Chapter 3

SIMPLEX METHOD

In this chapter, we put the theory developed in the last to practice. We develop the simplex method algorithm for LP problems given in feasible canonical form and standard form. We also discuss two methods, the M-Method and the Two-Phase Method, that deal with the situation that we have an infeasible starting basic solution.

3.1 Simplex Method for Problems in Feasible Canonical Form

The *Simplex method* is a method that proceeds from one BFS or extreme point of the feasible region of an LP problem expressed in tableau form to another BFS, in such a way as to continually increase (or decrease) the value of the objective function until optimality is reached. The simplex method moves from one extreme point to one of its neighboring extreme point. Consider the following LP in feasible canonical form, i.e. its right hand side vector $\mathbf{b} \geq \mathbf{0}$:

$$\begin{array}{ll} \max & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} \leq \mathbf{b}, \\ \quad \mathbf{x} \geq \mathbf{0}. \end{cases} \end{array}$$

Its initial tableau is

	x_1	x_2	$\cdots$	x_s	$\cdots$	x_n	x_{n+1}	$\cdots$	x_{n+r}	$\cdots$	x_{n+m}	$\mathbf{b}$
x_{n+1}	a_{11}	a_{12}	$\cdots$	a_{1s}	$\cdots$	a_{1n}	1	$\cdots$	0	$\cdots$	0	b_1
x_{n+2}	a_{21}	a_{22}	$\cdots$	a_{2s}	$\cdots$	a_{2n}	0	$\cdots$	0	$\cdots$	0	b_2
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_{n+r}	a_{r1}	a_{r2}	$\cdots$	a_{rs}	$\cdots$	a_{rn}	0	$\cdots$	1	$\cdots$	0	b_r
$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
x_{n+m}	a_{m1}	a_{m2}	$\cdots$	a_{ms}	$\cdots$	a_{mn}	0	$\cdots$	0	$\cdots$	1	b_r
x_0	$-c_1$	$-c_2$	$\cdots$	$-c_s$	$\cdots$	$-c_n$	0	$\cdots$	0	$\cdots$	0	0

Here x_{n+i}, $i = 1 \cdots, m$ are the slack variables. The original variables x_i, $i = 1, \cdots, n$ are called the *structural* or *decision variables*. Since all $b_i \geq 0$, we can read off directly from the tableau a starting

BFS given by $[0, 0, \cdots, 0, b_1, b_2, \cdots, b_m]^T$, i.e. all structural variables x_j are set to zero. Note that this corresponds to the origin of the n-dimensional subspace $\mathbb{R}^n$ of $\mathbb{R}^{n+m}$.

In matrix form, the original constraint $A\mathbf{x} \leq \mathbf{b}$ has be augmented to

$$[A \quad I] \begin{bmatrix} \mathbf{x} \\ \mathbf{x}_s \end{bmatrix} = A\mathbf{x} + I\mathbf{x}_s = \mathbf{b}. \tag{3.1}$$

Here $\mathbf{x}_s$ is the vector of slack variables. Since the columns of the augmented matrix $[A \vdots I]$ that correspond to the slack variables $\{x_{n+i}\}_{i=1}^m$ is an identity matrix which is clearly invertible, the slack variables $\{x_{n+i}\}_{i=1}^m$ are basic. We denote by $\mathcal{B}$ the set of current basic variables, i.e. $\mathcal{B} = \{x_{n+i}\}_{i=1}^m$. The set of non-basic variables, i.e. $\{x_i\}_{i=1}^n$ will be denoted by $\mathcal{N}$.

Consider now the process of replacing an $x_r \in \mathcal{B}$ by an $x_s \in \mathcal{N}$. We say that x_r is to *leave* the basis and x_s is to *enter* the basis. Consequently after this operation, x_r becomes non-basic, i.e. $x_r \in \mathcal{N}$ and x_s becomes basic, i.e. $x_s \in \mathcal{B}$. This of course amounts to a different (selection of columns of matrix A to give a different) basis $\mathcal{B}$. We shall achieve this *change of basis* by a *pivot operation* (or simply called a *pivot*). This pivot operation is designed to maintain an identity matrix as the basis in the tableau at all time.

3.1.1 Pivot Operation with Respect to the Element a_{rs}.

Once we have decided to replace $x_r \in \mathcal{B}$ by $x_s \in \mathcal{N}$, the a_{rs} in the tableau will be called the *pivot element*. We will see later that the feasibility condition implies that $a_{rs} > 0$. The r-th row and the s-th column of the tableau are called the *pivot row* and the *pivot column* respectively. The rules to update the tableau are:

(a) In pivot row, $a_{rj} \leftarrow a_{rj}/a_{rs}$ for $j = 1, \cdots n + m$.

(b) In pivot column, $a_{rs} \leftarrow 1$, $a_{is} \leftarrow 0$ for $i = 0, \cdots m, i \neq r$.

(c) For all other elements, $a_{ij} \leftarrow a_{ij} - a_{rj} * a_{is}/a_{rs}$.

Graphically, we have

	j	s
i	a_{ij}	a_{is}
r	a_{rj}	a_{rs}^*

becomes

	j	s
i	$a_{ij} - a_{rj}a_{is}/a_{rs}$	0
r	a_{rj}/a_{rs}	1

Or, simply,

a	b
c	d^*

becomes

$a - \frac{bc}{d}$	0
$\frac{c}{d}$	1

Notice that this pivot operation is simply the Gaussian elimination such that variable x_s is eliminated from all $m+1$ but the r-th equation, and in the r-th equation, the coefficient of x_s is equal to 1. In fact, Rule (a) above amounts to normalization of the pivot row such that the pivot element becomes 1. Rule (b) above amounts to eliminations of all the entries in the pivot column except the pivot element. Rule (c) is to compute the *Schur's complement* for the remaining entries in the tableau.

Example 3.1. Consider

$$\begin{cases} x_1 + x_2 - x_3 + x_4 = 5 \\ 2x_1 - 3x_2 + x_3 + x_5 = 3 \\ -x_1 + 2x_2 - x_3 + x_6 = 1 \end{cases}$$

The initial tableau is given by
Tableau 1:

	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_4	1^*	1	−1	1	0	0	5
x_5	2	−3	1	0	1	0	3
x_6	−1	2	−1	0	0	1	1

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

The current basic solution is $[0,0,0,5,3,1]^T$ which is clearly feasible. Suppose we choose $a_{1,1}$ as our pivot element. Then after one pivot operation, we have
Tableau 2:

	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_1	1	1	−1	1	0	0	5
x_5	0	-5^*	3	−2	1	0	−7
x_6	0	3	−2	1	0	1	6

$$B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

We note that the current basic solution is $[5,0,0,0,-7,6]^T$ which is infeasible. Using the new $(2,2)$ entry as pivot, we have
Tableau 3:

	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_1	1	0	$-\frac{2}{5}$	$\frac{3}{5}$	$\frac{1}{5}$	0	$\frac{18}{5}$
x_2	0	1	$-\frac{3}{5}$	$\frac{2}{5}$	$-\frac{1}{5}$	0	$\frac{7}{5}$
x_6	0	0	$-\frac{1}{5}^*$	$-\frac{1}{5}$	$\frac{3}{5}$	1	$\frac{9}{5}$

$$B_3 = \begin{bmatrix} 1 & 1 & 0 \\ 2 & -3 & 0 \\ -1 & 2 & 1 \end{bmatrix}$$

The current basic solution is $[18/5, 7/5, 0, 0, 0, 9/5]^T$ and is feasible. Finally, let us eliminate the last slack variable x_6 by replacing it by x_3.
Tableau 4:

	x_1	x_2	x_3	x_4	x_5	x_6	**b**
x_1	1	0	0	1	−1	−2	0
x_2	0	1	0	1	−2	−3	−4
x_3	0	0	1	1	−2	−5	−9

$$B_4 = \begin{bmatrix} 1 & 1 & -1 \\ 2 & -3 & 1 \\ -1 & 2 & -1 \end{bmatrix}$$

The current basic solution is $[0,-4,-9,0,0,0]^T$ which is infeasible and degenerate. Thus we see that one cannot choose the pivot arbitrarily. It has to be chosen according to some feasibility criterion.

There are three important observations that we should note here. First the pivot operations which amounts to *elementary row operations* on the tableaus, are being recorded in the tableaus at the columns that correspond to the slack variables. In the example above, one can easily check that Tableau i is obtained from Tableau 1 by pre-multiplying Tableau 1 by the matrix formed by the columns of x_4, x_5 and x_6 in Tableau i. In the tableaus, the inverse of these matrices are computed and are denoted by B_i. Let the columns in Tableau 1 be denoted as usual by $\mathbf{a}_j$ and the columns in Tableau i be denoted by $\mathbf{y}_j$, then since

$$[\mathbf{a}_1, \mathbf{a}_2, \cdots, \mathbf{a}_{m+n}] = [A \vdots I] = B_i[B_i^{-1}A \vdots B_i^{-1}] = B_i[\mathbf{y}_1, \mathbf{y}_2, \cdots, \mathbf{y}_{m+n}],$$

it is clear that $B_i\mathbf{y}_j = \mathbf{a}_j$. Comparing this with equation (2.20), we see that B_i are the change of basis matrices from Tableau i to Tableau 1.

Our second observation is the following one. Since the last column in Tableau 1 is given by $\mathbf{b}$, the last column in Tableau i, which we denote by $\mathbf{y}_0 = (y_{10}, \cdots, y_{m0})^T$, will be given by $B_i\mathbf{y}_0 = \mathbf{b}$. Since B_i is invertible, $\mathbf{y}_0$ gives the basic variables of the current basic solution, i.e. the basic solution $\mathbf{x}_B^i$ corresponding to B_i is given by

$$\mathbf{x}_B^i = \mathbf{y}_0 = B_i^{-1}\mathbf{b}. \tag{3.2}$$

For this reason, the last column of the tableau, i.e. $\mathbf{y}_0$, is called the *solution column.*

The third observation is that the columns of B_i are the columns of the initial tableau. For example, the columns of B_3 are the first, second and the sixth columns of Tableau 1. In fact, Tableau 3 is obtained by moving (via elementary row operations) the identity matrix in Tableau 1 to the first, second and the sixth columns in Tableau 3. It indicates that in each iteration of the simplex method, we are just choosing different selection of columns of the augmented matrix to give a different basic matrix B. In particular, the solution obtained in each tableau is indeed the basic solution to our original augmented matrix system (3.1). In fact, Tableau 3 means that

$$[A \vdots I] \begin{bmatrix} 18/5 \\ 7/5 \\ 0 \\ 0 \\ 0 \\ 9/5 \end{bmatrix} = B_3 \begin{bmatrix} 18/5 \\ 7/5 \\ 9/5 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} = \mathbf{b},$$

i.e. the current solution is given by $[18/5, 7/5, 0, 0, 0, 9/5]^T$. For Tableau 4, since $B_4 = A$, we have

$$[I \vdots A^{-1}] \begin{bmatrix} A^{-1}\mathbf{b} \\ \mathbf{0} \end{bmatrix} = A^{-1}\mathbf{b},$$

which is equivalent to

$$A \cdot (A^{-1}\mathbf{b}) + I \cdot \mathbf{0} = \mathbf{b},$$

i.e. the current solution in Tableau 4 is given by $[A^{-1}\mathbf{b}, \mathbf{0}]^T$.

In the following, we consider the criteria that guarantee the feasibility and optimality of the solutions.

3.1.2 Feasibility Condition.

Suppose that the entering variable x_s has been chosen according to some optimality conditions, i.e. the pivot column is the s-th column. Then the leaving basic variable x_r must be selected as the basic variable corresponding to the *smallest* positive ratio of the values of the current right hand side to the current positive constraint coefficients of the entering non-basic variable x_s.

To determine row r	x_s	$\mathbf{y}$	Ratio
	y_{1s}	y_{10}	$\frac{y_{10}}{y_{1s}}$
	y_{2s}	y_{20}	$\frac{y_{20}}{y_{2s}}$
	$\vdots$	$\vdots$	$\vdots$
$\frac{y_{r0}}{y_{rs}} = \min_i\{\frac{y_{i0}}{y_{is}} \mid y_{is} > 0\}$	y_{is}	y_{i0}	$\frac{y_{i0}}{y_{is}}$
	$\vdots$	$\vdots$	$\vdots$
	y_{ms}	y_{m0}	$\frac{y_{m0}}{y_{ms}}$

This follows directly from equation (2.24) and the fact that the current basic solution $\mathbf{x}_B$ defined in (2.24) is given by the solution column $\mathbf{y}_0$ here, see (3.2) above.

3.1.3 Optimality Condition.

For simplicity, we consider a maximization problem. We first denote the entries in the row that correspond to x_0 by y_{0j}. The $(m+1, m+n+1)$-th entry in the tableau is denoted by y_{00}. We will show in the next section that

$$y_{0j} = -(c_j - z_j), \quad j = 1, \cdots, n+m, \tag{3.3}$$

the negation of the reduced cost coefficients that appeared in Theorem 2.6. Here z_j is defined in (2.25). Moreover, we will show also that

$$y_{00} = \mathbf{c}_B^T \mathbf{x}_B, \tag{3.4}$$

i.e. y_{00} is the current objective function value associated with the current BFS in the tableau. Thus according to Theorem 2.6, the entering variable $x_s \in \mathcal{N}$ can be selected as a non-basic variable x_s having a *negative* coefficient. Usual choices are the first negative y_{0s} or the most negative y_{0s}. If all coefficients y_{0j} are non-negative, then by Theorem 2.7, an optimal solution has been reached.

3.1.4 Summary of Computation Procedure.

Once the initial tableau has been constructed, the simplex procedure calls for the successive iteration of the following steps.

1. Testing of the coefficients of the objective function row to determine whether an optimal solution has been reached, i.e., whether the optimality condition that all coefficients are non-negative in that row is satisfied.

2. If not, select a currently non-basic variable x_s to enter the basis. For example, the first negative coefficient or the most negative one.

3. Then determine the currently basic variable x_r to leave the basis using the feasibility condition, i.e. select x_r where $y_{r0}/y_{rs} = \min_i\{y_{i0}/y_{is} \mid y_{is} > 0\}$.

4. Perform a pivot operation with pivot row corresponding to x_r and pivot column corresponding to x_s. Return to 1.

Example 3.2. Consider the LP problem:

$$\text{Max} \quad x_0 = 3x_1 + x_2 + 3x_3$$

$$\text{Subject to} \begin{cases} 2x_1 + x_2 + x_3 & \le 2 \\ x_1 + 2x_2 + 3x_3 & \le 5 \\ 2x_1 + 2x_2 + x_3 & \le 6 \\ x_1, x_2, x_3 & \ge 0 \end{cases}$$

By adding slack variables x_4, x_5 and x_6, we have the following initial tableau.
Tableau 1: Initial tableau, current BFS is $\mathbf{x} = [0, 0, 0, 2, 5, 6]^T$ and $x_0 = 0$.

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}$	Ratio
x_4	2	1^*	1	1	0	0	2	$\frac{2}{1} = 2^*$
x_5	1	2	3	0	1	0	5	$\frac{5}{2} = 2.5$
x_6	2	2	1	0	0	1	6	$\frac{6}{2} = 3$
x_0	-3	-2	-3	0	0	0	0	

We choose x_2 as the entering variable to illustrate that *any* nonbasic variable with negative coefficient can be chosen as entering variable. The smallest ratio is given by x_4 row. Thus x_4 is the leaving variable.
Tableau 2: Current BFS is $\mathbf{x} = [0, 2, 0, 0, 1, 2]^T$ and $x_0 = 2$.

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}$	Ratio
x_2	2	1	1	1	0	0	2	$\frac{2}{1} = 2$
x_5	-3	0	1^*	-2	1	0	1	$\frac{1}{1} = 1^*$
x_6	-2	0	-1	-2	0	1	2	–
x_0	-1	0	-2	1	0	0	2	

Tableau 3: Current BFS is $\mathbf{x} = [0, 1, 1, 0, 0, 3]^T$ and $x_0 = 4$.

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}$	Ratio
x_2	5^*	1	0	3	-1	0	1	$\frac{1}{5}$
x_3	-3	0	1	-2	1	0	1	–
x_6	-5	0	0	-4	1	1	3	–
x_0	-7	0	0	-3	2	0	4	

Tableau 4: Optimal tableau, optimal BFS $\mathbf{x}^* = [1/5, 0, 8/5, 0, 0, 4]^T$, $x_0^* = 27/5$.

	x_1	x_2	x_3	x_4	x_5	x_6	$\mathbf{b}$
x_1	1	$\frac{1}{5}$	0	$\frac{3}{5}$	$-\frac{1}{5}$	0	$\frac{1}{5}$
x_3	0	$\frac{3}{5}$	1	$-\frac{1}{5}$	$\frac{2}{5}$	0	$\frac{8}{5}$
x_6	0	1	0	-1	0	1	4
x_0	0	$\frac{7}{5}$	0	$\frac{6}{5}$	$\frac{3}{5}$	0	$\frac{27}{5}$

We note that the extreme point sequence that the simplex method passes through are $\{x_4, x_5, x_6\} \to \{x_2, x_5, x_6\} \to \{x_2, x_3, x_6\} \to \{x_1, x_3, x_6\}$.

3.2 Simplex Methods for Problems in Standard Form

Our previous method is based upon the existence of an initial BFS to the problem. It is desirable to have an identity matrix as the initial basic matrix. For LP in feasible canonical form, the initial basic matrix is the matrix associated with the slack variables, and is an identity matrix. Consider an LP in standard form:

$$\begin{array}{ll} \max & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases} \end{array}$$

where we assume that $\mathbf{b} \geq \mathbf{0}$. There is no obvious initial starting basis $\mathcal{B}$ such that $B = I_m$. For notational simplicity, assume that we pick B as the last m (linearly independent) columns of A, i.e. A is of the form $A = [N \vdots B]$. We then have for the augmented system:

$$\begin{cases} \phantom{x_0 - \mathbf{c}_N^T}N\mathbf{x}_N & + \quad B\mathbf{x}_B = \mathbf{b} \\ x_0 - \mathbf{c}_N^T\mathbf{x}_N & - \quad \mathbf{c}_B^T\mathbf{x}_B = \mathbf{0} \end{cases}$$

Multiplying by B^{-1} to the first equation yields,

$$B^{-1}N\mathbf{x}_N + \mathbf{x}_B = B^{-1}\mathbf{b}$$

or

$$\mathbf{x}_B = B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_N.$$

Hence the x_0 equation becomes

$$x_0 - \mathbf{c}_N^T\mathbf{x}_N - \mathbf{c}_B^T(B^{-1}\mathbf{b} - B^{-1}N\mathbf{x}_N) = 0.$$

Thus we have

$$\begin{cases} B^{-1}N\mathbf{x}_N & + \quad \mathbf{x}_B = B^{-1}\mathbf{b} \\ x_0 - (\mathbf{c}_N^T - \mathbf{c}_B^T B^{-1}N)\mathbf{x}_N & \qquad\quad = \mathbf{c}_B^T B^{-1}\mathbf{b} \end{cases}$$

Denoting $\mathbf{z}_N^T = \mathbf{c}_B^T B^{-1}N$ (an $(n-m)$ row vector) gives

$$\begin{cases} B^{-1}N\mathbf{x}_N & + \quad \mathbf{x}_B = B^{-1}\mathbf{b} \\ x_0 - (\mathbf{c}_N^T - \mathbf{z}_N^T)\mathbf{x}_N & \qquad\quad = \mathbf{c}_B^T B^{-1}\mathbf{b} \end{cases}$$

which is called the *general representation* of an LP in standard form with respect to the basis $\mathcal{B}$. Its initial simplex tableau is then

	$\mathbf{x}_N$	$\mathbf{x}_B$	$\mathbf{b}$
$\mathbf{x}_B$	$B^{-1}N$	I	$B^{-1}\mathbf{b}$
x_0	$-(\mathbf{c}_N^T - \mathbf{z}_N^T)$	$\mathbf{0}$	$\mathbf{c}_B^T B^{-1}\mathbf{b}$

We note that the j-th entry of $\mathbf{z}_N$ is given by

$$\mathbf{c}_B^T B^{-1} N_{\cdot j} = \mathbf{c}_B^T B^{-1} \mathbf{a}_j = \mathbf{c}_B^T \mathbf{y}_j = z_j$$

where z_j is defined as in (2.21). Thus in the table, we see that the entries in the x_0 row are given by $-(c_j - z_j)$ for $x_j \in \mathcal{N}$ and zero for $x_j \in \mathcal{B}$. Thus they are the negation of the reduced cost coefficients. This verifies equation (3.3) that we have assumed earlier. Moreover, by (3.2), we see that

$$y_{00} = \mathbf{c}_B^T B^{-1} \mathbf{b} = \mathbf{c}_B^T \mathbf{x}_B,$$

which is the same as (3.4).

We remark that x_0 is now expressed in terms of the non-basic variables,

$$x_0 = \mathbf{c}_B^T B^{-1} \mathbf{b} + \sum_{x_j \in \mathcal{N}} (c_j - z_j) x_j. \tag{3.5}$$

Hence it is easy to see that for maximization problem, the current BFS is optimal when $c_j - z_j \leq 0$ for all j. For minimization problem, the current BFS will be optimal when $c_j - z_j \geq 0$ for all j.

Example 3.3. Consider the following LP.

$$\begin{array}{ll} \max & x_0 = x_1 + x_2 \\ & 2x_1 + x_2 \geq 4 \\ \text{subject to} & x_1 + 2x_2 = 6 \\ & x_1, x_2 \geq 0 \end{array}$$

Putting into standard form by adding the surplus variable x_3, the augmented system is:

$$\begin{cases} \quad\; 2\,x_1 + x_2 - x_3 = 4 \\ \quad\;\; x_1 + 2x_2 \quad\;\; = 6 \\ x_0 - x_1 - x_2 \quad\;\; = 0 \end{cases}$$

The simplex tableau for the problem is:
Tableau 0:

	x_1	x_2	x_3	$\mathbf{b}$
	2	1	-1	4
	1	2	0	6
x_0	-1	-1	0	0

Here we do not have a starting identity matrix. Suppose we let x_1 and x_2 to be our starting basic variables, then

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}, \quad N = \begin{bmatrix} -1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{c}_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

In this case

$$\begin{aligned} B^{-1} &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \\ x_3 = B^{-1} N &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \end{bmatrix}, \\ \mathbf{b} = B^{-1} \mathbf{b} &= \frac{1}{3} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ 6 \end{bmatrix} = \begin{bmatrix} \frac{2}{3} \\ \frac{8}{3} \end{bmatrix}. \end{aligned}$$

It is also easily check that

$$z_3 = \mathbf{c}^T B^{-1}\mathbf{a}_j = [-1,-1]\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} -1 \\ 0 \end{bmatrix} = \frac{1}{3}$$

and the current value of the objective function is given by

$$\mathbf{c}_B^T B^{-1}\mathbf{b} = [-1,-1]\frac{1}{3}\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}\begin{bmatrix} 4 \\ 6 \end{bmatrix} = 10/3.$$

Hence the starting tableau is:
Tableau 1:

	x_1	x_2	x_3	$\mathbf{b}$
x_1	1	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_2	0	1	$\frac{1}{3}^*$	$\frac{8}{3}$
x_0	0	0	$-\frac{1}{3}$	$\frac{10}{3}$

Thus $\mathbf{x} = [2/3, 8/3, 0]^T$ is an initial BFS. We can now apply the simplex method as discussed in §1 to find the optimal solution. The next iteration gives:
Tableau 2:

	x_1	x_2	x_3	$\mathbf{b}$
x_1	1	2	0	6
x_3	0	3	1	8
x_0	0	1	0	6

Thus the optimal solution is $\mathbf{x}^* = [6, 0, 8]^T$ with $x_0^* = 6$.

We note that if we choose x_1 and x_3 as our starting basis variables, then we get Tableau 2 immediately and no iteration is required. However, if x_2 and x_3 are chosen as starting variables, then we have
Tableau 1':

	x_1	x_2	x_3	$\mathbf{b}$
x_1	$\frac{1}{2}$	1	0	3
x_3	$-\frac{3}{2}$	0	1	-1
x_0	$-\frac{1}{2}$	0	0	3

Hence the starting basic solution is not feasible and we cannot use the simplex method to find our optimal solution.

3.3 The M-Method

Example 3.3 illustrates that the starting basic solution may sometimes be infeasible. The M-method and the Two-phase method discussed in this and the next sections are methods that can find a starting basic feasible solution whenever it exists. Consider again an LPP where there is no desirable starting identity matrix.

$$\begin{array}{ll} \max & x_0 = \mathbf{c}^T\mathbf{x} \\ \text{subject to} & \begin{cases} A\mathbf{x} = \mathbf{b}, \\ \mathbf{x} \geq \mathbf{0}. \end{cases} \end{array}$$

where $\mathbf{b} \geq \mathbf{0}$. We may add suitable number of artificial variables $x_{a_1}, x_{a_2}, \cdots, x_{a_m}$ to it to get a starting identity matrix. The corresponding prices for the artificial variables are $-M$ for maximization problem, where M is sufficiently large. The effect of the constant M is to penalize any artificial variables that will occur with positive values in the final optimal solutions. Using the idea, the LPP becomes

$$\begin{array}{ll} \max & z = \mathbf{c}^T\mathbf{x} - M \cdot \mathbf{1}^T\mathbf{x}_a \\ \text{subject to} & \begin{cases} A\mathbf{x} + I_m\mathbf{x}_a = \mathbf{b} \\ \mathbf{x} \geq \mathbf{0}, \end{cases} \end{array}$$

where $\mathbf{x}_a = (x_{a_1}, x_{a_2}, \cdots, x_{a_m})^T$ and $\mathbf{1}$ is the vector of all ones. We observe that $\mathbf{x} = \mathbf{0}$ and $\mathbf{x}_a = \mathbf{b}$ is a feasible starting BFS. Moreover, any solution to $A\mathbf{x} + I_m\mathbf{x}_a = \mathbf{b}$ which is also a solution to $A\mathbf{x} = \mathbf{b}$ must have $\mathbf{x}_a = \mathbf{0}$. Thus, we have to drive $\mathbf{x}_a = 0$ if possible.

Example 3.4. Consider the LP in Example 3.3 again.

$$\begin{array}{ll} \max & x_0 = x_1 + x_2 \\ \text{subject to} & \begin{cases} 2x_1 + x_2 \geq 4 \\ x_1 + 2x_2 = 6 \\ x_1, x_2 \geq 0 \end{cases} \end{array}$$

Introducing surplus variable x_3 and artificial variables x_4 and x_5 yields,

$$\begin{cases} 2x_1 + x_2 - x_3 + x_4 = 4 \\ x_1 + 2x_2 + x_5 = 6 \\ x_0 - x_1 - x_2 + Mx_4 + Mx_5 = 0 \end{cases}$$

Now the columns corresponding to x_4 and x_5 form an identity matrix. In tableau form, we have

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	2	1	-1	1	0	4
x_5	1	2	0	0	1	6
x_0	-1	-1	0	M	M	0

Notice that in the x_0 row, the reduced cost coefficients that correspond to the basic variables x_4 and x_5 are not zero. These nonzero entries are to be eliminated first before we have our starting tableau. After eliminations of those M, we have the initial tableau:

	x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_4	2^*	1	-1	1	0	4
x_5	1	2	0	0	1	6
x_0	$-(1+3M)$	$-(1+3M)$	M	0	0	$-10M$

We note that once an artificial variable becomes non-basic, it can be dropped from consideration in subsequent calculations.

	x_1	x_2	x_3	x_5	$\mathbf{b}$
x_1	1	$\frac{1}{2}$	$-\frac{1}{2}$	0	2
x_5	0	$\frac{3}{2}^*$	$\frac{1}{2}$	1	4
x_0	0	$-\frac{1+3M}{2}$	$-\frac{1+M}{2}$	0	$2-4M$

After we eliminate all the artificial variables we have

	x_1	x_2	x_3	$\mathbf{b}$
x_1	1	0	$-\frac{2}{3}$	$\frac{2}{3}$
x_2	0	1	$\frac{1}{3}^*$	$\frac{8}{3}$
x_0	0	0	$-\frac{1}{3}$	$\frac{10}{3}$

At this point all artificial variables are dropped from the problem, and $\mathbf{x} = [2/3, 8/3, 0]^T$ is an initial BFS. Notice that this is the same as Tableau 1 in Example 3.3. After one iteration, we get the final optimal tableau.

	x_1	x_2	x_3	$\mathbf{b}$
x_1	1	2	0	6
x_3	0	3	1	8
x_0	0	1	0	6

Thus the optimal solution is $\mathbf{x}^* = (6, 0, 8)^T$ with $x_0^* = 6$.

3.4 The Two-Phase Method

The M-method is sensitive to round-off error when being implemented on computers. The two-phase method is used to circumvent this difficulty.

PHASE I: (Search for a Starting BFS)

Instead of considering the actual objective function in the M-Method

$$z = \sum_{i=1}^{n} c_i x_i - M \sum_{i=1}^{m} x_{a_i} \ ,$$

we maximize the function

$$z^* = -\sum_{i=1}^{m} x_{a_i}.$$

Since $\mathbf{b} \geq \mathbf{0}$, the initial BFS satisfies $\mathbf{x}_a \geq \mathbf{0}$. Notice that $z^* \leq 0$ and the possible maximum value of z^* is zero. Moreover, z^* will be zero only if each artificial variable is zero. If the maximum of z^* is zero, we have driven all artificial variables to zero. If the maximum of z^* is not zero, then the artificial variables cannot be driven to zero and the original problem has no feasible solution. In Phase I, we stop as soon as z^* becomes zero, because we know that this is the maximum value of z^*. We need not continue until the optimality criterion is satisfied if z^* becomes zero before this happens. During Phase I, the sequence of vectors to enter and leave the basis is the same as the M-method except when the vectors are tied. In fact, the reduced cost coefficients of both methods are given by

$$\begin{aligned} M\text{-method:} \qquad & z_j - c_j = -M\sum_r y_{rj} + \beta \\ \text{Phase I:} \qquad & z_j - c_j = -\sum_r y_{rj} \end{aligned}$$

where β is a small quantity compared with M. At the end of Phase I, i.e. when the optimality condition is satisfied or $z^* = 0$, we have one of the following three possibilities:

(i) $\max z^* < 0$, in this case, no feasible solution exists for our original problem, see §4.1.

(ii) $\max z^* = 0$ and no artificial variable appears in the basis, i.e. we have found a BFS to the original problem.

(iii) $\max z^* = 0$ and one or more artificial variables appear in the basis at zero level. In this case, we have also found a basic degenerate "feasible solution" to the original problem. Degenerate solutions are discussed in §4.5.

Phase II: (Conclude with an Optimal BFS)

When Phase I ends in (ii) or (iii), we go to Phase II to find an optimal solution. In Phase II, we assign the actual price c_j to each structural variable and a price of zero to any artificial variables which still appear in the basis at zero level. Thus the objective function to be optimized in Phase II is the actual objective function $z = \sum_{i=1}^{n} c_i x_i$.

When Phase I ends in (ii), we are back to the situation discussed in §2, and there should be no problem. When Phase I ends in (iii), we must give special attentions to the artificial variables which appear in the basis at zero level. We must make sure that the artificial variables never become positive again in Phase II. We will return to this case in §4.6.

Example 3.5. Consider the following LP.

$$\begin{array}{ll} \min & x_0 = -2x_1 + 4x_2 + 7x_3 + x_4 + 5x_5 \\ \text{subject to} & \begin{cases} -x_1 + \ \ x_2 + 2x_3 + \ \ x_4 + 2x_5 = 7 \\ -x_1 + 2x_2 + 3x_3 + \ \ x_4 + \ \ x_5 = 6 \\ -x_1 + \ \ x_2 + \ \ x_3 + 2x_4 + \ \ x_5 = 4 \\ \qquad x_1 \text{ free}, \quad x_2, x_3, x_4, x_5 \geq 0 . \end{cases} \end{array}$$

Since x_1 is free, it can be eliminated by solving for x_1 in terms of the other variables from the first equation and substituting everywhere else. This can be done nicely using our pivot operation on the following simplex tableau:

x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$
-1^*	1	2	1	2	7
-1	2	3	1	1	6
-1	1	1	2	1	4
2	-4	-7	-2	-5	0

Initial tableau

We select any non-zero element in the first column as our pivot element – this will eliminate x_1 from all other rows:-

x_1	x_2	x_3	x_4	x_5	$\mathbf{b}$	
1	-1	-2	-1	-2	-7	$\leftarrow (*)$
0	1	1	0	-1	-1	
0	0	-1	1	-1	-3	
0	-2	-3	1	-1	14	

Equivalent Problem

Saving the first row $(*)$ for future reference only, we carry on only the sub-tableau with the first row *and* the first column deleted. There is no obvious basic feasible solution, so we use the two-phase method: After making $\mathbf{b} \geq \mathbf{0}$, we introduce artificial variables $y_1 \geq 0$ and $y_2 \geq 0$ to give the artificial problem:-

x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{b}$
-1	-1	0	1	1	0	1
0	1	-1	1	0	1	3
0	0	0	0	-1	-1	0

$$c_B = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Initial Tableau for Phase I

The cost coefficients of the artificial variables are $+1$ because we are dealing with a minimization problem. Transforming (by adding the first two rows to the last row) the last row to give a tableau in canonical form, we get

	x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{b}$
y_1	-1	-1	0	1	1	0	1
y_2	0	1	-1	1	0	1	3
x_0	-1	0	-1	2	0	0	4

First tableau for Phase I

which is in canonical form. Recall that this is a minimization problem, entering variable is chosen with positive entry (rather than negative) in the x_0-row. We carry out the pivot operations with the indicated pivot elements:-

	x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{b}$
x_5	-1	-1	0	1	1	0	1
y_2	1^*	2	-1	0	-1	1	2
x_0	1	2	-1	0	-2	0	2

Second tableau for Phase I

	x_2	x_3	x_4	x_5	y_1	y_2	$\mathbf{b}$
x_5	0	1	-1	1	0	1	3
x_1	1	2	-1	0	-1	1	2
x_0	0	0	0	0	-1	-1	0

Final tableau for Phase I

At the end of Phase I, we go back to the equivalent reduced problem (i.e. discarding the artificial variables y_1, y_2):-

	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_5	0	1	-1	1	3
x_1	1	2	-1	0	2
x_0	-2	-3	1	-1	14

$$c_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Initial problem for Phase II

This is transform into

	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_5	0	1	-1	1	3
x_1	1	2^*	-1	0	2
x_0	0	2	-2	0	21

$$c_B = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

Initial tableau for Phase II

Pivoting as shown gives

	x_2	x_3	x_4	x_5	$\mathbf{b}$
x_5	$-\frac{1}{2}$	0	$-\frac{1}{2}$	1	2
x_3	$\frac{1}{2}$	1	$-\frac{1}{2}$	0	1
x_0	-1	0	-1	0	19

Final tableau for Phase II

The solution $x_3 = 1$, $x_5 = 2$ can be inserted in the expression $(*)$ for x_1 giving

$$x_1 = -7 + 2(1) + 2(2) = -1 \ .$$

Thus the final solution is $\mathbf{x}^* = [-1, 0, 1, 0, 2]^T$ with $x_0^* = 19$.