required coefficient of friction	**0.64**	
All wheel drive required coefficient of friction	**0.36**	

Loadings: *Axial*, *Bending*, *Torsion*, & *Shear*



- There are four basic types of loads:
  - *Axial tension* or *compression*: the applied force directly acts on the material to cause tension or compression
  - *Bending*: an applied force acts via a lever to bend a beam, causing tension on one side and compression on the other side of the structure
  - *Torsion*: a torque (e.g., twisting or two equal and opposite forces applied about a point) causes twist of the structure
  - *Shear*: two equal and opposite essentially collinear forces act perpendicular to a structure
    - Structures that fail in torsion are actually also failing in shear
    - Glue joints in laminates, subjected to bending, actually fail in shear
- Boundary conditions are critical!

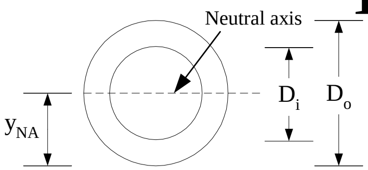


For equal diameter and length beams: $\frac{\delta_{\text{ends of beams joined}}}{\delta_{\text{one beam resting on the other}}} = \frac{3+4\nu}{4\nu}$

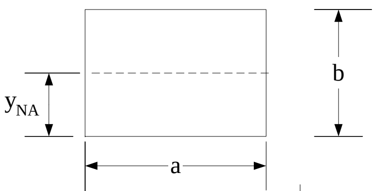
Be thankful for spreadsheets!

Loadings: *Structural Cross-Sections*

$$K_{\text{torsional stiffness}} = \frac{G I_P}{L}$$

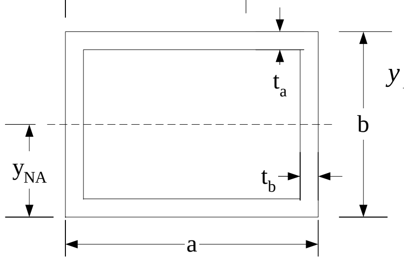


$$y_{NA} = \frac{D_o}{2} \qquad I_{bending} = \frac{\pi\left(D_o^4 - D_i^4\right)}{64} \qquad I_{torsion} = \frac{\pi\left(D_o^4 - D_i^4\right)}{32} \qquad \tau_{max} = \frac{16\Gamma}{\pi D_o^4}$$



$$y_{NA} = \frac{b}{2} \qquad I_{bending} = \frac{ab^3}{12} \qquad I_{torsion} = ab^3\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right] \qquad \tau_{max} = \frac{4.8\Gamma}{a^3}$$

$$I_{\text{torsion with a = b}} = 2.25b^4$$

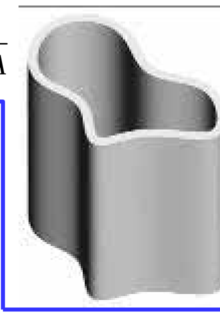


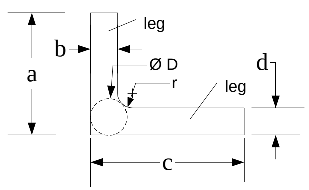
$$y_{NA} = \frac{b}{2} \qquad I_{bending} = \frac{ab^3 - \left(a - 2t_b\right)\left(b - 2t_a\right)^3}{12} \qquad I_{torsion} = \frac{2t_a t_b\left(a - t_b\right)^2\left(b - t_a\right)^2}{a t_b + b t_a - t_a^2 - t_b^2} \qquad \tau_{\text{max short side}} = \frac{\Gamma}{2t_b\left(a - t_b\right)\left(b - t_a\right)}$$

$$I_{\text{torsion with a = b and uniform thickness t}} = t\left(b - t\right)^3 \qquad \tau_{\text{max long side}} = \frac{\Gamma}{2t_a\left(a - t_b\right)\left(b - t_a\right)}$$

$$I_p = \frac{4A^2_{\text{mean of areas enclosed by boaundaries}}\, t_{\text{thickness}}}{P_{\text{length of median boundary}}}$$

$$\tau_{\text{average shear stress}} = \frac{\Gamma}{2tA}$$





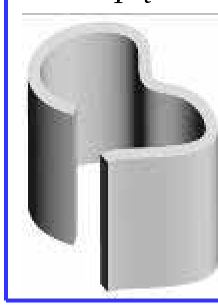
$$y_{NA} = \frac{a^2 b + d^2\left(c - b\right)}{2\left(ab + d\left(c - b\right)\right)} \qquad I_{bending} = ab\left[\frac{a^2}{12} + \left(\frac{a}{2} - y_{NA}\right)^2\right] + d\left(c - b\right)\left[\frac{d^2}{12} + \left(\frac{d}{2} - y_{NA}\right)^2\right]$$

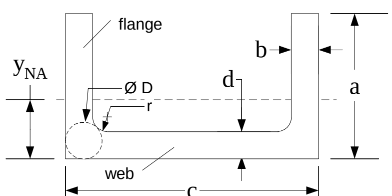
Ignore r in $I_{bending}$ calculations)

$$I_{torsion} = ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right] + \left(c - b\right)d^3\left[\frac{1}{3} - 0.105\frac{d}{c - b}\left(1 - \frac{b^4}{192\left(c - b\right)^4}\right)\right] + D^4\frac{d}{b}\left(0.07 + 0.076\frac{r}{b}\right)$$

$$I_p = \frac{P_{\text{length of median boundary}}\, t^3_{\text{thickness}}}{3}$$

$$\tau_{\text{average shear stress}} = \frac{\Gamma\left(3P + 1.8t\right)}{P^2 t^2}$$

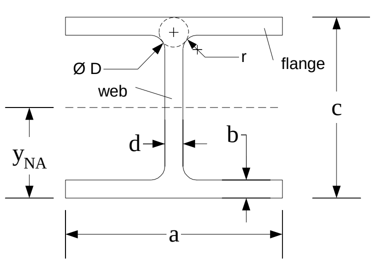




$$y_{NA} = \frac{2a^2 b + d^2\left(c - 2b\right)}{2\left(2ab + d\left(c - 2b\right)\right)} \qquad I_{bending} = 2ab\left[\frac{a^2}{12} + \left(\frac{a}{2} - y_{NA}\right)^2\right] + d\left(c - 2b\right)\left[\frac{d^2}{12} + \left(\frac{d}{2} - y_{NA}\right)^2\right]$$

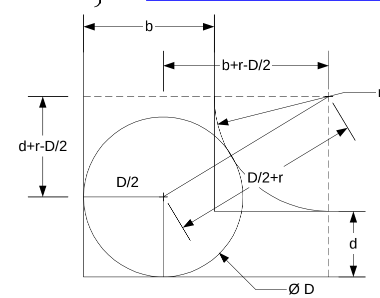
Ignore r in $I_{bending}$ calculations)

$$I_{torsion} = 2\left\{ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right] + \left(\frac{c}{2} - b\right)d^3\left[\frac{1}{3} - 0.105\frac{d}{c/2 - b}\left(1 - \frac{b^4}{192\left(c/2 - b\right)^4}\right)\right] + D^4\frac{d}{b}\left(0.07 + 0.076\frac{r}{b}\right)\right\}$$

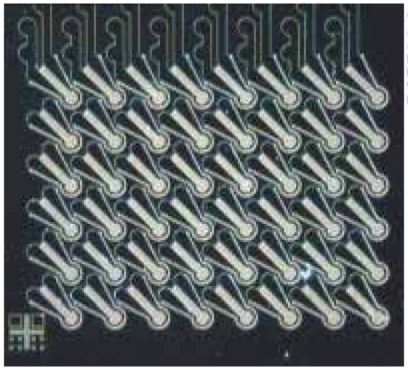


$$y_{NA} = \frac{c}{2} \qquad I_{bending} = \frac{ac^3 - 2\left(a - 2d\right)\left(c - 2b\right)^3}{12}$$

$$I_{torsion} = 2\left\{ab^3\left[\frac{1}{3} - 0.21\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right] + \left(\frac{c}{2} - b\right)d^3\left[\frac{1}{3} - 0.105\frac{d}{c/2 - b}\left(1 - \frac{b^4}{192\left(c/2 - b\right)^4}\right)\right] + D^4\frac{d}{b}\left(0.15 + 0.10\frac{r}{b}\right)\right\}$$

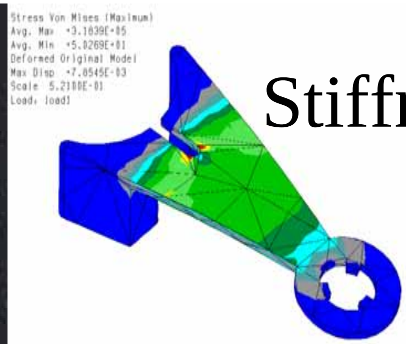


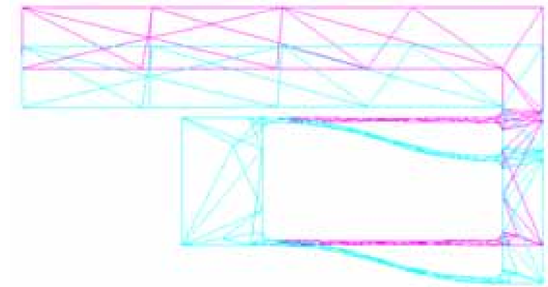
Stiffness (1/Compliance)

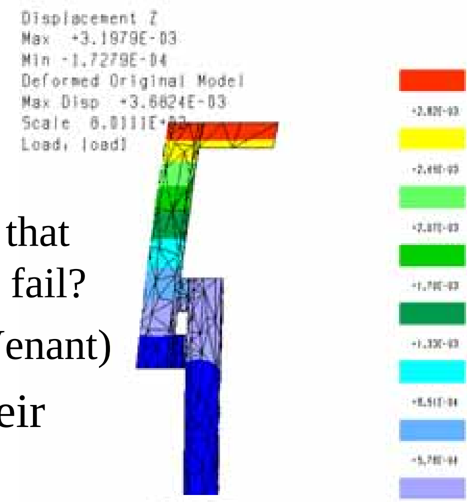


1 mm long silicon microspring electrical contacts. See US Patent 6,497,581

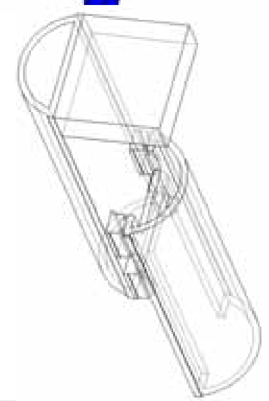
Width-tapered silicon springs

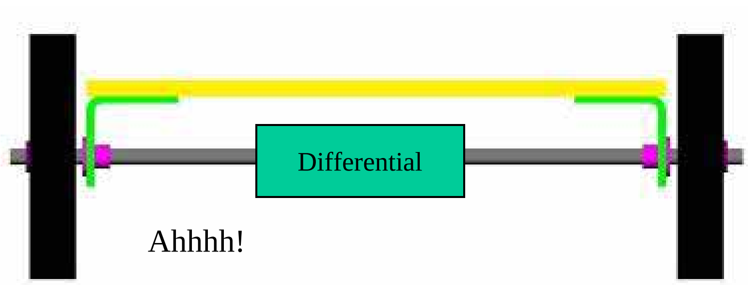


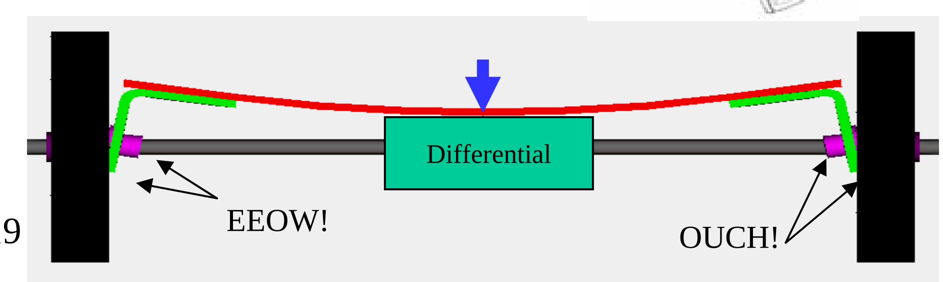




- All structures deform under load
  - Will the deformations create translational and angular displacements that will cause other elements to become overloaded or interfere and then fail?
    - Make the deflection 3-5x LESS than critical clearances (Saint-Venant)
- Where are forces transmitted between members with respect to their neutral axes"?
  - Position interface contact points at neutral axis planes!
- System compliance = sum of structural and element compliances
  - Machine elements (e.g., bearings) and joints should have a stiffness on the order of the structure itself
  - During early design stage, before bearings and joints are designed, assume net stiffness will thus be structural stiffness/3

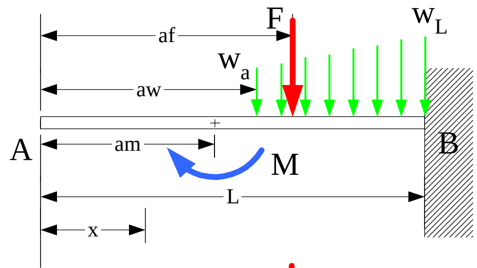






Stiffness: *First-Order Analysis*

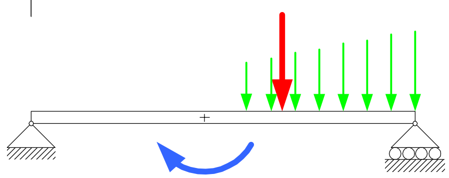
- Complex systems can often be modeled by superimposing simple models



$$R_B = F + \frac{(w_L+w_a)(L-a)}{2} \qquad M_B = -F(L-a) - \frac{(L-a)^2}{2}\left[w_a + \frac{(w_L-w_a)}{3}\right] + M$$

$$\theta_A = \frac{F(L-a)^2}{2EI} + \frac{(L-a)^3}{6EI}\left[w_a + \frac{(w_L-w_a)}{4}\right] - \frac{M(L-a)}{EI}$$

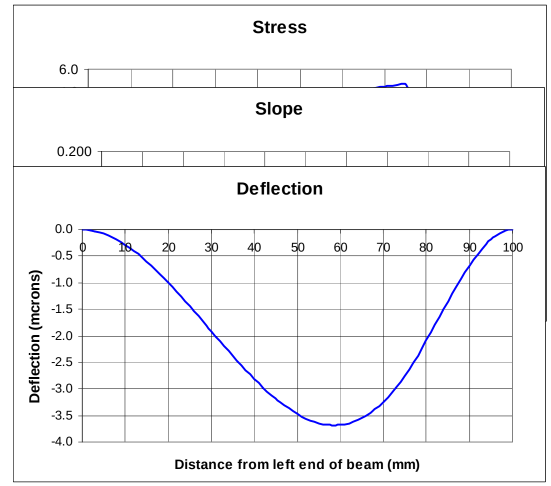
$$\delta_A = \frac{-F(2L^3-3aL^2+a^3)}{6EI} - \frac{(L-a)^3}{24EI}\left[w_a(3L+a) + \frac{(w_L-w_a)(4L+a)}{5}\right] + \frac{M(L^2-a^2)}{2EI}$$

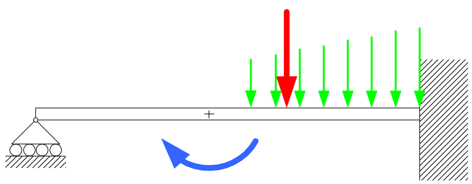


$$R_A = \frac{F(L-a)}{L} + \frac{(L-a)^2(w_L+2w_a)}{6L} - \frac{M}{L} \qquad R_B = \frac{Fa}{L} + \frac{(L-a)}{2}\left[w_L + w_a - \frac{(L-a)(w_L+2w_a)}{3L}\right] + \frac{M}{L}$$

$$\theta_A = \frac{-Fa(L-a)(2L-a)}{6EIL} - \frac{(L-a)^2}{24EIL}\left[w_a(L^2+2aL-a^2) + \frac{(w_L-w_a)(7L^2+6aL-3a^2)}{15}\right] - \frac{M(2L^2-6aL+3a^2)}{6EIL}$$

$$\theta_B = \frac{Fa(L^2-a^2)}{6EIL} + \frac{1}{24EIL}\left[w_a(L^2-a^2)^2 + \frac{(w_L-w_a)(L-a)^2(8L^2+9aL+3a^2)}{15}\right] + \frac{M(L^2-3a^2)}{6EIL}$$



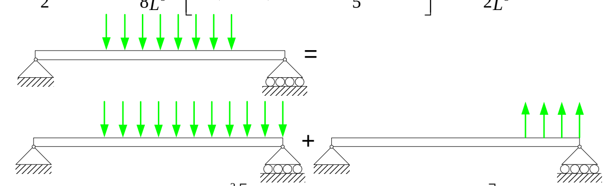


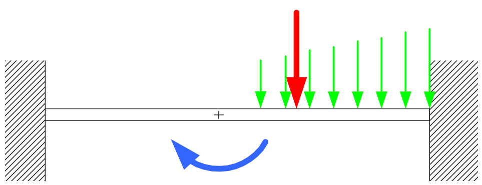
$$R_A = \frac{F(L-a)^2(2L+a)}{2L^3} + \frac{(L-a)^3}{8L^3}\left[w_a(3L+a) + \frac{(w_L-w_a)(4L+a)}{5}\right] - \frac{3M(L^2-a^2)}{2L^3}$$

$$R_B = \frac{Fa(3L^2-a^2)}{2L^3} + \frac{(w_L+w_a)(L-a)}{2} - \frac{(L-a)^3}{8L^3}\left[w_a(3L+a) + \frac{(w_L-w_a)(4L+a)}{5}\right] + \frac{3M(L^2-a^2)}{2L^3}$$

$$M_B = \frac{-Fa(L^2-a^2)}{2L^2} + \frac{(L-a)^3}{8L^2}\left[w_a(3L+a) + \frac{(w_L-w_a)(4L+a)}{5}\right] - \frac{(L-a)^2}{2}\left[w_a + \frac{(w_L-w_a)}{3}\right] + \frac{M(3a^2-L^2)}{2L^2}$$

$$\theta_A = \frac{-Fa(L-a)^2}{4EIL} - \frac{(L-a)^3}{48EIL}\left[w_a(L+3a) + \frac{(w_L-w_a)(2L+3a)}{5}\right] + \frac{M(L-a)(3a-L)}{4EIL}$$



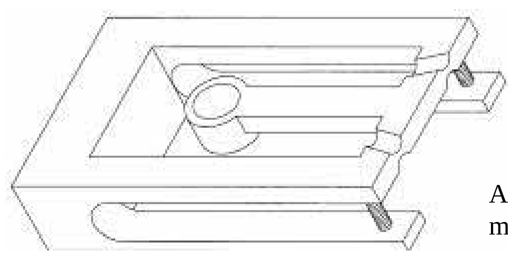


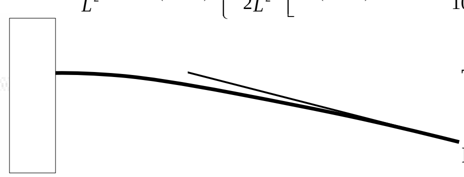
$$R_A = \frac{F(L-a)^2(L+2a)}{L^3} + \frac{(L-a)^3}{2L^3}\left[w_a(L+a) + \frac{(w_L-w_a)(3L+2a)}{10}\right] - \frac{6Ma(L-a)}{L^3}$$

$$R_B = \frac{Fa^2(3L-2a)}{L^3} + \frac{(w_L+w_a)(L-a)}{2} - \frac{(L-a)^3}{2L^3}\left[w_a(L+a) + \frac{(w_L-w_a)(3L+2a)}{10}\right] + \frac{6Ma(L-a)}{L^3}$$

$$M_A = \frac{-Fa(L-a)^2}{L^2} - \frac{(L-a)^3}{12L^2}\left[w_a(L+3a) + \frac{(w_L-w_a)(2L+3a)}{5}\right] - \frac{M(L^2-4aL+3a^2)}{L^2}$$

$$M_B = \frac{-Fa^2(L-a)}{L^2} + (L-a)^2\left\{\frac{(L-a)}{2L^2}\left[w_a(L+a) + \frac{(w_L-w_a)(3L+2a)}{10} - \frac{w_a(L+3a)}{6} - \frac{(w_L-w_a)(2L+3a)}{30}\right] - \frac{w_a}{2} - \frac{(w_L-w_a)}{6}\right\} + \frac{Ma(3a-2L)}{L^2}$$





Transverse shear $V = R_A - F\langle x-a\rangle^0 - w_a\langle x-a\rangle - \frac{(w_L-w_a)\langle x-a\rangle^2}{2(L-a)}$

Moment $M = M_A + R_A x - F\langle x-a\rangle - \frac{w_a\langle x-a\rangle^2}{2} - \frac{(w_L-w_a)\langle x-a\rangle^3}{6(L-a)} + M\langle x-a\rangle^0$

Abbe errors exist, but reciprocity says they might be usable… Do you see a pattern?

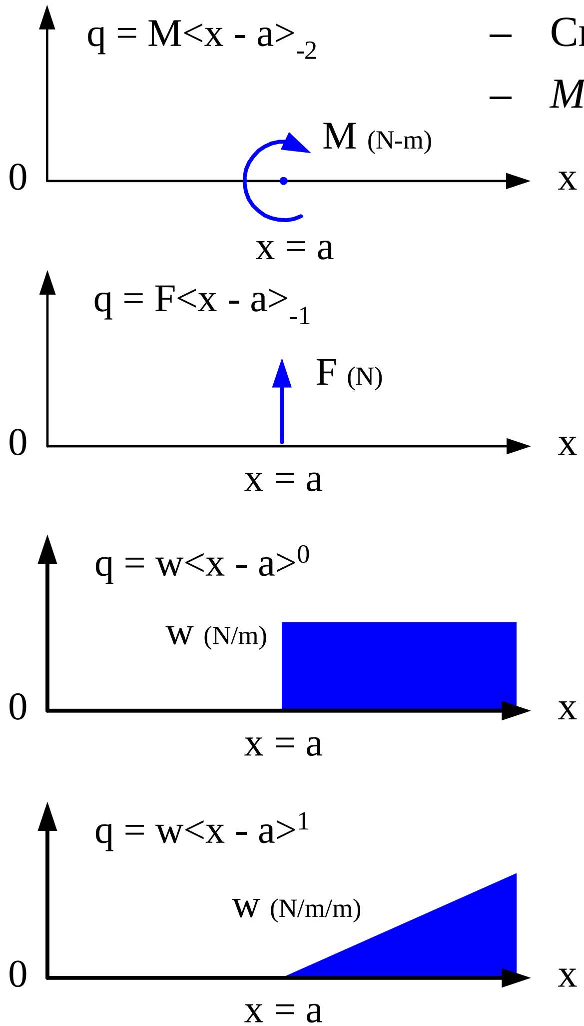
Slope $\theta = \theta_A + \frac{M_A x}{EI} + \frac{R_A x^2}{2EI} - \frac{F\langle x-a\rangle^2}{2EI} - \frac{w_a\langle x-a\rangle^3}{6EI} - \frac{(w_L-w_a)\langle x-a\rangle^4}{24EI(L-a)} + \frac{M\langle x-a\rangle}{EI}$

Displacement $\delta = \delta_A + \theta_A x + \frac{M_A x^2}{2EI} + \frac{R_A x^3}{6EI} - \frac{F\langle x-a\rangle^3}{6EI} - \frac{w_a\langle x-a\rangle^4}{24EI} - \frac{(w_L-w_a)\langle x-a\rangle^5}{120EI(L-a)} + \frac{M\langle x-a\rangle^2}{2EI}$

$$\delta = \frac{FL^3}{3EI} \qquad \alpha = \frac{FL^2}{2EI} \qquad \delta = \frac{ML^2}{2EI} \qquad \alpha = \frac{ML}{EI}$$

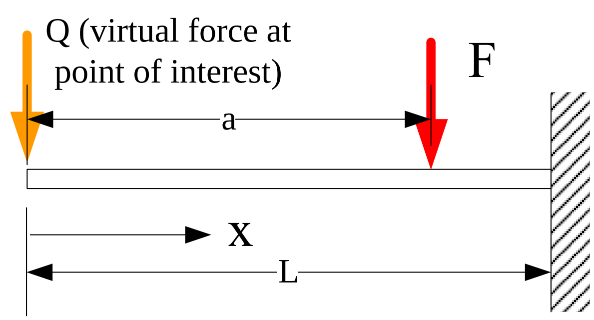
Stiffness: *Advanced Analysis Methods*

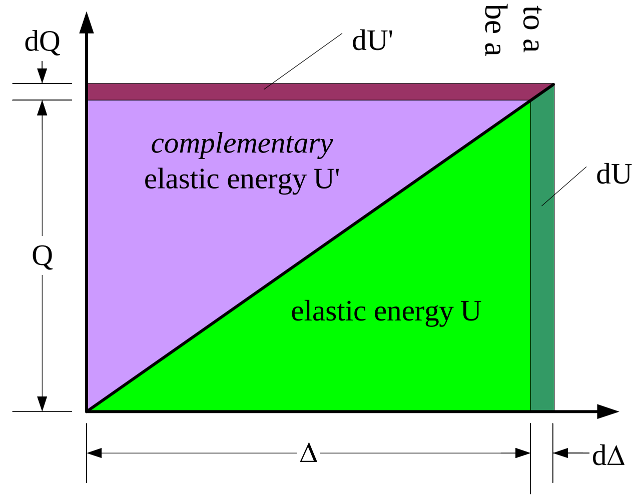
- Singularity functions enable loads to be "turned on"
  - Create expressions for the moment *M(x)* as a function of position
  - *M(x)* is used in moment-curvature & energy method calculations



Loading	Variables	Energy	Deflection
Axial	P, E , A	$\int_0^L \frac{P^2}{2EA} ds$	$\int_0^L \frac{F(\partial P/\partial Q)}{EA} ds$
Torsion	Γ, G, I_{polar}	$\int_0^L \frac{\Gamma^2}{2GI_{polar}} ds$	$\int_0^L \frac{\Gamma(\partial \Gamma/\partial Q)}{GI_{polar}} ds$
Transverse Shear (approximate for non-rectangular sections, else ½ => 3/5)	V, G, A	$\int_0^L \frac{V^2}{2GA} ds$	$\int_0^L \frac{V(\partial V/\partial Q)}{GA} ds$
Bending	M, E, I	$\int_0^L \frac{M^2}{2EI} ds$	$\int_0^L \frac{M(\partial M/\partial Q)}{EI} ds$

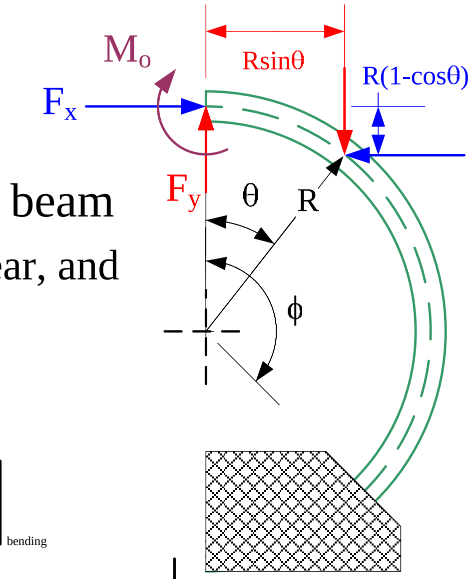
ds is the distance along the beam (e.g., *ds* = *dx* for straight beams, *ds* = *Rdq* for curved beam). For determining the slope, differentiate not with respect to a force *Q*, but an applied moment *M* (which may also be a virtual applied moment at the point of interest)





A most useful website is http://www.integrals.com/index.en.cgi

Stiffness: *Energy Methods Examples*



- Energy methods can be used to determine the deflections in a curved beam
  - They can also be used to calculate the relative contributions of axial, shear, and bending; E.g., for a curved beam:

$$P = F_x\cos\theta - F_y\sin\theta \qquad V = F_x\sin\theta + F_y\cos\theta \qquad M = F_xR(1-\cos\theta) + F_yR\sin\theta + M_o \qquad ds = Rd\theta$$

$$\frac{\partial P}{\partial F_x} = \cos\theta \quad \frac{\partial P}{\partial F_y} = -\sin\theta \quad \frac{\partial P}{\partial M_o} = 0 \qquad \frac{\partial V}{\partial F_x} = \sin\theta \quad \frac{\partial V}{\partial F_y} = \cos\theta \quad \frac{\partial V}{\partial M_o} = 0 \qquad \frac{\partial M}{\partial F_x} = R(1-\cos\theta) \quad \frac{\partial M}{\partial F_y} = R\sin\theta \quad \frac{\partial M}{\partial M_o} = 1$$

$$\delta_x = \frac{\partial U}{\partial F_x} = \int_0^\phi \frac{F_x\cos^2\theta - F_y\sin\theta\cos\theta}{EA}Rd\theta\Bigg|_{\text{axial}} + \int_0^\phi \frac{F_x\sin^2\theta + F_y\cos\theta\sin\theta}{GA}Rd\theta\Bigg|_{\text{shear}} + \int_0^\phi \frac{F_xR^2(1-\cos\theta)^2 + F_yR^2\sin\theta(1-\cos\theta) + M_oR(1-\cos\theta)}{EI}Rd\theta\Bigg|_{\text{bending}}$$

$$\delta_x = \frac{R}{EA}\left[F_x\left(\frac{2\phi+\sin 2\phi}{4}\right) - F_y\left(\frac{1-\cos^2\phi}{2}\right)\right]\Bigg|_{\text{axial}} + \frac{R}{GA}\left[F_x\left(\frac{2\phi-\sin 2\phi}{4}\right) + F_y\left(\frac{1-\cos^2\phi}{2}\right)\right]\Bigg|_{\text{shear}} + \frac{1}{EI}\left[R^3F_x\left(\frac{6\phi - 8\sin\phi + \sin 2\phi}{4}\right) + R^3F_y\left(\frac{1-2\cos\phi+\cos^2\theta}{2}\right) + M_oR^2(\phi-\sin\phi)\right]\Bigg|_{\text{bending}}$$

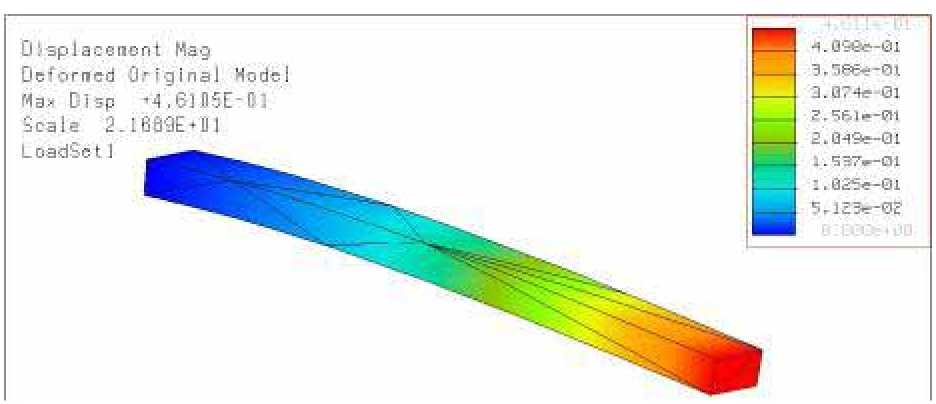
See *Beam_curved.xls*

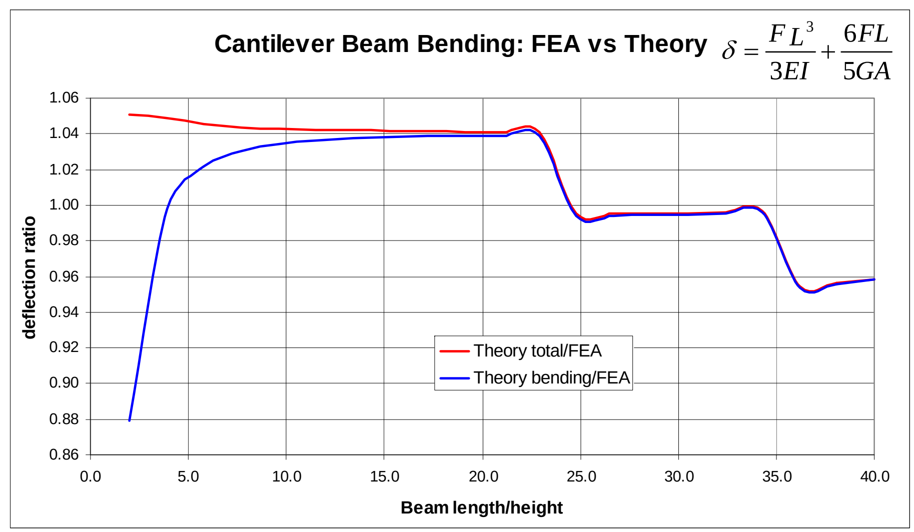
$$\delta_y = \frac{\partial U}{\partial F_y} = \int_0^\phi \frac{-F_x\cos\theta\sin\theta + F_y\sin^2\theta}{EA}Rd\theta\Bigg|_{\text{axial}} + \int_0^\phi \frac{F_x\cos\theta\sin\theta + F_y\cos^2\theta}{GA}Rd\theta\Bigg|_{\text{shear}} + \int_0^\phi \frac{F_xR^2\sin\theta(1-\cos\theta) + F_yR^2\sin^2\theta + M_oR\sin\theta}{EI}Rd\theta\Bigg|_{\text{bending}}$$

$$\delta_y = \frac{R}{EA}\left[-F_x\left(\frac{1-\cos^2\phi}{2}\right) + F_y\left(\frac{2\phi-\sin 2\phi}{4}\right)\right]\Bigg|_{\text{axial}} + \frac{R}{GA}\left[F_x\left(\frac{1-\cos^2\phi}{2}\right) + F_y\left(\frac{2\phi+\sin 2\phi}{4}\right)\right]\Bigg|_{\text{shear}} + \frac{1}{EI}\left[R^3F_x\left(\frac{1-2\cos\phi+\cos^2\theta}{2}\right) + R^3F_y\left(\frac{2\phi-\sin 2\phi}{4}\right) + M_oR^2(1-\cos\phi)\right]\Bigg|_{\text{bending}}$$

$$\theta = \frac{\partial U}{\partial M_o} = \int_0^\phi \frac{F_xR(1-\cos\theta) + F_yR\sin\theta + M_o}{EI}Rd\theta\Bigg|_{\text{bending}}$$

$$\theta = \frac{1}{EI}\left[R^2F_x(\phi-\sin\phi) + R^2F_y(1-\cos\phi) + M_oR\phi\right]\Bigg|_{\text{bending}}$$





Stiffness: *Plates*

- A few simple loading cases give insight into the nature of stresses & deformations in plates
- *Check out the many different spreadsheets!*

$$r_o' = \sqrt{1.6r_o^2 + t^2} - 0.675t \quad \text{if } r_o < 0.5t$$

$$M_r = \frac{W}{4\pi}\left[(1+\nu)\ln\frac{a}{r} - 1 + \frac{(1-\nu)r_o'^2}{4r^2}\right]$$

$$M_t = \frac{W}{4\pi}\left[(1+\nu)\ln\frac{a}{r} - \nu + \frac{\nu(1-\nu)r_o'^2}{4r^2}\right]$$

$$\theta = \frac{Wr}{4\pi D}\ln\frac{a}{r}$$

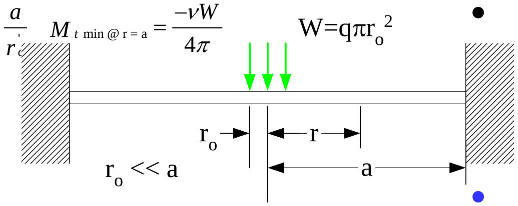
$$\delta = \frac{-W}{16\pi D}\left(a^2 - r^2\left(1 + 2\ln\frac{a}{r}\right)\right)$$

$$M_{r\ \max\ @\ r=0} = \frac{W}{4\pi}(1+\nu)\ln\frac{a}{r_o'} \qquad M_{r\ \min\ @\ r=a} = \frac{-W}{4\pi}$$

$$M_{t\ \max\ @\ r=0} = \frac{W}{4\pi}(1+\nu)\ln\frac{a}{r_o'} \qquad M_{t\ \min\ @\ r=a} = \frac{-\nu W}{4\pi}$$

$$\theta_{\max\ @\ r=0.368a} = 0.0293\frac{Wa}{D}$$

$$\delta_{\max\ @\ r=0} = \frac{-Wa^2}{16\pi D}$$



$$r_o' = \sqrt{1.6r_o^2 + t^2} - 0.675t \quad \text{if } r_o < 0.5t$$

$$M_r = \frac{W}{16\pi}\left[4(1+\nu)\ln\frac{a}{r} + (1-\nu)\left(\frac{a^2-r^2}{a^2}\right)\frac{r_o'^2}{r^2}\right]$$

$$M_t = \frac{W}{16\pi}\left[4(1+\nu)\ln\frac{a}{r} + (1-\nu)\left(4 - \frac{r_o'^2}{r^2}\right)\right]$$

$$\theta = \frac{Wr}{4\pi D}\left(\frac{1}{1+\nu} + \ln\frac{a}{r}\right)$$

$$\delta = \frac{-W}{16\pi D}\left(\frac{3+\nu}{1+\nu}(a^2-r^2) - 2r^2\ln\frac{a}{r}\right)$$

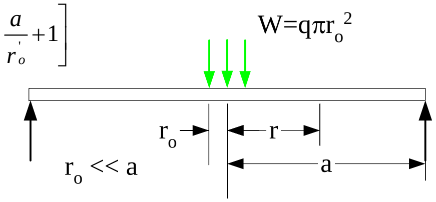
$$M_{r\ \max\ @\ r=0} = \frac{W}{4\pi}\left[4(1+\nu)\ln\frac{a}{r_o'} + 1\right]$$

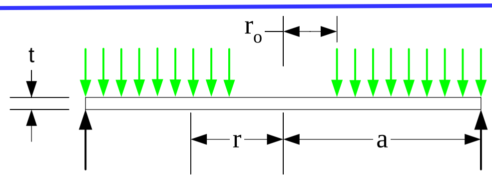
$$M_{t\ \max\ @\ r=0} = \frac{W}{4\pi}\left[4(1+\nu)\ln\frac{a}{r_o'} + 1\right]$$

$$\theta_{\max\ @\ r=a} = \frac{Wa}{4\pi D(1+\nu)}$$

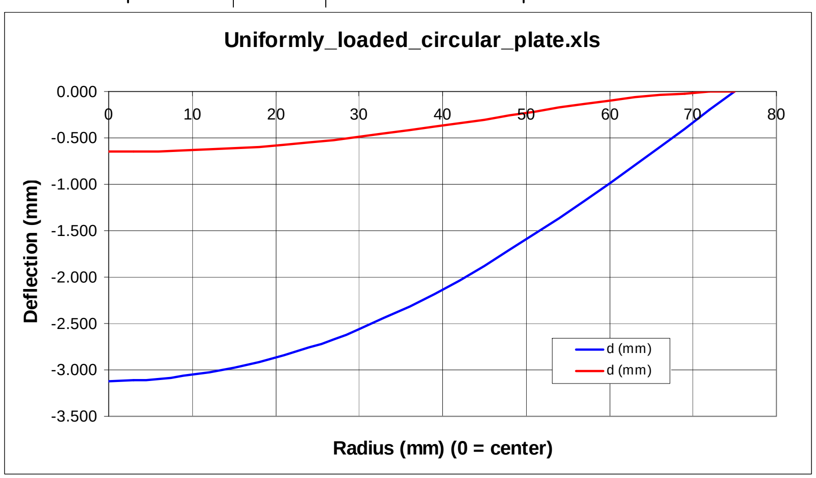
$$\delta_{\max\ @\ r=0} = \frac{-Wa^2(3+\nu)}{16\pi D(1+\nu)}$$

Circular plates with support around entire outer boundary

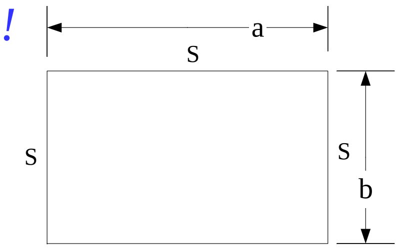




Circular plates with support (simply supported or fixed) around entire outer boundary



$$D = \frac{Et^3}{12(1-\nu^2)}$$

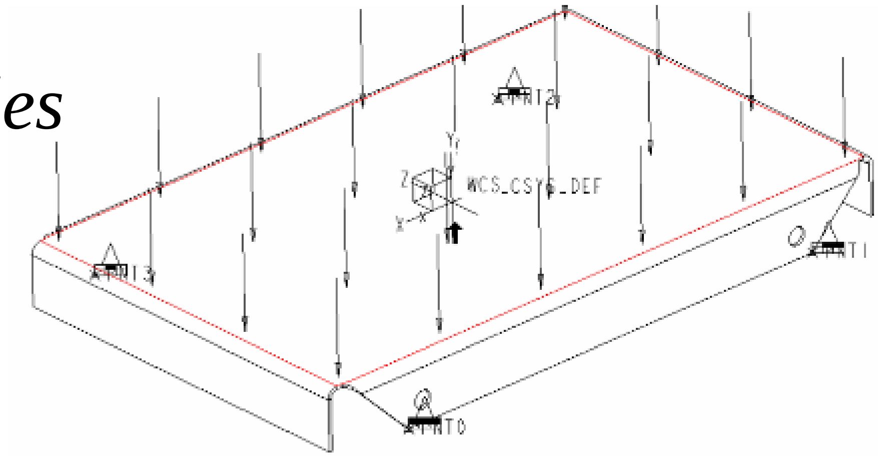


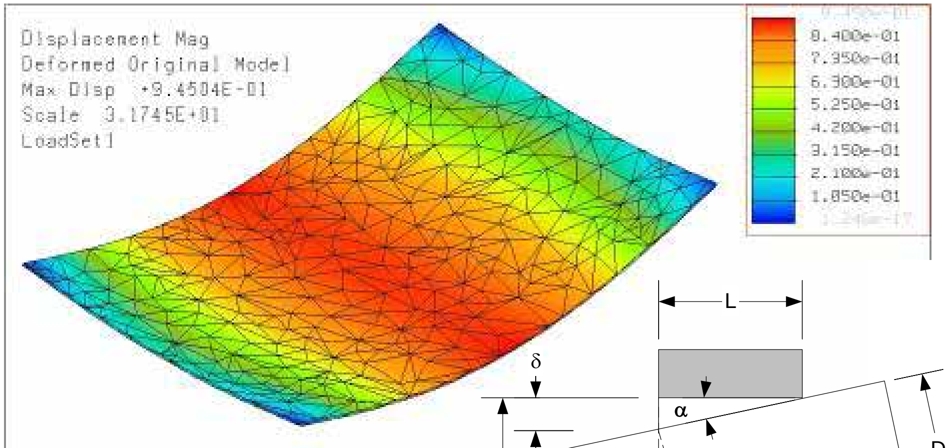
simply supported = S

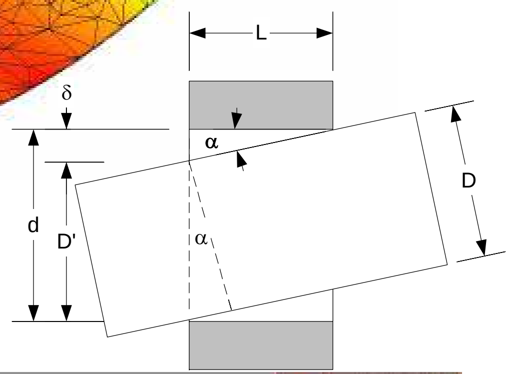
Plate_Rectangular.xls	
To determine deflection of a rectangular plate	
From Roark & Young, Formulas for Stress and Strain, 5th edition, pag	
By Alex Slocum, Last modified 1/1/04 by Alex Slocum	
Enters numbers in **BOLD,** Results in **RED**	
Plate	
length a (mm)	**300**
width, b (mm)	**200**
thickness, t (mm)	**1.5**
Modulus of elasticity, E (N/mm^2)	**68947.6**
Poisson ratio, v	**0.3**
Total area	**60000**
a/b, aob	**1.50**
Loading	
uniform loading pressure over entire plate, q (N/mm^2)	**0.00017**
total load applied to the plate, W (N)	**10**
uniform over small concentric circle of ro, qo (N/mm^2)	**10**
load application radius, ro (mm)	**2**
total load applied to center of plate, Wo (N)	**126**
Simply supported outer boundary:	
Distributed load:	
Center displacement, dc (mm)	**-0.096**
Bending stress at center (N/mm^2)	**1**
Reaction load at center of long side (N/mm)	**0**
Centrally applied concentrated load:	
Center displacement, fdc (mm)	**-3.621**
Bending stress at center (N/mm^2)	**190**

Stiffness: *Plate Examples*

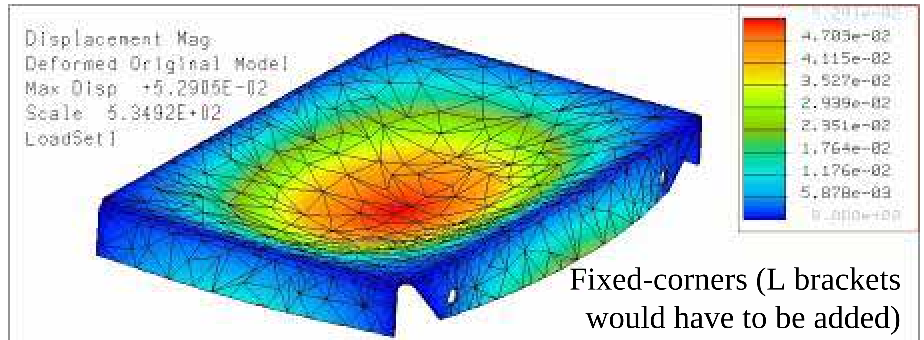
- Al plate 200 x 300 x 1.5mm uniformly loaded with 10N



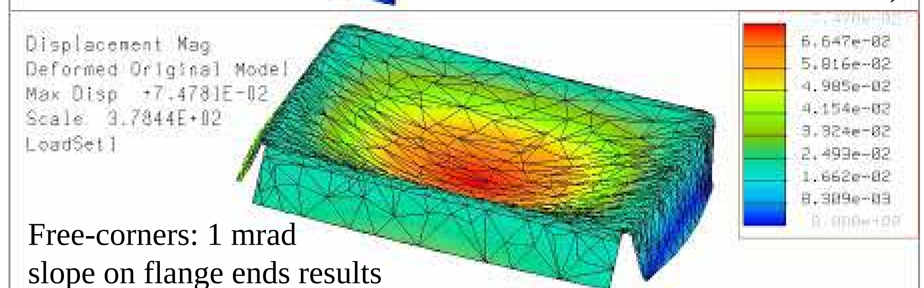




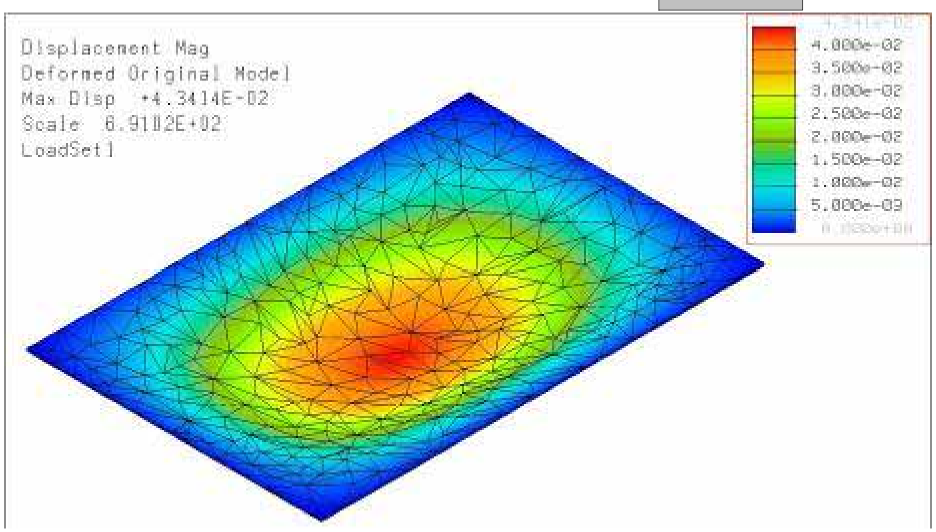
Shaft misaligned in bearing from slope in plate

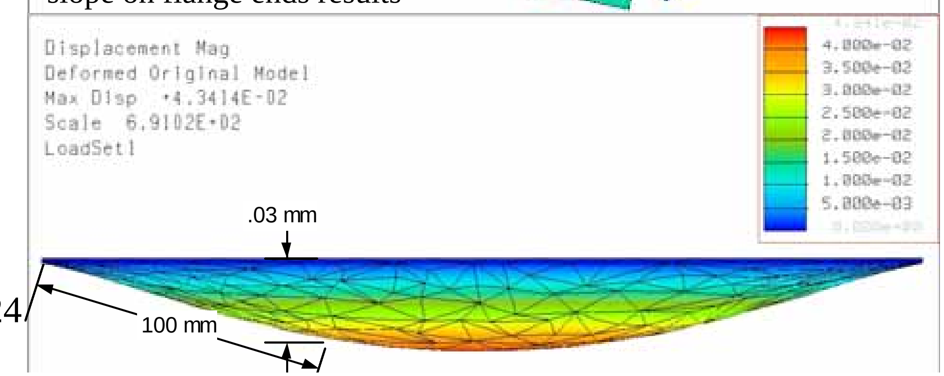


Fixed-corners (L brackets would have to be added)



Free-corners: 1 mrad slope on flange ends results





Strength

- Stress=Moment*distance of farthest fiber from Neutral axis/Moment of Inertia:

$$\sigma = \frac{Mc}{I}$$

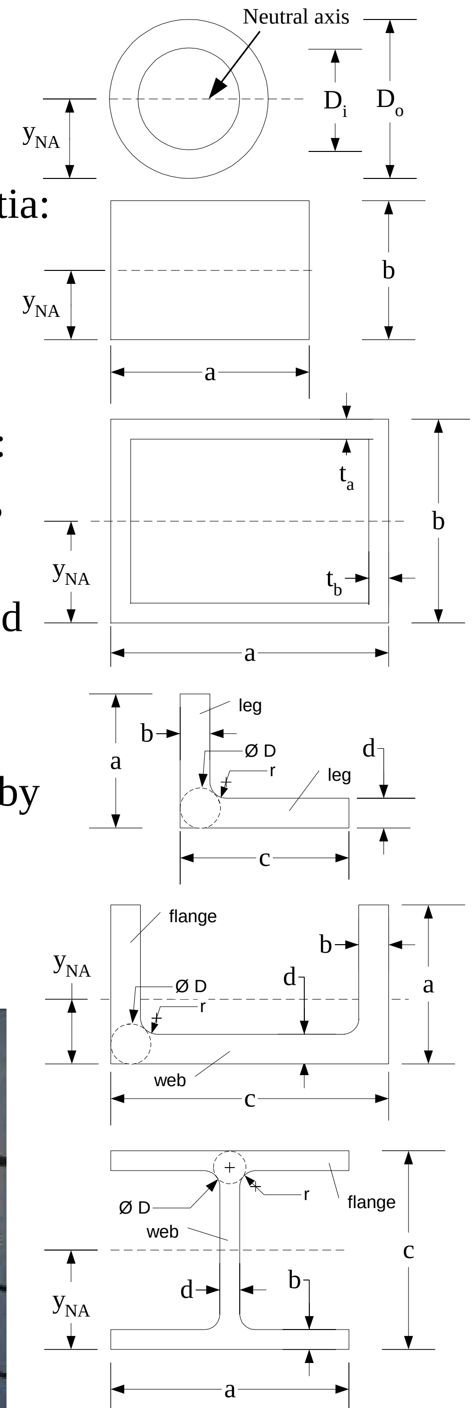
  - Stress ratio = Applied stress / Maximum allowable stress
- The *parallel axis theorem* can be used to evaluate any cross section's inertia:
  - Most important: The parallel axis theorem tells us that a section stiffens with the square of the distance from the neutral axis!
    - E.g., when designing a laminate (1/16" AL sheet separated by wood core), double the core thickness and quadruple the panel stiffness!
- Torsional shear stress=Torque*radius/polar moment of inertia
- Stresses caused by multiple loads can be combined into an equivalent stress by the von Mises equivalent stress formula:

$$\sigma_{equivalent} = \sqrt{\frac{(\sigma_x - \sigma_y)^2 + (\sigma_x - \sigma_z)^2 + (\sigma_y - \sigma_z)^2}{2} + 3\tau_{xy}^2 + 3\tau_{xz}^2 + 3\tau_{yz}^2}$$

Funny image found on www, photographer not credited, would like to, email slocum@mit.edu

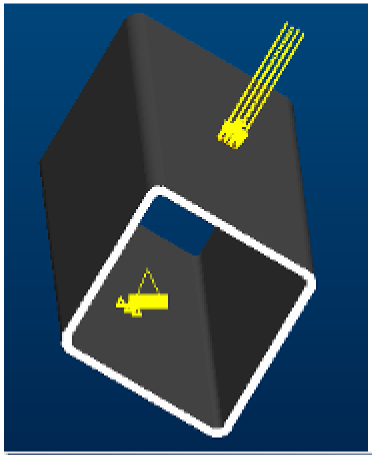
Ten 80 lb bags of concrete

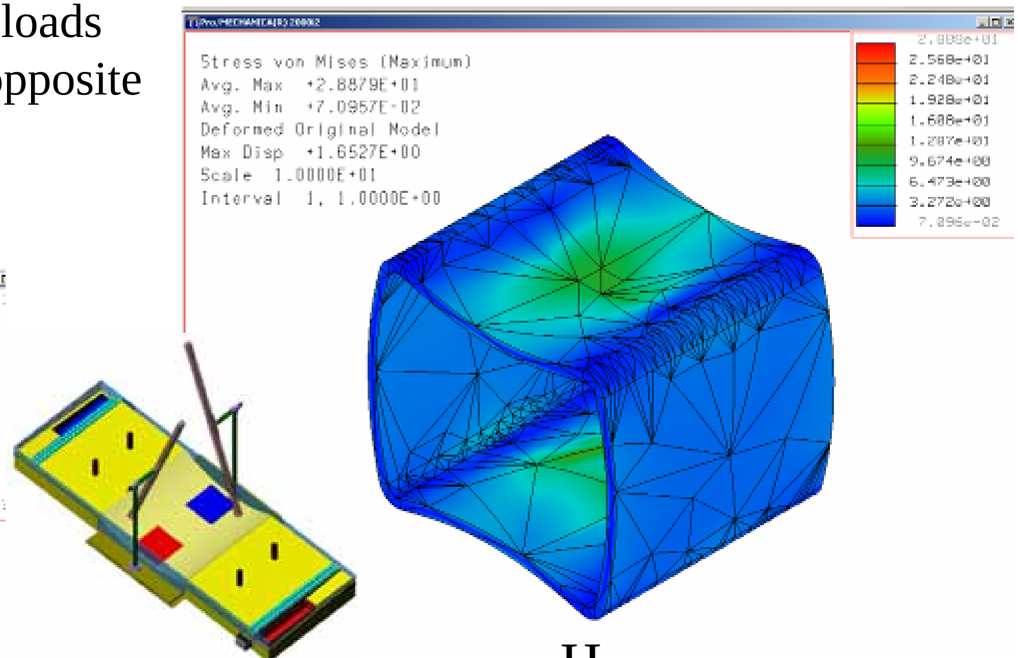
OK, what's the ratio of the stress ratio in the bar to the stress ratio in the knees?

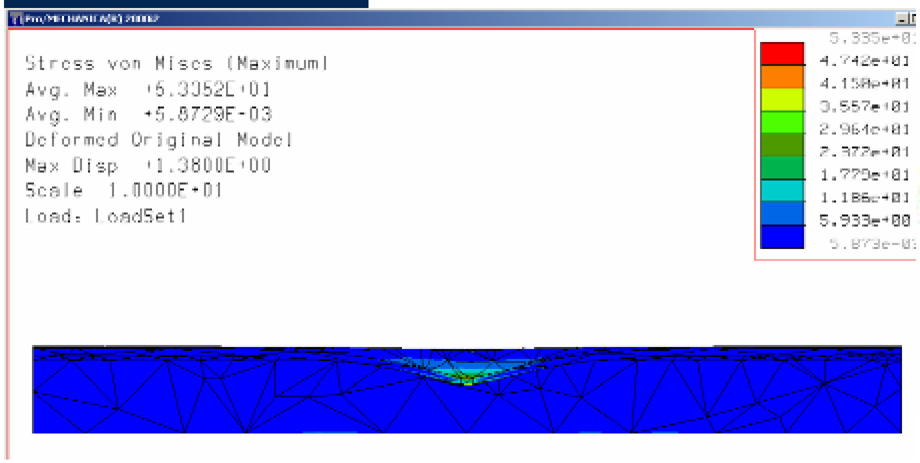


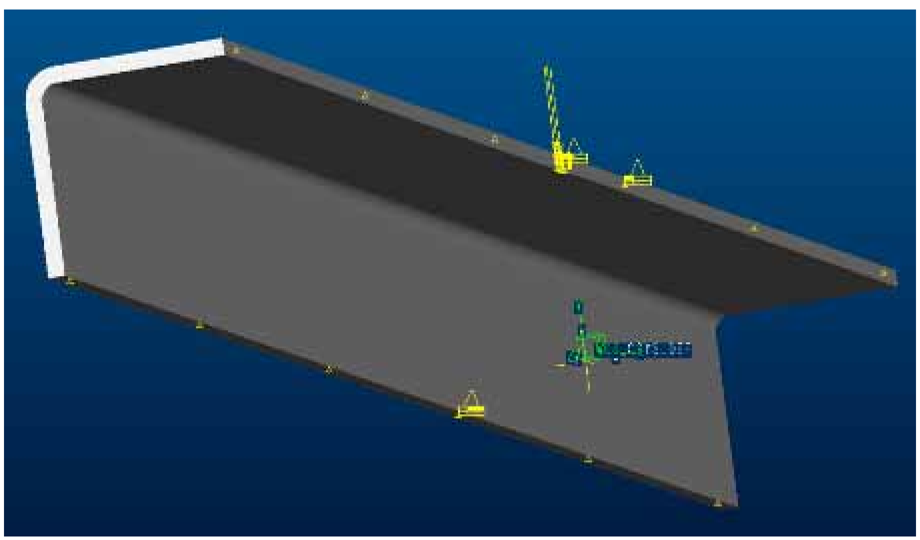
Strength: *1st Order Analysis*

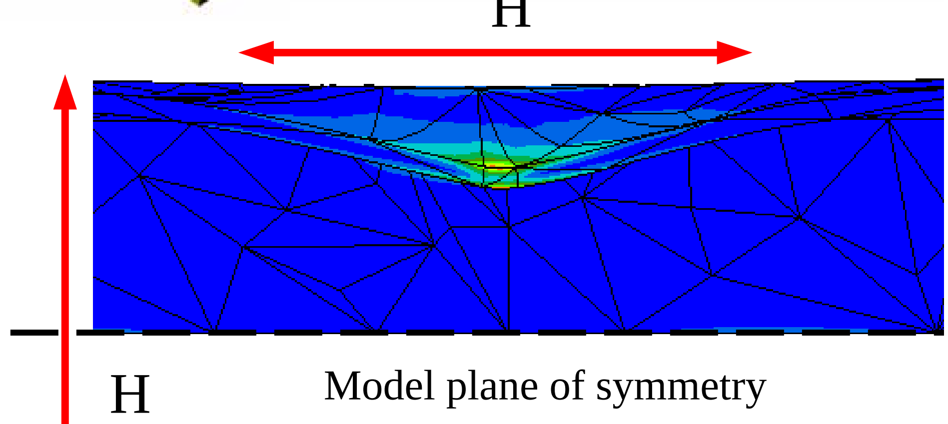
- Square tube with point loads (from wheels) on two opposite sides







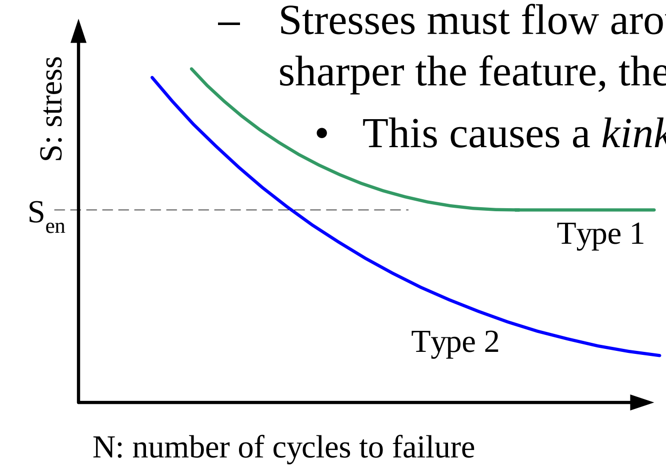


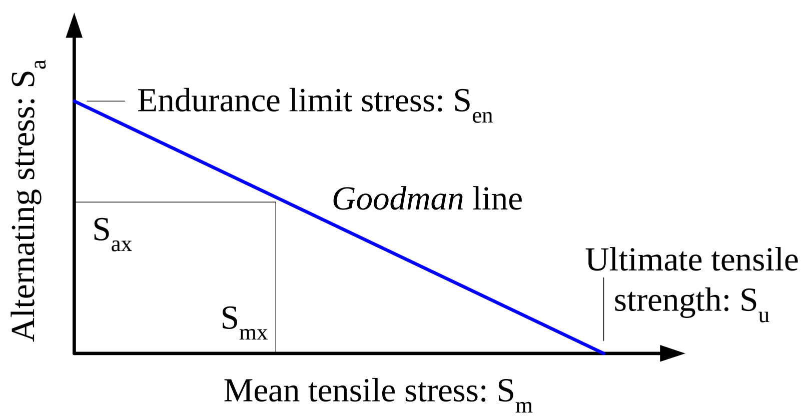


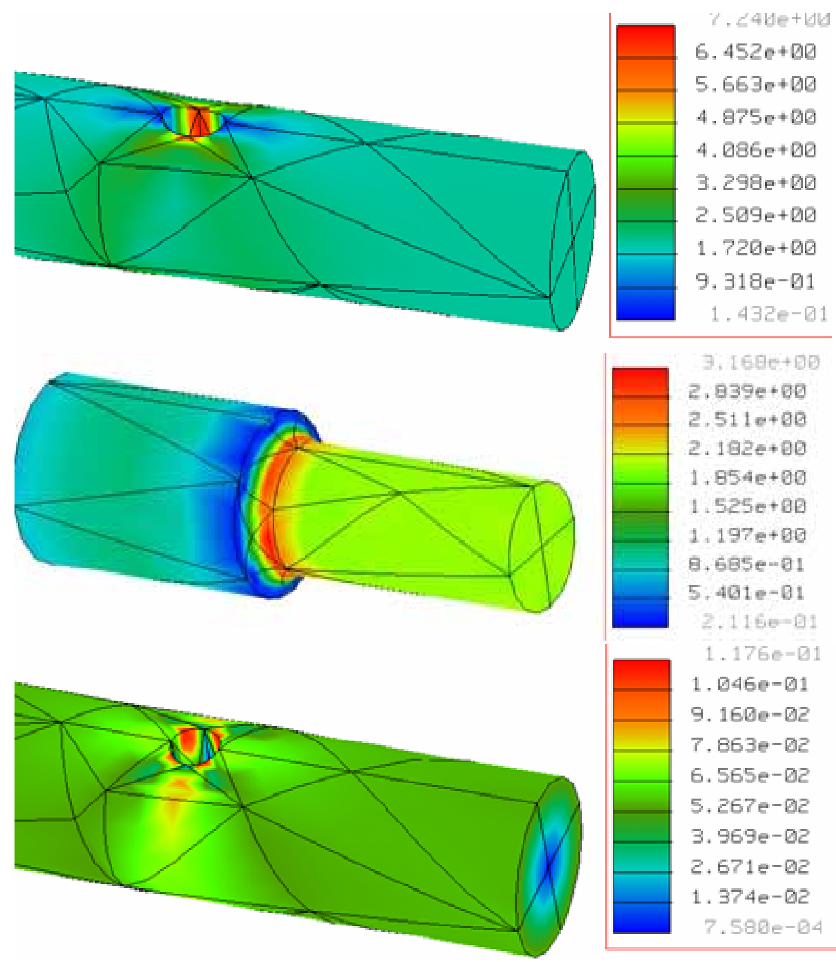
Yes, yes, yes, for all you FEA avocados out there, only a 1/8 models with zero slope constraints at the planes of symmetry is actually needed

Strength: *Life, Fatigue and Stress Concentration*

- Fatigue:
  - To obtain “infinite” life (endurance limit) in a steel structure
    - Applied stress should be less than ½ yield stress
    - Aluminum has no endurance limit: Infinite life requires zero stress
  - A structure can be subjected to simultaneous continuous and alternating stresses
- Stress concentration:
  - Stresses must flow around features, and the sharper the feature, the more severe the *turn*
    - This causes a *kink*, and the stress rises





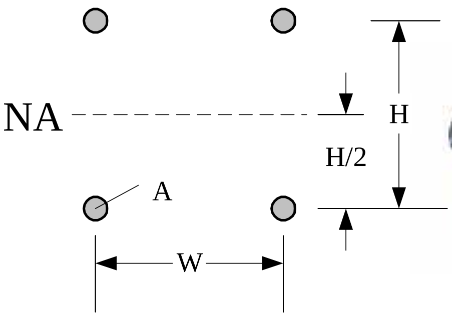


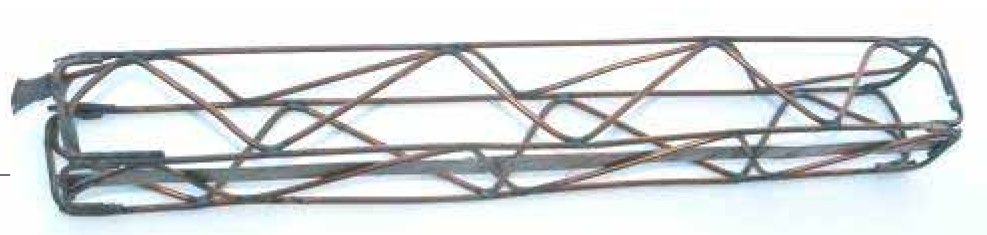
$$I_{NA} = I_{zz} + A y^2_{z-NA}$$

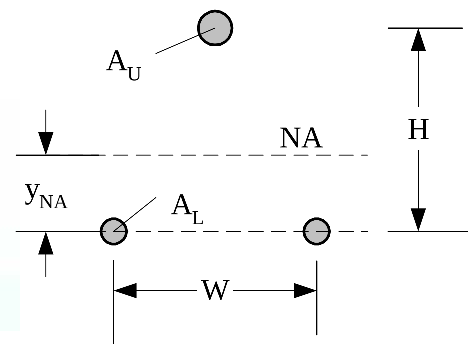
Trusses

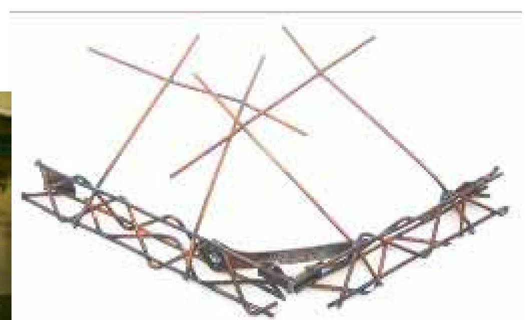
$$y_{NA} = \frac{\sum_{i=1}^{N} E_i A_i y_i}{\sum_{i=1}^{N} E_i A_i}$$

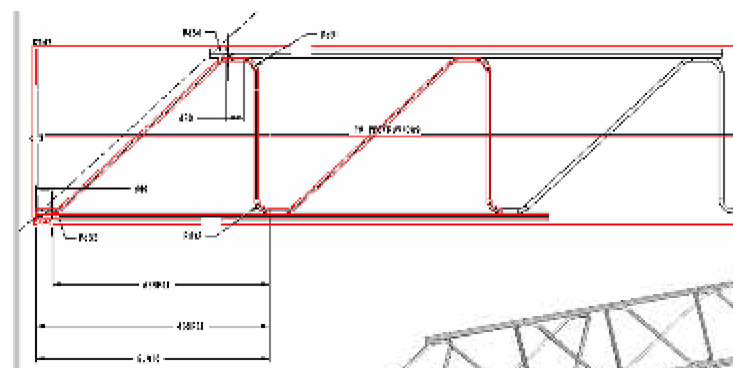
- Trusses can carry huge loads while being very light weight
  - Saint-Venant & trusses
    - Web member (braces) spacing is typically on the order of the truss height
    - Greater spacing leads to greater deflections and greater chance of chord buckling
  - Fundamentally, the farther away from the neutral axis you can add area, the better squared you will be!
  - Trusses are easily fabricated from spot-welded steel welding rod!

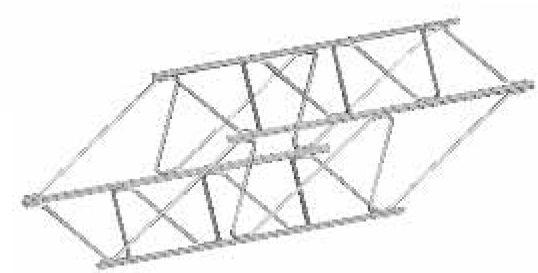










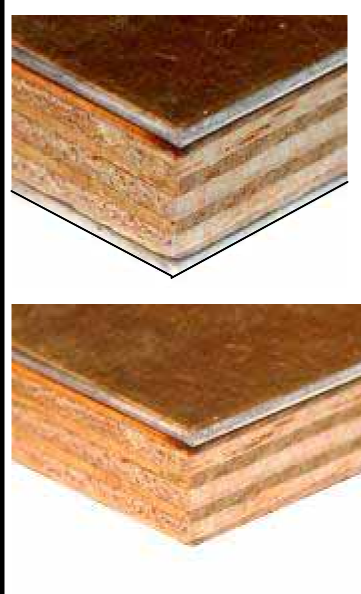


Laminates & Composites

- Laminates are made of stress-carrying elements bonded to a core (see *Laminate.xls*)
  - They have tensile/compressive carrying members on the outside (chords), and a shear carrying member (core) on the inside
    - Metal chords with a wood core can give nearly the strength of an I beam
    - The shear stresses in the adhesive and core materials must be carefully considered
- Composites use a matrix, typically a polymer such as epoxy, to bond together structural fibers, cloth…
  - Thin composite members are then often laminated to a core…

Simple Plate		
t	**0.0625**	thickness
w	**1**	width
L	**10**	length
E	**1.00E+07**	modulus
maxstress	**20000**	
I	2.03E-05	
A	6.25E-02	
Ioc	6.51E-04	
Fmax	1.30	
deflection	2.13	
Sandwhich beam (2 plates with wood core)		
Assume I and E of wood core are ignored		
tcore	**0.5**	
Icore	0.0099	
Ioccore	0.0318	
Fmaxcore	63.5	
deflection	0.21	

Drill holes in plywood to reduce weight of laminate core



Solid plywood