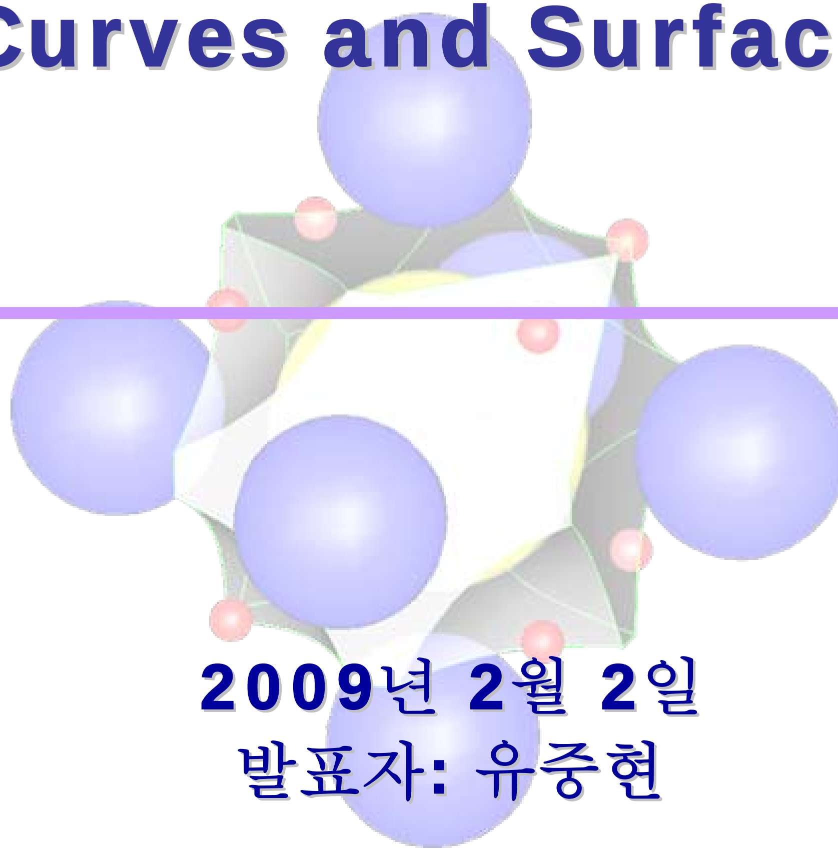
Architectural geometry: Curves and Surfaces



2009년 2월 2일

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Types of curve representations

- Explicit form
- Implicit form
- Parametric form

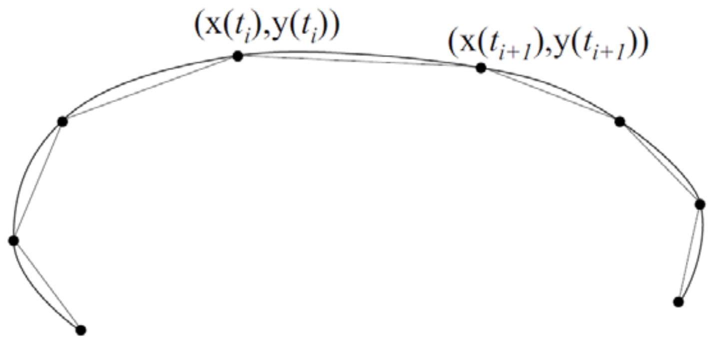
Explicit form

- y = f(x) $y = \pm\sqrt{r^2 - x^2},\ z = 0$
- Coordinate system dependent
- Single valued
- No vertical tangent allowed
- Can not represent closed curves
- Difficult to define a boundary curve
- Easy to check a point (x, y) lies on the curve
- Difficult to get a point on the curve

Implicit form

- f(x,y)=0 $x^2 + y^2 - r^2 = 0, z = 0$
- Coordinate system dependent
- Multiple valued
- Vertical tangent allowed
- Can not represent closed curves
- Difficult to define a boundary curve
- Easy to check a point (x, y) lies on the curve
- Difficult to get a point on the curve

Parametric form



- **x = x(t), y = y(t)**

  $$x = r\cos t, y = r\sin t, z = 0$$

  $$(0 \le t \le 2\pi)$$

- **Easy to evaluate a point**
- **Easy to evaluate orderly sequences of points**
- **Easy to draw a curve**
- **Easy to define a bounded curve ex) 0≤ t ≤1**
- **Difficult to see if a point lies on the curve**
- **Representation is not unique**

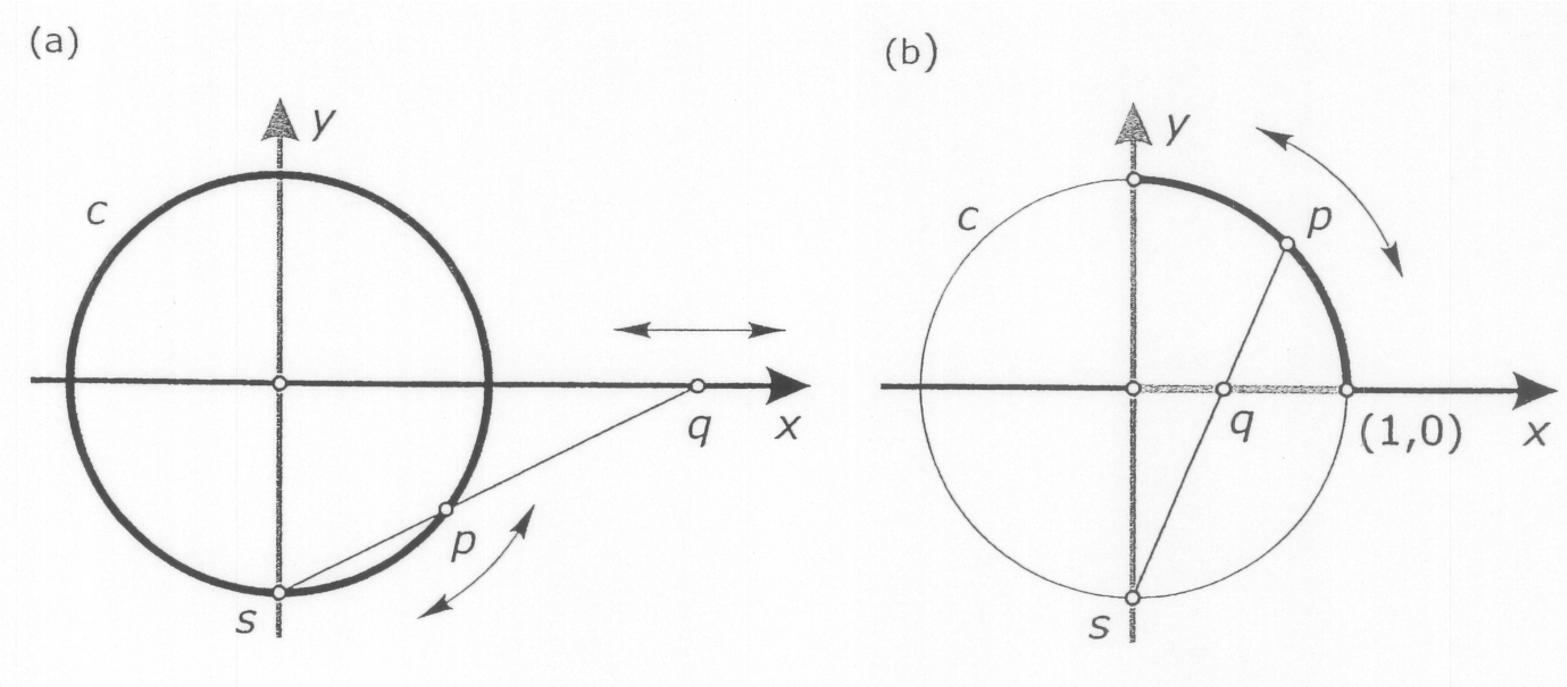
  $$x = r\frac{1-t^2}{1+t^2}, y = r\frac{2t}{1+t^2}, z = 0$$

  $$t = \tan\frac{\theta}{2} \quad (0 \le \theta \le 2\pi)$$

Parameterization

- **Line pass through s(0, -1) and q(t, 0)**

$$\frac{x}{t} + \frac{y}{-1} = 1$$ **(2D stereographic projection)**



Polynomial curves/surfaces

- Coordinate function
  - Curve: $C(t) = (x(t), y(t), z(t))$
  - Surface: $S(s, t) = (x(s,t), y(s,t), z(s,t))$
- Degree of curves and surfaces
  - Highest degree of parameter occurring in coordinate functions
- Polynomial vs. rational
  - Rational: polynomial + common denominator
  - Curve: $C(t) = (x(t) / d(t), y(t) / d(t), z(t) / d(t))$
  - Surface: $S(s, t) = (x(s,t) / d(s,t), y(s,t) / d(s,t), z(s,t) / d(s,t))$

Why cubic curve?

- Minimum degree for Spatial curve
- Minimum degree for Interior inflection point
- Computation cost
- Numerical error
- Minimum degree for C2 continuity
- Curve controllability

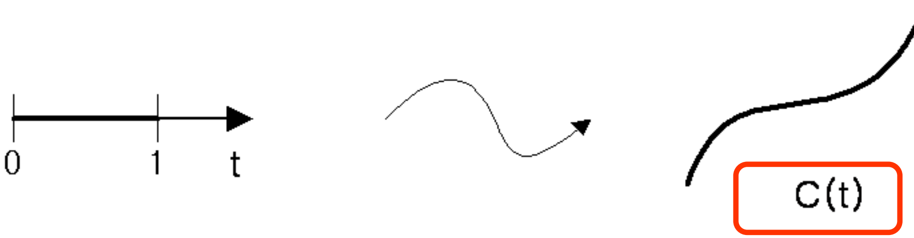
Why polynomials?

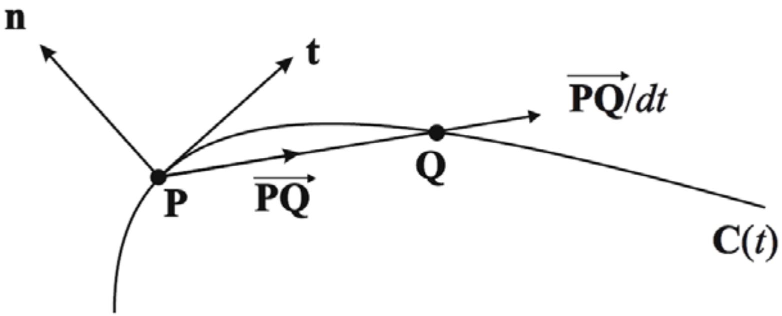
- **Can be easily, efficiently, and accurately processed in a computer**
  - Ex. Polynomial vs. trigonometric functions
- **Simple and mathematically well understood**
  - Ex. derivatives and integration
- **Capable of precisely representing all the curves the users of the system need**
- **Widely used class of functions**
  - Ex. **Power basis form**

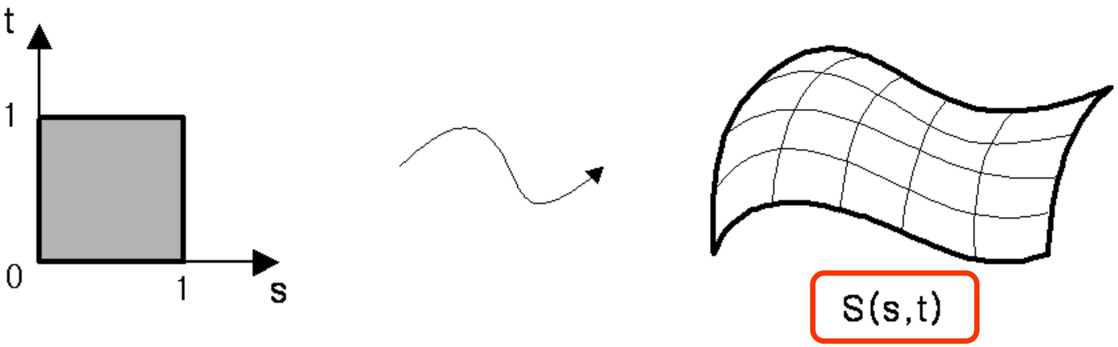
$$P(t) = \left[x(t), y(t), z(t)\right] = \sum_{i=0}^{n} \mathbf{a_i} t^i \quad (0 \le t \le 1) \qquad \mathbf{a_i} = (x_i, y_i, z_i)$$

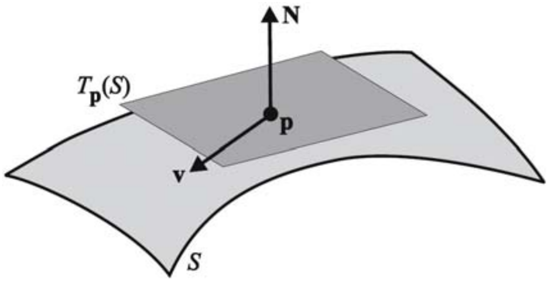
Parametric curves and surfaces

- Parameter space
- Tangent?
- Normal?







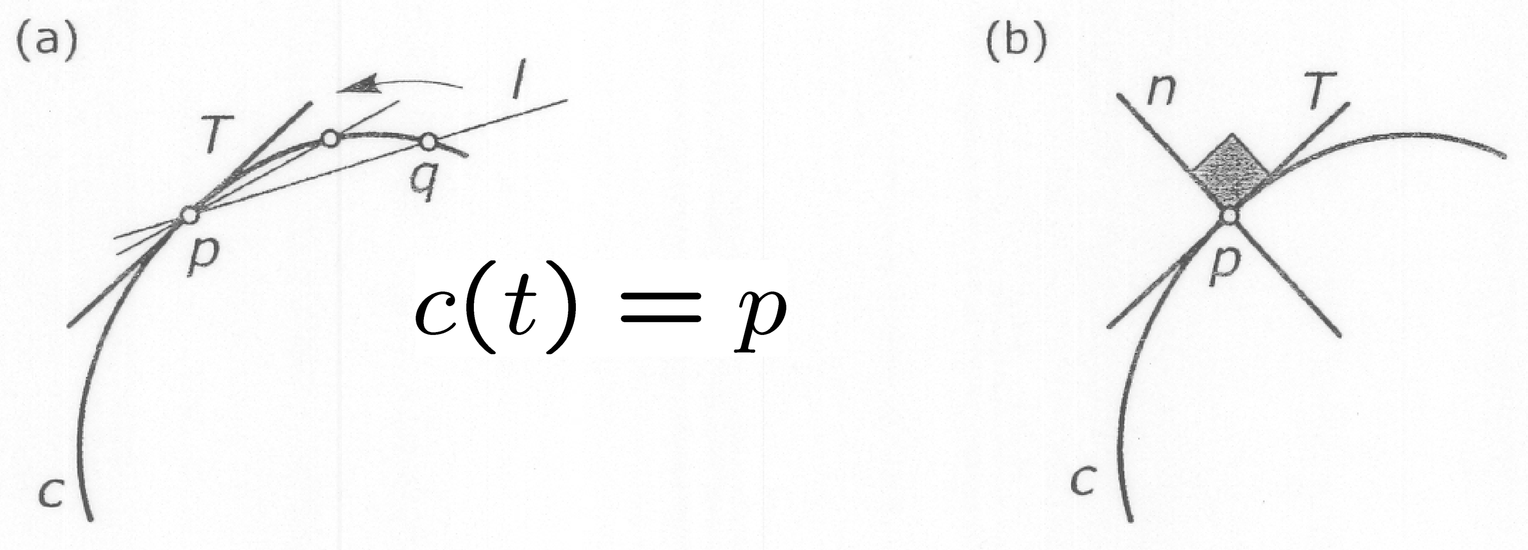


Tangent and normal

- **Tangent**
  - **Regular point / singular point**

$$\lim_{h\to 0}\frac{c(t+h)-c(t)}{h} = c'(t) = (x'(t), y'(t), z'(t))$$

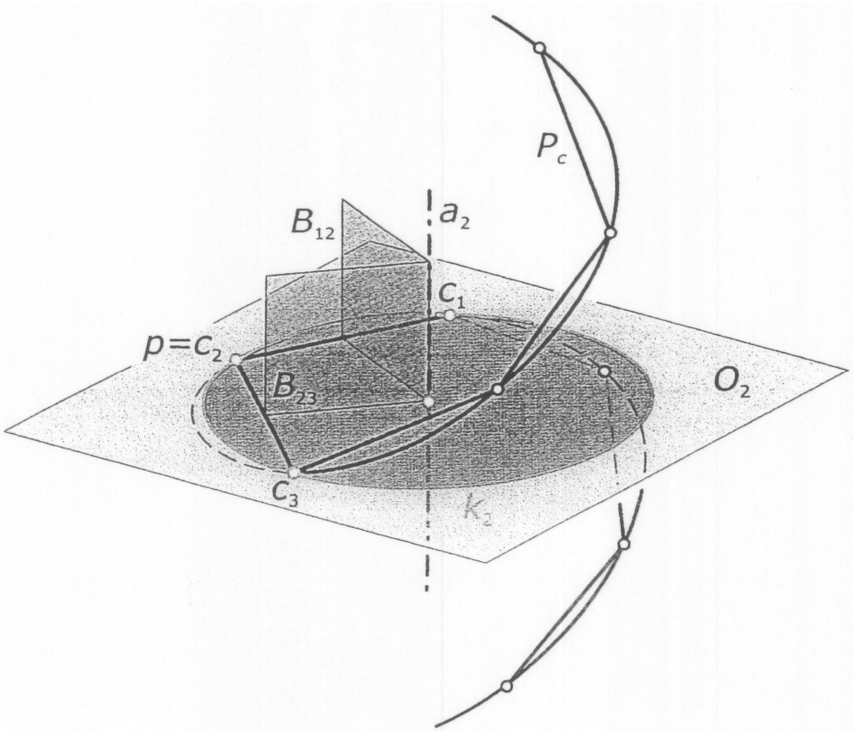
- **Normal**
  - **Spatial curve: normal plane**

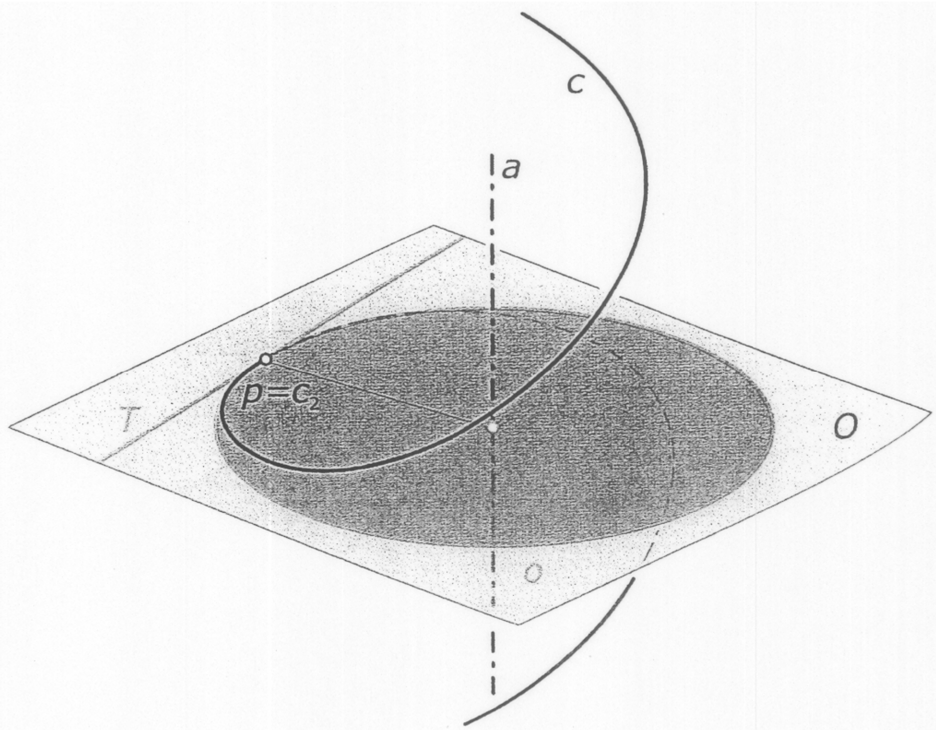


$$c(t) = p$$

Curvature

- Osculating plane / circle
  - Curvature center
  - Curvature radius (radius of osculating circle): r
    - Curvature: $k = 1/r$





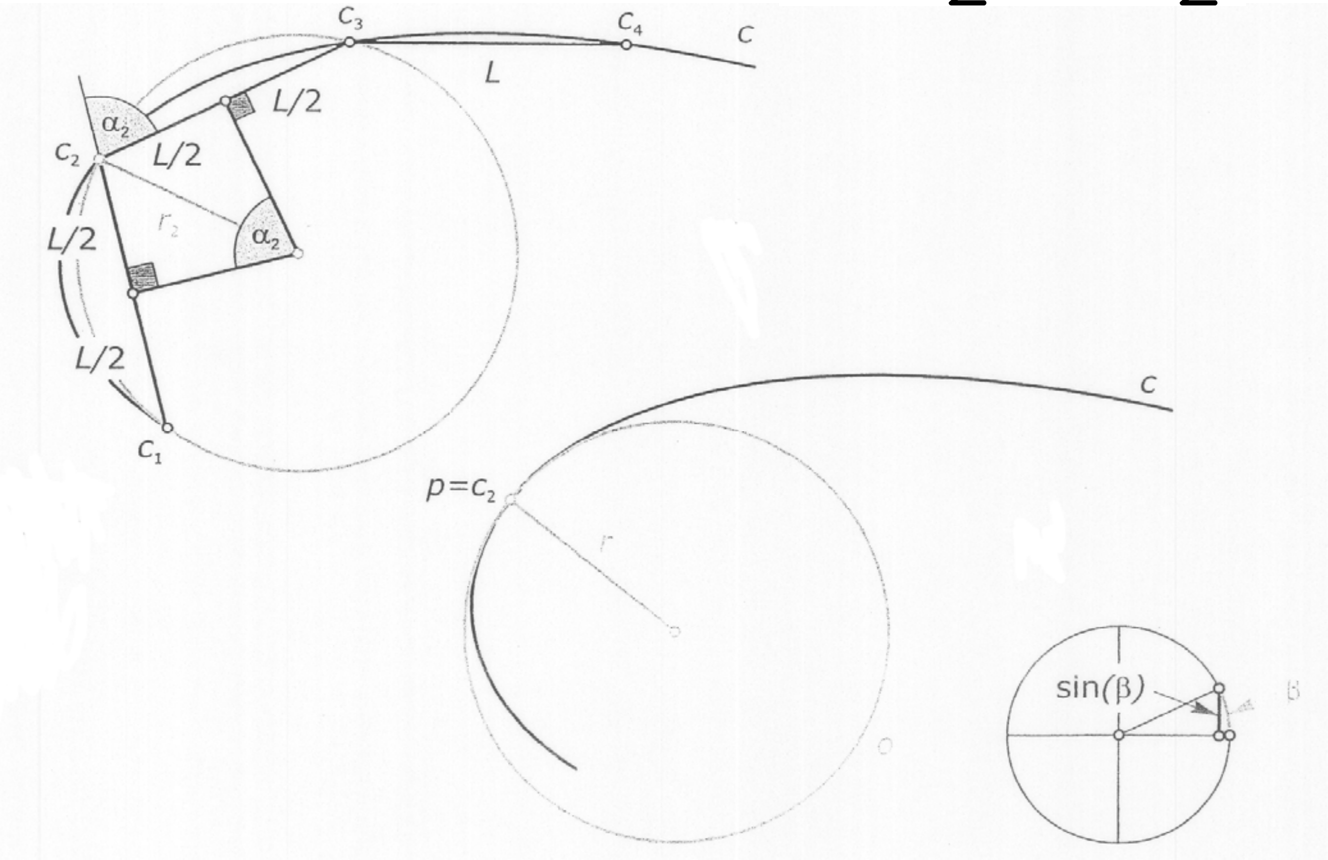
Curvature

- **Local directional change of the tangent**

$$k = \lim_{L \to 0} \frac{1}{r_2} = \frac{\alpha_2}{L}$$

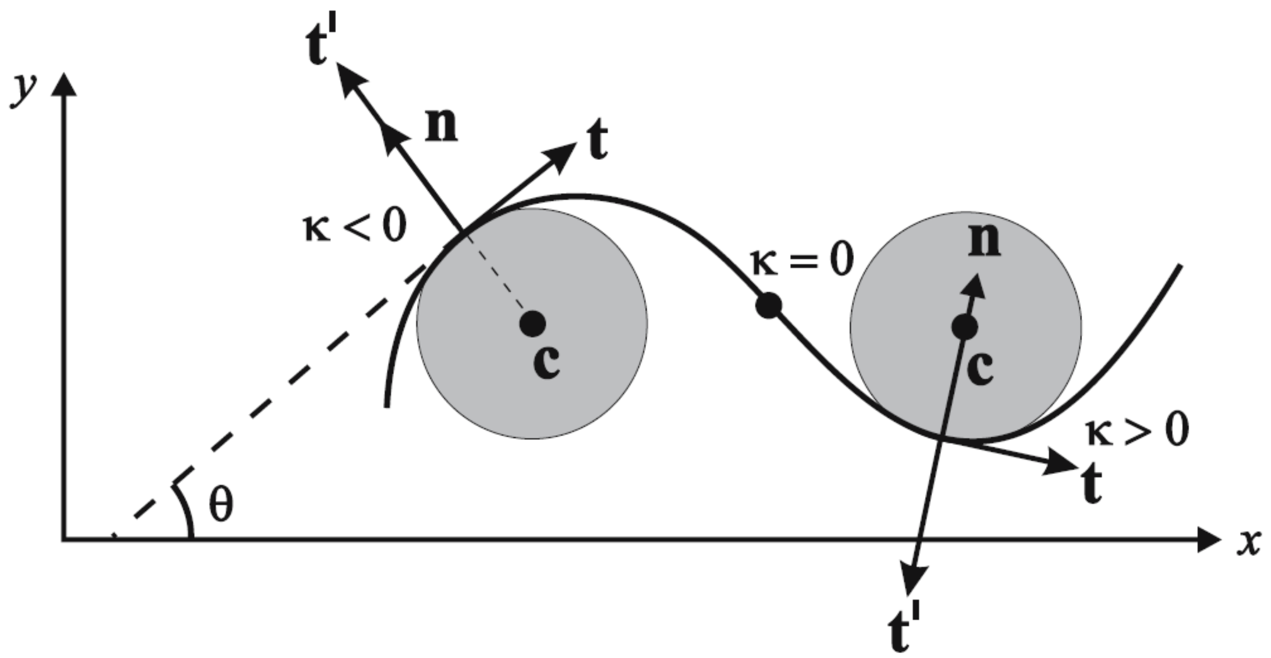
$$\sin \frac{\alpha_2}{2} = \frac{\frac{L}{2}}{r_2}$$

$$\sin \frac{\alpha_2}{2} \approx \frac{\alpha_2}{2}$$



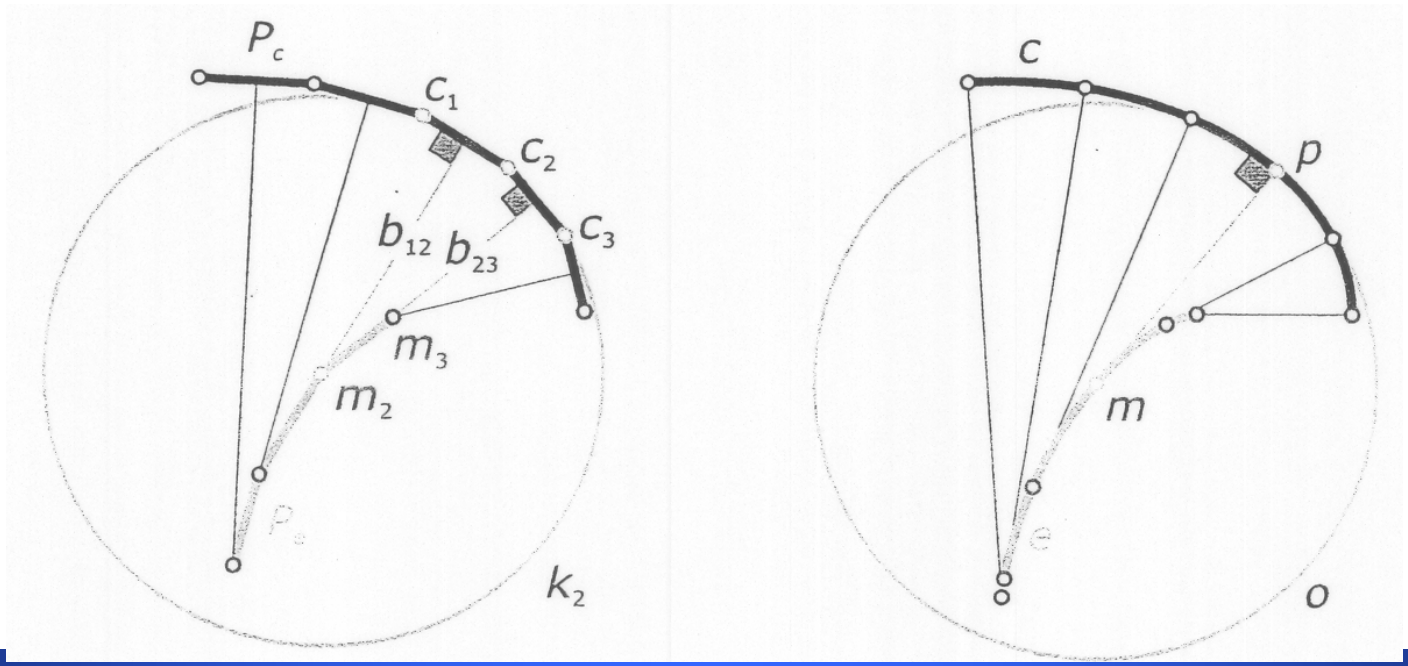
Curvature of a curve

- Inflection point
  - Curvature $k = 0$ (Osculating circle → tangent line)
- Sign of curvature is meaningful for only planar curve

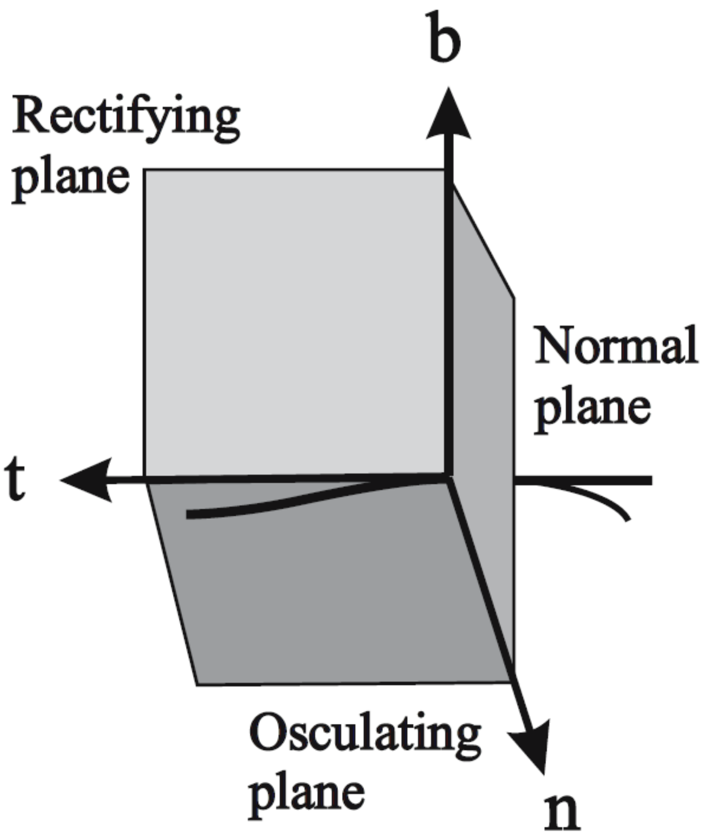


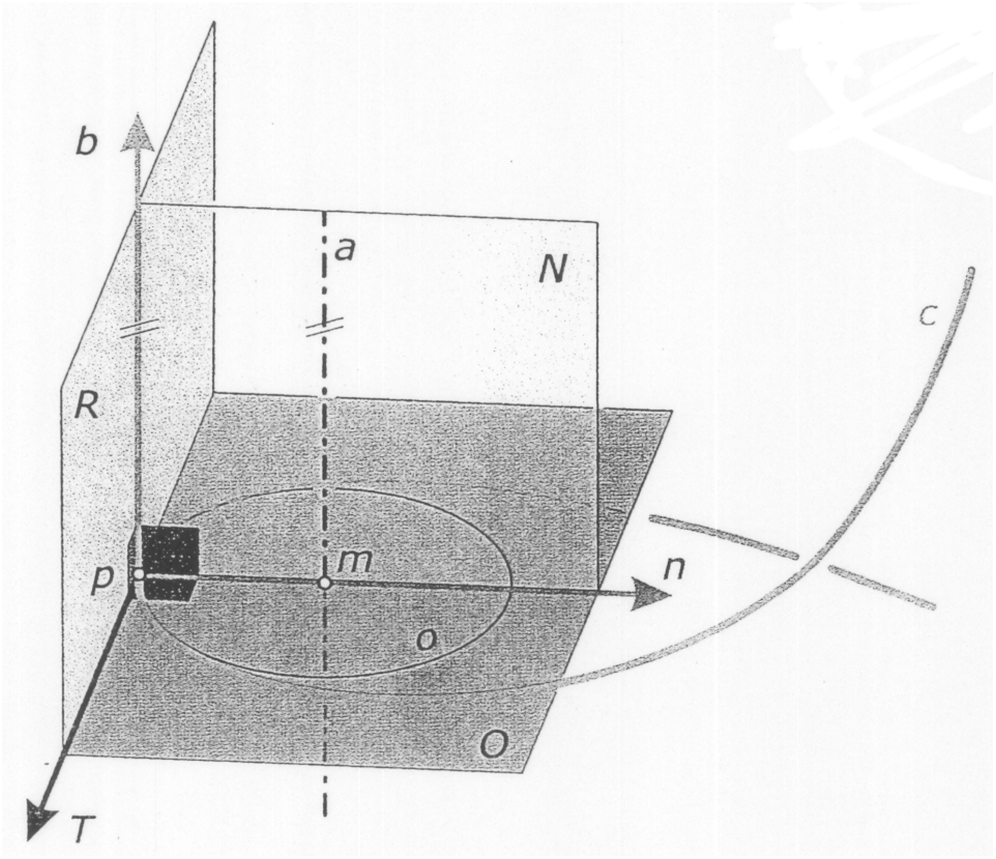
Evolute of a curve

- The locus of the centers of all osculating circles of a planar curve *c*
  - Evolute *e* can be generated as envelope of the curve normals
- The normals of *c* corresponds to the tangents of *e*



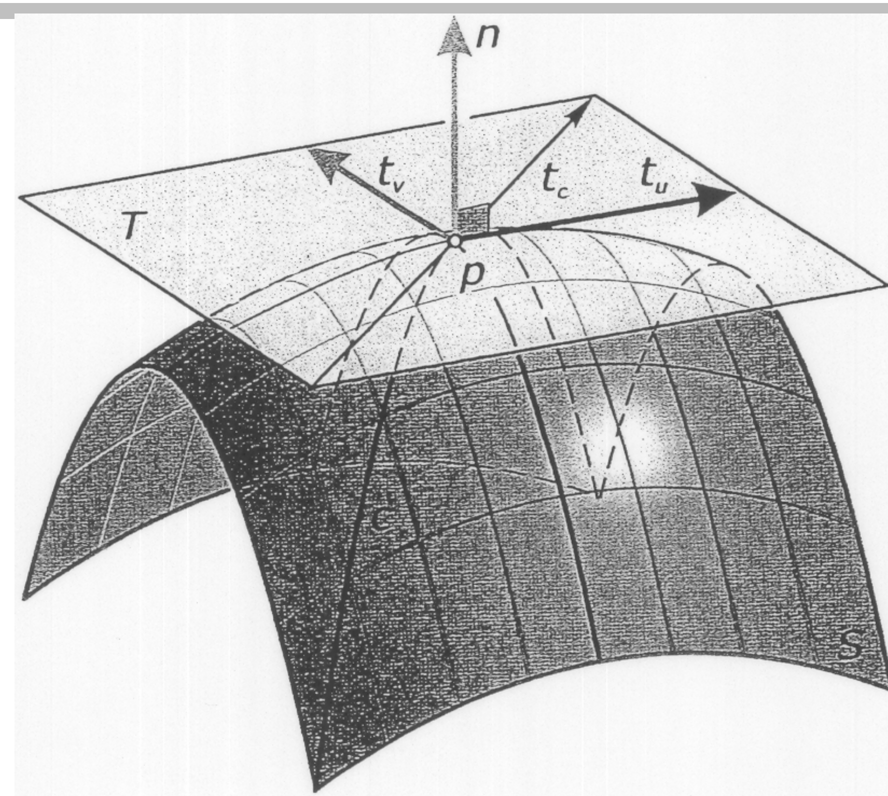
Frenet frame

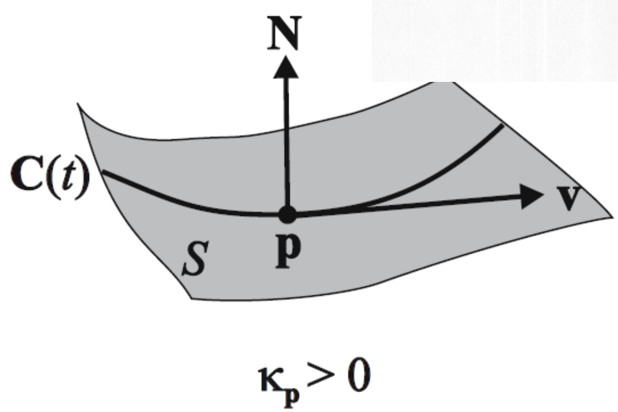


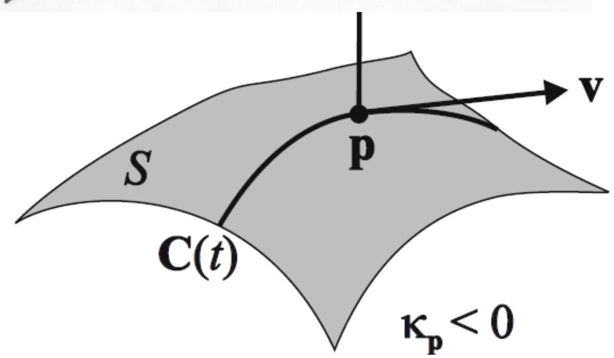


Curvature of a surface

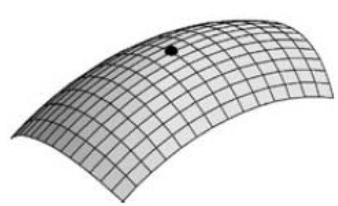
- Normal curvature
- Principal curvature
  - K_{max} and K_{min}
- Gaussian curvature
  - $K = K_{max} * K_{min}$
- Mean curvature
  - $H = (K_{max} + K_{min}) / 2$



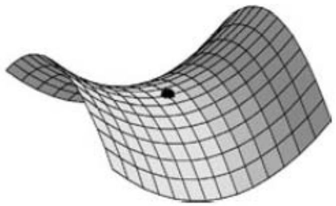




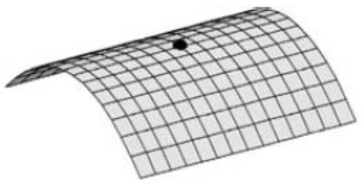
Classification of surface points



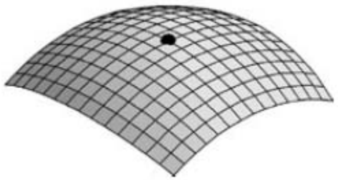
(a) Elliptic point $K > 0, H \neq 0$



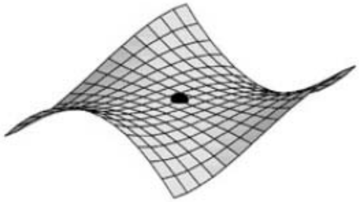
(b) Hyperbolic point $K < 0, H \neq 0$



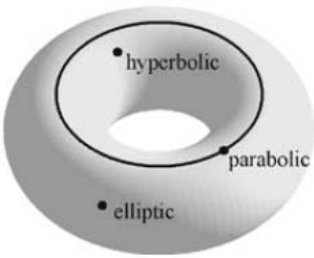
(c) Parabolic point $K = 0, H \neq 0$



(d) Umbilic point $\kappa_{\max} = \kappa_{\min} \neq 0$



(e) Planar point $\kappa_{\max} = \kappa_{\min} = 0$



(f) Elliptic, hyperbolic, and parabolic points on a torus

Classification of surface points

The principal, mean, and Gaussian curvatures distinguish the local geometry of a surface at a point $\mathbf{p}$ as follows.

Elliptic Point: $H \neq 0, K > 0$. At an elliptic point $\kappa_{\min}$ and $\kappa_{\max}$ have the same sign. Therefore the normal sections have the same profile, implying the surface near $\mathbf{p}$ has the shape of an ellipsoid.

Hyperbolic Point: $H \neq 0, K < 0$. At a hyperbolic point $\kappa_{\min}$ and $\kappa_{\max}$ have opposite signs. So the surface near $\mathbf{p}$ has the shape of a saddle.

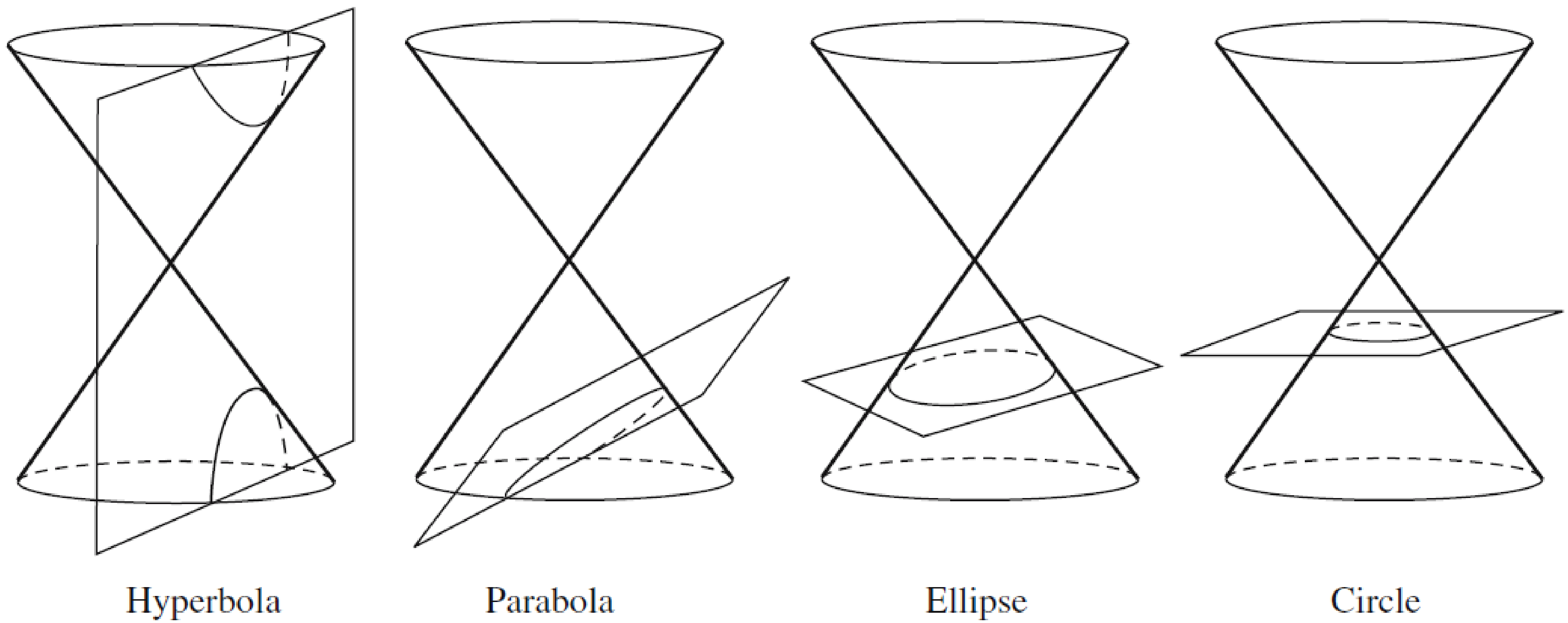
Parabolic Point: $H \neq 0, K = 0$. So either $\kappa_{\min} = 0$ or $\kappa_{\max} = 0$. Therefore the surface is linear in one principal direction, and near $\mathbf{p}$ the surface has the shape of a parabolic cylinder. In computer vision applications the surface is said to be a *ridge* or a *trough*.

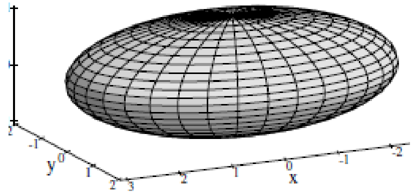
Umbilic Point: $\kappa_{\min} = \kappa_{\max} \neq 0$ $(H \neq 0, K > 0)$. An umbilic point is a special case of an elliptic point. The normal curvature is constant (non-zero) and near $\mathbf{p}$ the surface has the shape of a sphere.

Flat or Planar Point: $\kappa_{\min} = \kappa_{\max} = 0$ $(H = K = 0)$. The normal curvature is identically zero and the surface near $\mathbf{p}$ is flat.

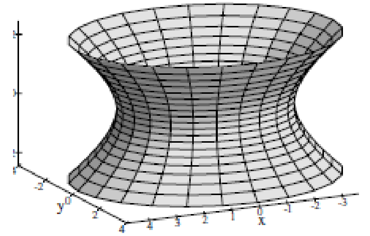
Conic sections

- The curves, or portions of the curves, obtained by cutting a cone with a plane are conic sections.

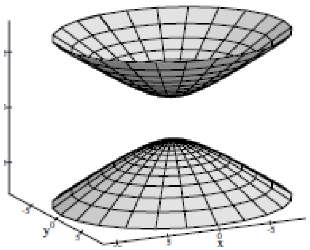




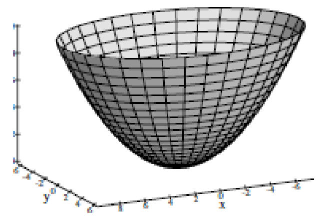
(a) Ellipsoid



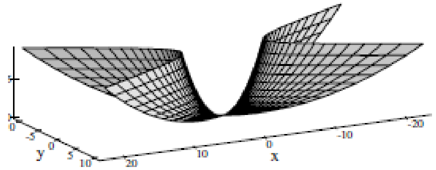
(b) Hyperboloid of one sheet



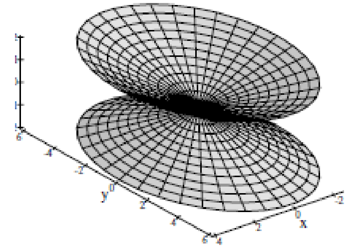
(c) Hyperboloid of two sheets



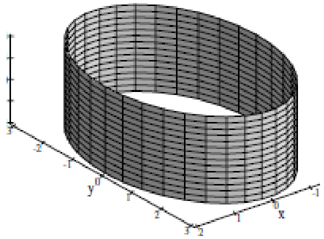
(d) Elliptic paraboloid



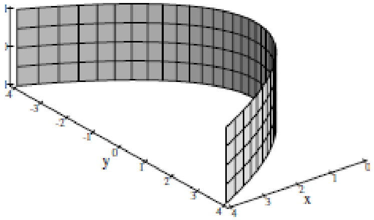
(e) Hyperbolic paraboloid



(f) Elliptic cone



(g) Elliptic cylinder



(h) Parabolic cylinder

Intersection curves of surfaces

- **Solving the nonlinear equation by numerical method.**

$$\mathbf{P}(u,v)-\mathbf{Q}(s,t)=0$$

  - **3 equations with 4 unknowns, u, v, s and t**
  - **Assigning an arbitrary value to one of the unknowns**
  - **Convergence depends on the initial values of unknowns**
  - **May not provide all the intersection curves**

Intersection curves of surfaces

- Subdivision based method
  - Recursively subdivide each surface *S1* and *S2* until all of surface segments are close to the planar quadrilaterals
  - Quadrilaterals of *S1* are tested for intersection *I* with those of *S2*
  - Use the intersection curves from the intersection *I* as initial guesses
  - Require more computations than previous method but have less chance of missing some intersection curves