Q function and error function

We first note that

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi} \qquad ; \qquad \int_{-\infty}^{\infty} e^{-\frac{ax^2}{2}} dx = \sqrt{\frac{2\pi}{a}}$$

For our needs in Digital Communication course, we define:

$$Q(\alpha) \triangleq \frac{1}{\sqrt{2\pi}} \int_{\alpha}^{\infty} e^{-\frac{x^2}{2}} dx$$

The $Q(\cdot)$ function is monotonically decreasing. Some features:

$$Q(-\infty) = 1 \qquad ; \qquad Q(0) = \frac{1}{2} \qquad ; \qquad Q(\infty) = 0 \qquad ; \qquad Q(-x) = 1 - Q(x)$$

Known bounds (valid for $x > 0$):

$$\begin{aligned} \frac{1}{\sqrt{2\pi}x}\left(1 - \frac{1}{x^2}\right) e^{-x^2/2} < \quad Q(x) \quad &< \frac{1}{\sqrt{2\pi}x} e^{-x^2/2} \\ Q(x) \quad &\leq \frac{1}{2} e^{-x^2/2} \end{aligned}$$

Matlab does not have a build-in function for $Q(\cdot)$. Instead, we use its *erf* function:

$$\mathrm{erf}(\alpha) \triangleq \frac{2}{\sqrt{\pi}} \int_{0}^{\alpha} e^{-x^2} dx$$

Note that *erf* function is defined over $[0, \infty)$ only, and

$$\mathrm{erf}(0) = 0 \qquad ; \qquad \mathrm{erf}(\infty) = 1$$

The relations between the two functions are

$$Q(\alpha) = \frac{1}{2} - \frac{1}{2}\mathrm{erf}\left(\frac{\alpha}{\sqrt{2}}\right) \qquad ; \qquad \mathrm{erf}(\alpha) = 1 - 2Q(\sqrt{2}\alpha)$$

If we have a normal variable $X \sim N(\mu, \sigma^2)$, the probability that $X > x$ is

$$\Pr\{X > x\} = Q\left(\frac{x - \mu}{\sigma}\right)$$

Now, if we want to know the probability of X to be away from its expectation μ by at least a (either to the left or to the right) we have:

$$\Pr\{X > \mu + a\} = \Pr\{X < \mu - a\} = Q\left(\frac{a}{\sigma}\right)$$

The probability to be away from the center where we *don't matter* in which direction is $2 \cdot Q(\frac{a}{\sigma})$.

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