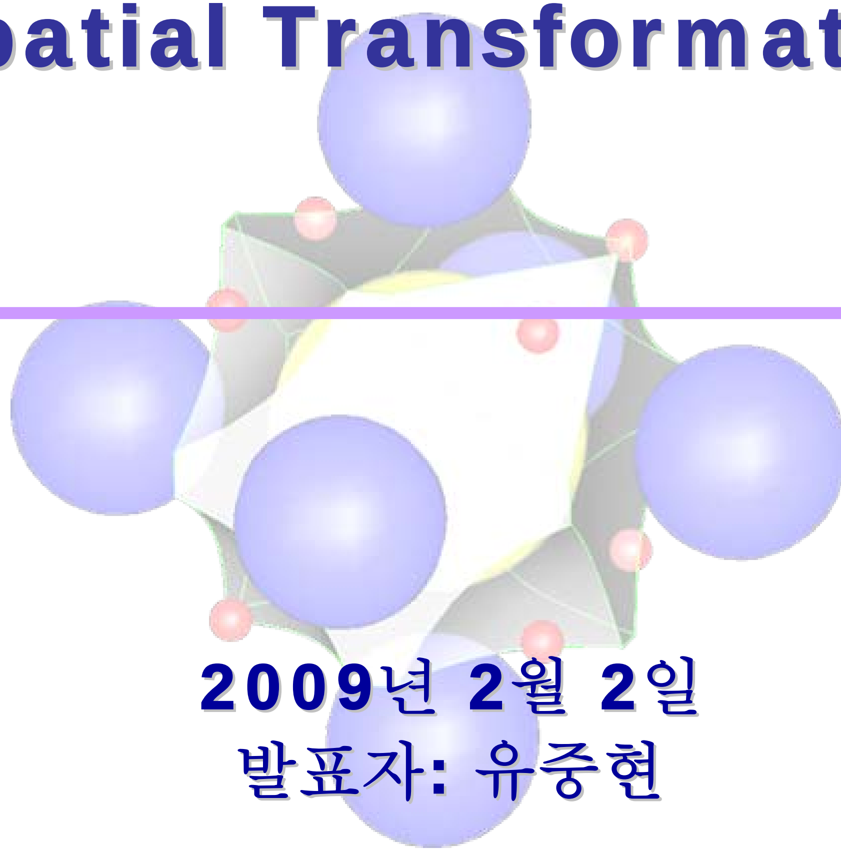
Architectural geometry: Spatial Transformation



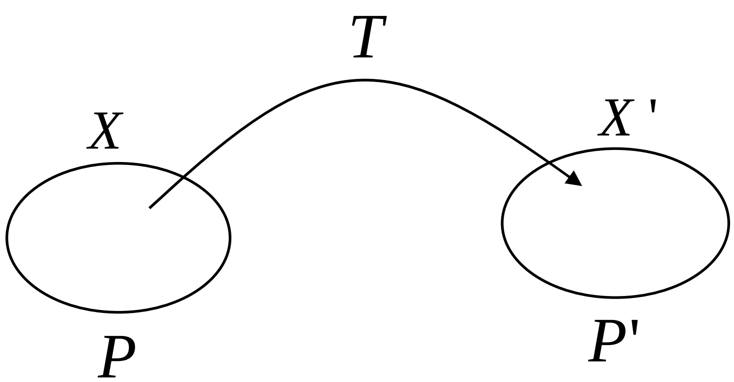
2009년 2월 2일

발표자: 유중현

Planar transformation matrices

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$$

rotation
scaling
translation
perspective transform



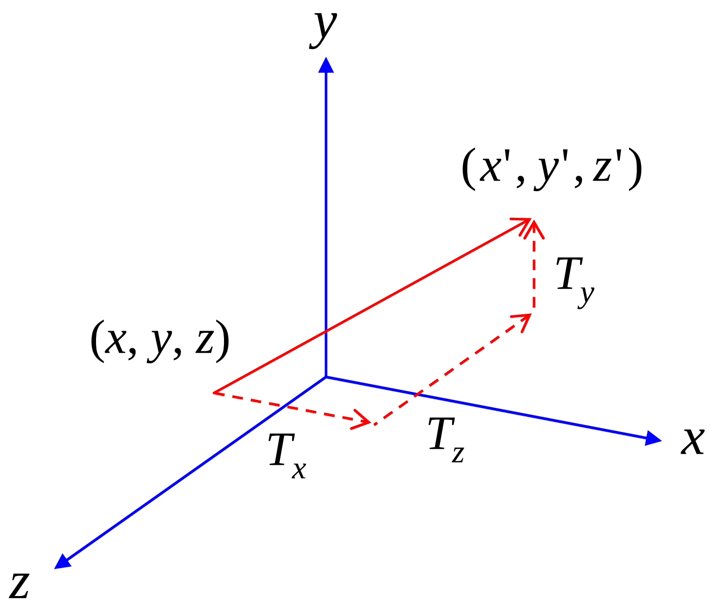
$$P' = TP$$

where

$$P' = [x', \; y', \; w']^T,$$

$$P = [x, \; y, \; w]^T.$$

Translation



$$x' = x + T_x$$

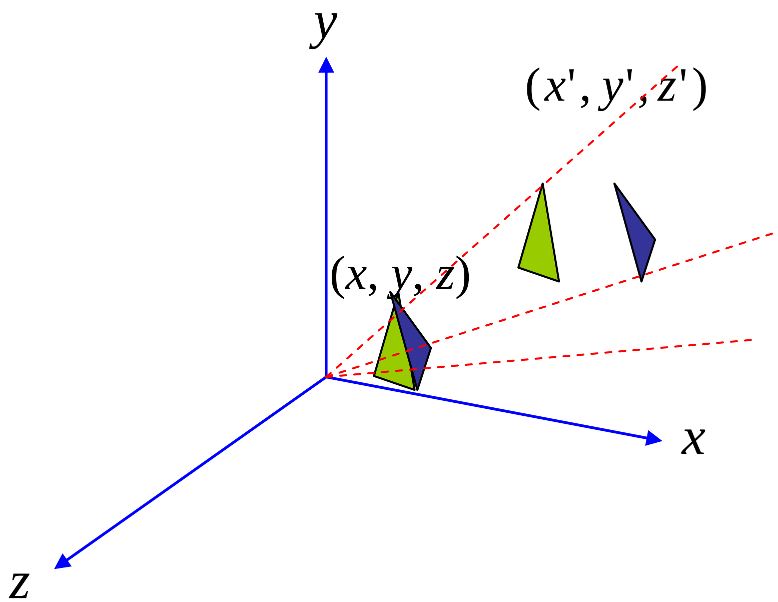
$$y' = y + T_y$$

$$z' = z + T_z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & T_x \\ 0 & 1 & 0 & T_y \\ 0 & 0 & 1 & T_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = TP$$

Scaling



$\therefore$

$$x' = x \cdot S_x$$
$$y' = y \cdot S_y$$
$$z' = z \cdot S_z$$

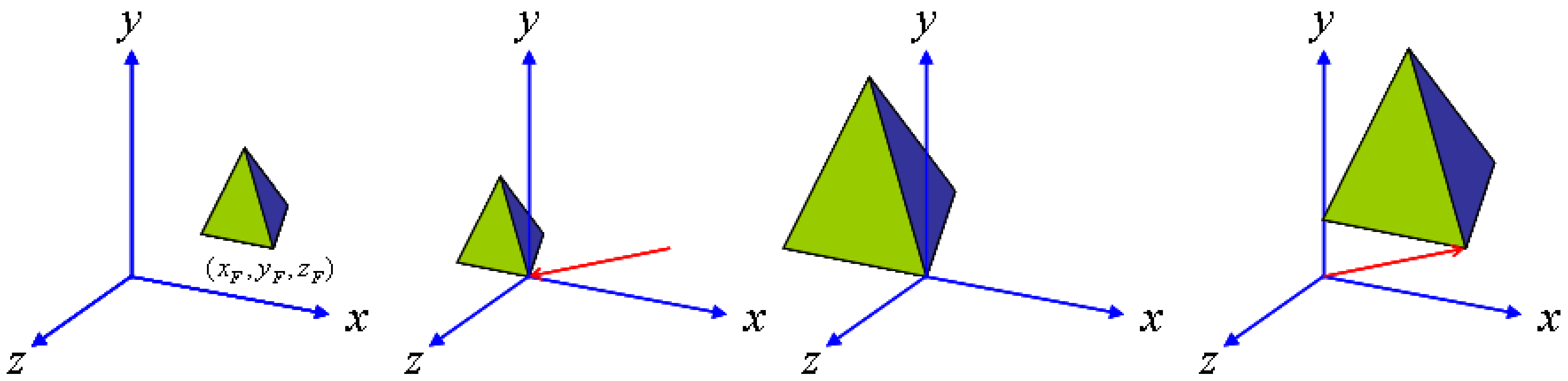
$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$\boxed{P' = SP}$$

$S_x = S_y = S_z \quad \Rightarrow \quad$ Uniform scaling

Otherwise, non-uniform scaling

Relative Scaling



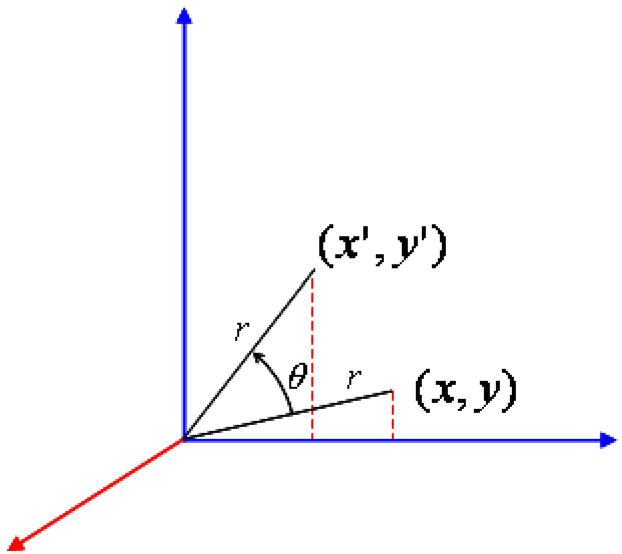
$$P' = [T^{-1}][S][T]P$$

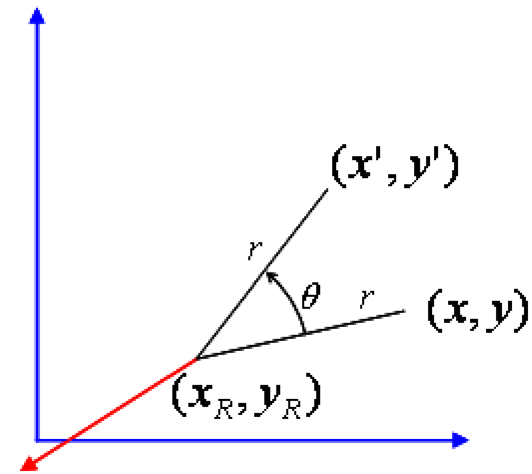
where

$$T = \begin{bmatrix} 1 & 0 & 0 & -x_F \\ 0 & 1 & 0 & -y_F \\ 0 & 0 & 1 & -z_F \\ 0 & 0 & 0 & 1 \end{bmatrix}, S = \begin{bmatrix} S_x & 0 & 0 & 0 \\ 0 & S_y & 0 & 0 \\ 0 & 0 & S_z & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \text{ and } T^{-1} = \begin{bmatrix} 1 & 0 & 0 & x_F \\ 0 & 1 & 0 & y_F \\ 0 & 0 & 1 & z_F \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Rotation

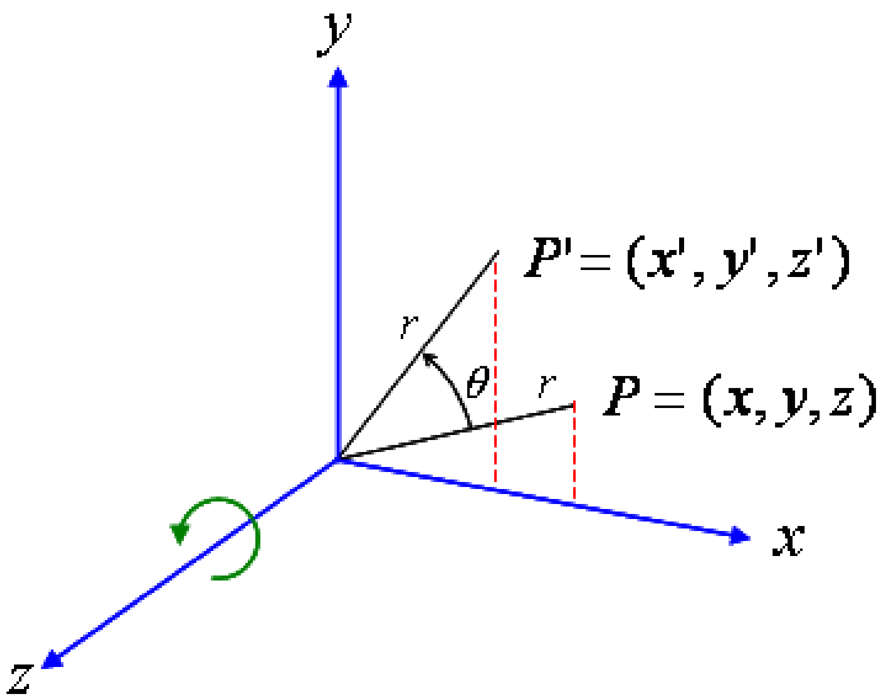
Planar rotation





Rotation about *z*-axis is implicitly assumed !!!

Rotation about z-axis

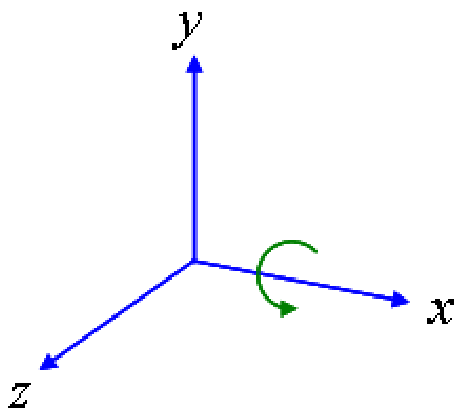


$$\therefore \quad x' = x\cos\theta - y\sin\theta$$
$$y' = x\sin\theta + y\cos\theta$$
$$z' = z$$

$$\begin{bmatrix} x' \\ y' \\ z' \\ 1 \end{bmatrix} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix}$$

$$P' = R_z P$$

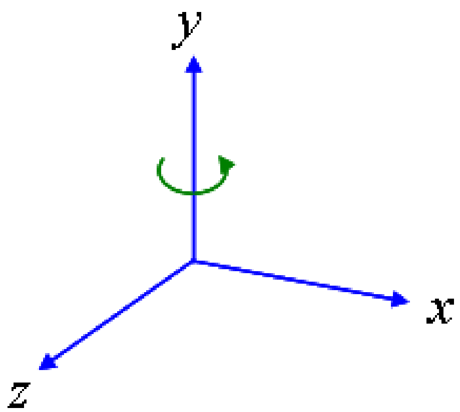
Rotation about x- and y-axis



$P' = R_X P$, where

$$R_X = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta & 0 \\ 0 & \sin\theta & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x' = x$$
$$y' = y\cos\theta - z\sin\theta$$
$$z' = y\sin\theta + z\cos\theta$$



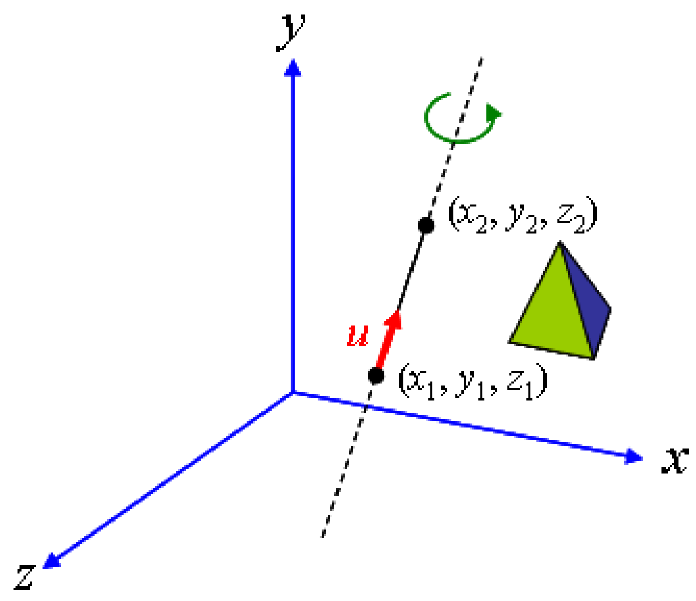
$P' = R_Y P$, where

$$R_Y = \begin{bmatrix} \cos\theta & 0 & \sin\theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\theta & 0 & \cos\theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$x' = x\cos\theta + z\sin\theta$$
$$y' = y$$
$$z' = -x\sin\theta + z\cos\theta$$

Rotation about an arbitrary axis

- Translate the object so that the rotation axis passes through the origin. (T)
- Rotate the object so that the rotation axis coincides with one of the coordinate axis. (R)
- Perform the specified rotation.
- R^{-1}
- T^{-1}



Reflection

- **Reflection about plane**
  - xy **plane :** $(x, y, z) \rightarrow (x, y, -z)$
  - yz **plane :** $(x, y, z) \rightarrow (-x, y, z)$
  - zx **plane :** $(x, y, z) \rightarrow (x, -y, z)$

- **Reflection about line**
  - x **axis :** $(x, y, z) \rightarrow (x, -y, -z)$
  - y **axis :** $(x, y, z) \rightarrow (-x, y, -z)$
  - z **axis :** $(x, y, z) \rightarrow (-x, -y, z)$

Shearing

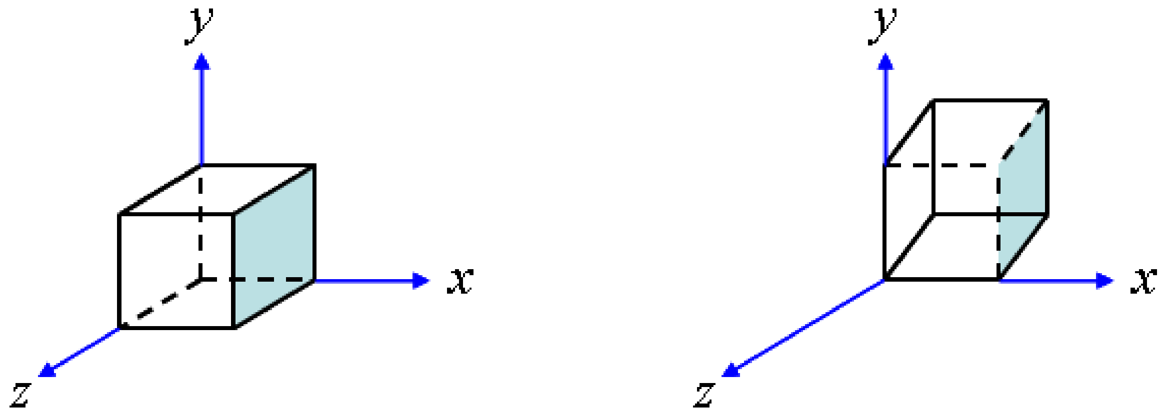
z-axis shearing

$$P' = \begin{bmatrix} 1 & 0 & a & 0 \\ 0 & 1 & b & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} P$$

$$x' = x + az$$
$$y' = y + bz$$

x-axis shearing?

y-axis shearing?



Composition of transformations

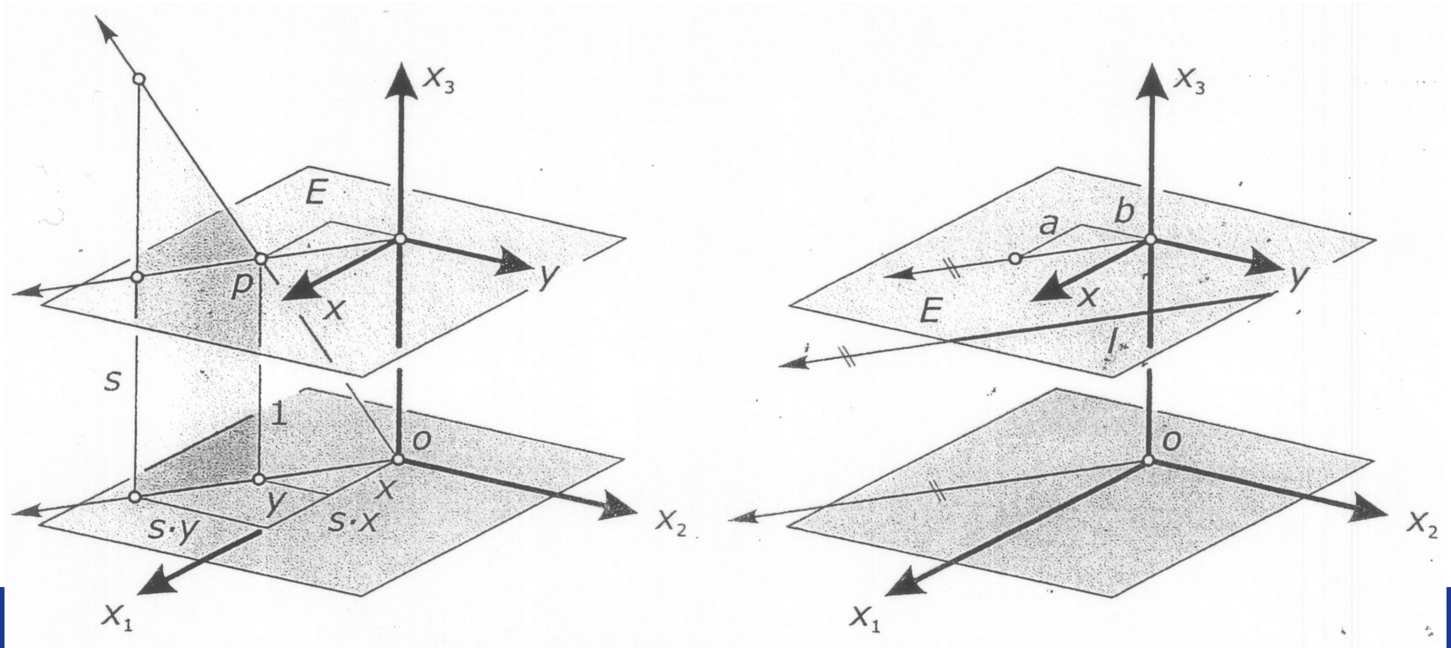
- The composition of two direct or two opposite congruence transformations generate a direct congruence transformation.

- The composition of a direct and an opposite congruence transformation results in an opposite congruence transformation.

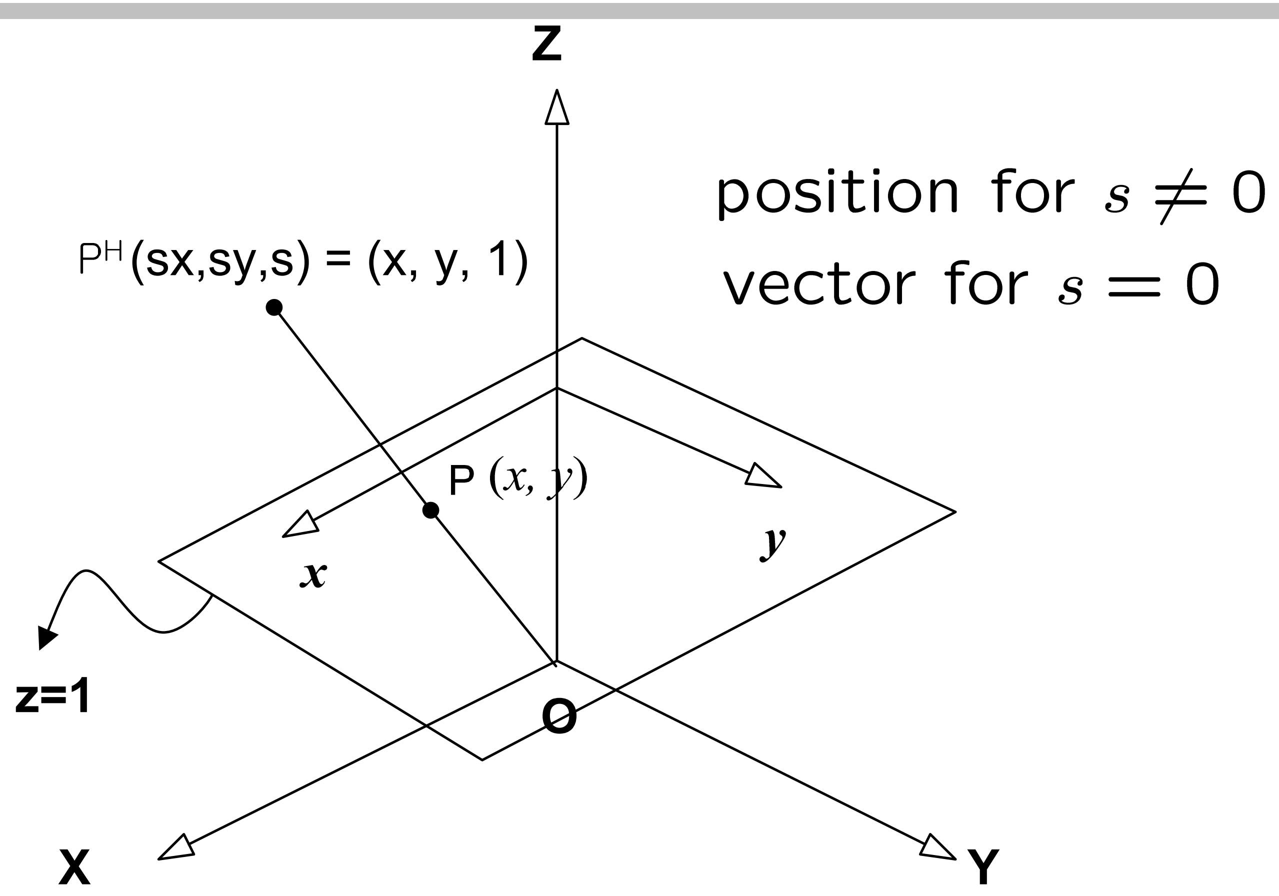
Projective extension of the plane

- **Any two lines in the plane have one point in common**
  - **A proper point if they intersect each other**
  - **A point at infinity if they are parallel (parallelism as a special form of intersection)**

- **Projective space = points in Euclidean space (position) + points at infinity (vector)**



Homogeneous coordinate



- Point, proper point (positions) and points at infinity (vectors) are homogeneously handled!
- All points at infinity form the line at infinity

Planar projective transformation

- Planar projective transformation between P and P’
  - Transformation between plane P’(x1’:x2’:x3’) between plane P(x1:x2:x3) via perspective projection

$$x_1' = a_{11} \cdot x_1 + a_{12} \cdot x_2 + a_{13} \cdot x_3,$$

$$x_2' = a_{21} \cdot x_1 + a_{22} \cdot x_2 + a_{23} \cdot x_3,$$

$$x_3' = a_{31} \cdot x_1 + a_{32} \cdot x_2 + a_{33} \cdot x_3.$$

  - A planar figure (a planar object) and its perspective view (a photo) are related by projective transformation

Projective transformation in 3D

homogenous coordinates $(x_1{:}x_2{:}x_3{:}x_4)$

$x = x_1/x_4, y = x_2/x_4, z = x_3/x_4$

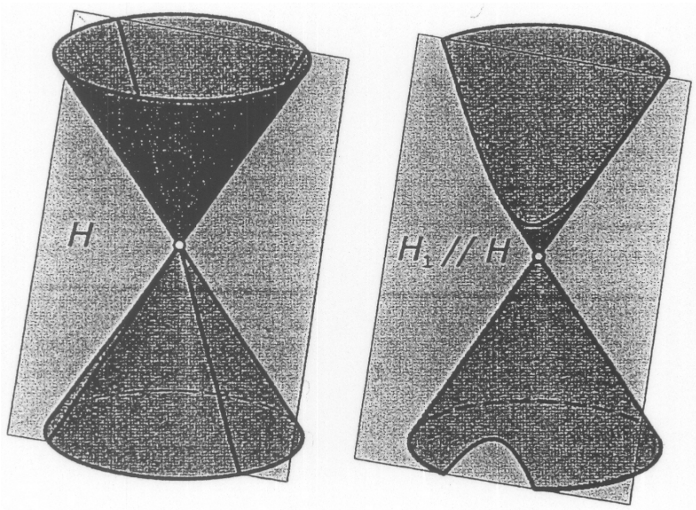
All points at infinity form the *plane at infinity*.

Projective images of circles

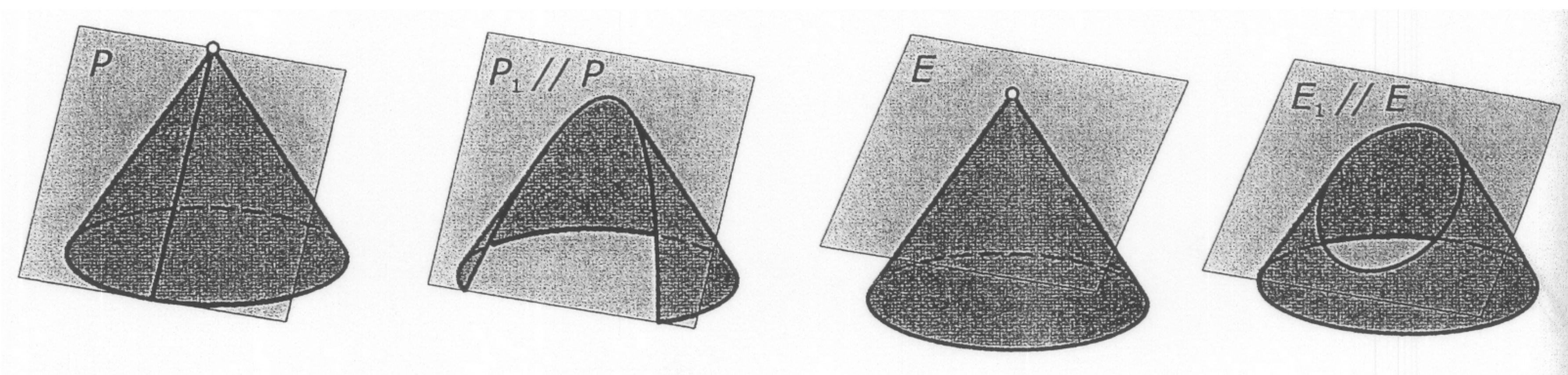
- Projective / Perspective projection of a circle leads to a conic

- Type of conic is determined by the number of points at infinity
  - Hyperbola: two points at infinity
  - Parabola: exactly one point at infinity
  - Ellipse: only proper points

Conic section



- H: intersect along two rulings
- P: touch along one ruling
- E: intersect only at the vertex of cone



Summary of linear transformations

	Euclidean	similarity	affine	projective
transformations				
translation	✓	✓	✓	✓
rotation	✓	✓	✓	✓
uniform scaling		✓	✓	✓
scaling			✓	✓
shear			✓	✓
perspective projection				✓
invariants				
length	✓			
angle	✓	✓		
ratio of lengths	✓	✓	✓	
parallelism	✓	✓	✓	