Two involutions on vertices of ordered trees

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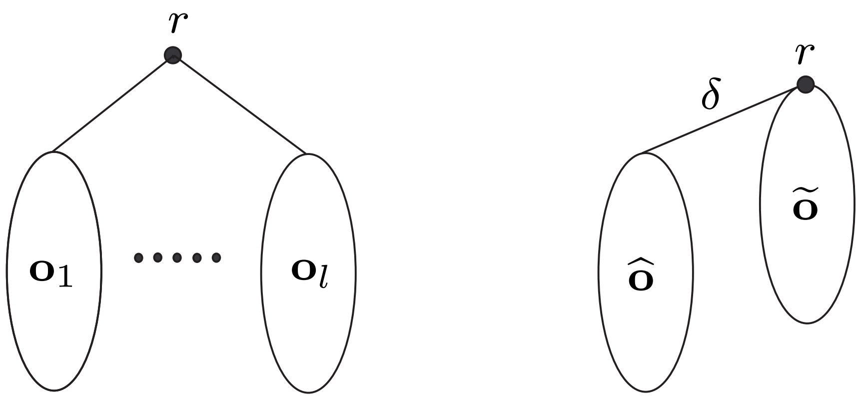
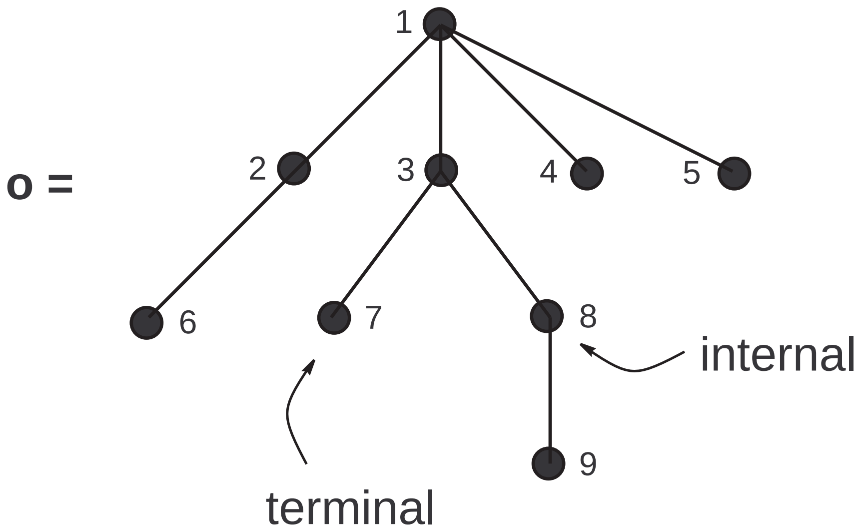
Ordered Trees

Formal Definition (Recursive form) : Either the tree consisting of its root r alone, or an *ordered* tuple $[r; \mathbf{o}_1, \ldots, \mathbf{o}_l]$, where $l \geq 1$ and $\mathbf{o}_1, \ldots, \mathbf{o}_l$ are smaller ordered trees.

Geometric Definition (Intuitive form) : Either the tree consisting of its root r alone, or a plane tree which has a root and a distinguished edge δ which is incident with the root.





$$I(\mathbf{o}) = \{1, 2, 3, 8\}, \quad T(\mathbf{o}) = \{4, 5, 6, 7, 9\}.$$

Let $\mathcal{O}_n$ be the set of all ordered trees with n edges.

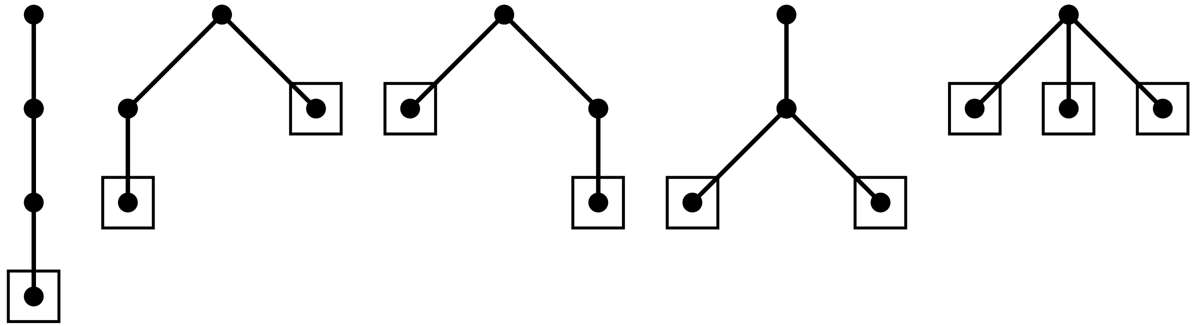


Figure 1: Ordered trees with 3 edges

Observe that

$$\sum_{\mathbf{o}\in\mathcal{O}_3} |I(\mathbf{o})| = 10 = \sum_{\mathbf{o}\in\mathcal{O}_3} |T(\mathbf{o})|\,.$$

Main Theorem

Theorem 1 *For all* $n \geq 1$*,*

$$\sum_{\mathbf{o}\in\mathcal{O}_n} |I(\mathbf{o})| = \sum_{\mathbf{o}\in\mathcal{O}_n} |T(\mathbf{o})| = \frac{1}{2}\binom{2n}{n}.$$

Theorem 1 can be proved by various *tools* as follows :

- Using the generating function technique.
- Changing the object set.
  - – Binary trees. (Dasarathy and Yang, 1980)
  - – Dyck paths. (Deutsch, 1999; Seo, 2001)
- Giving an *involution* on the vertex set of $\mathcal{O}_n$. ($*$)

Proof of Therem 1 using Binary trees (Dasarathy and Yang, 1980).

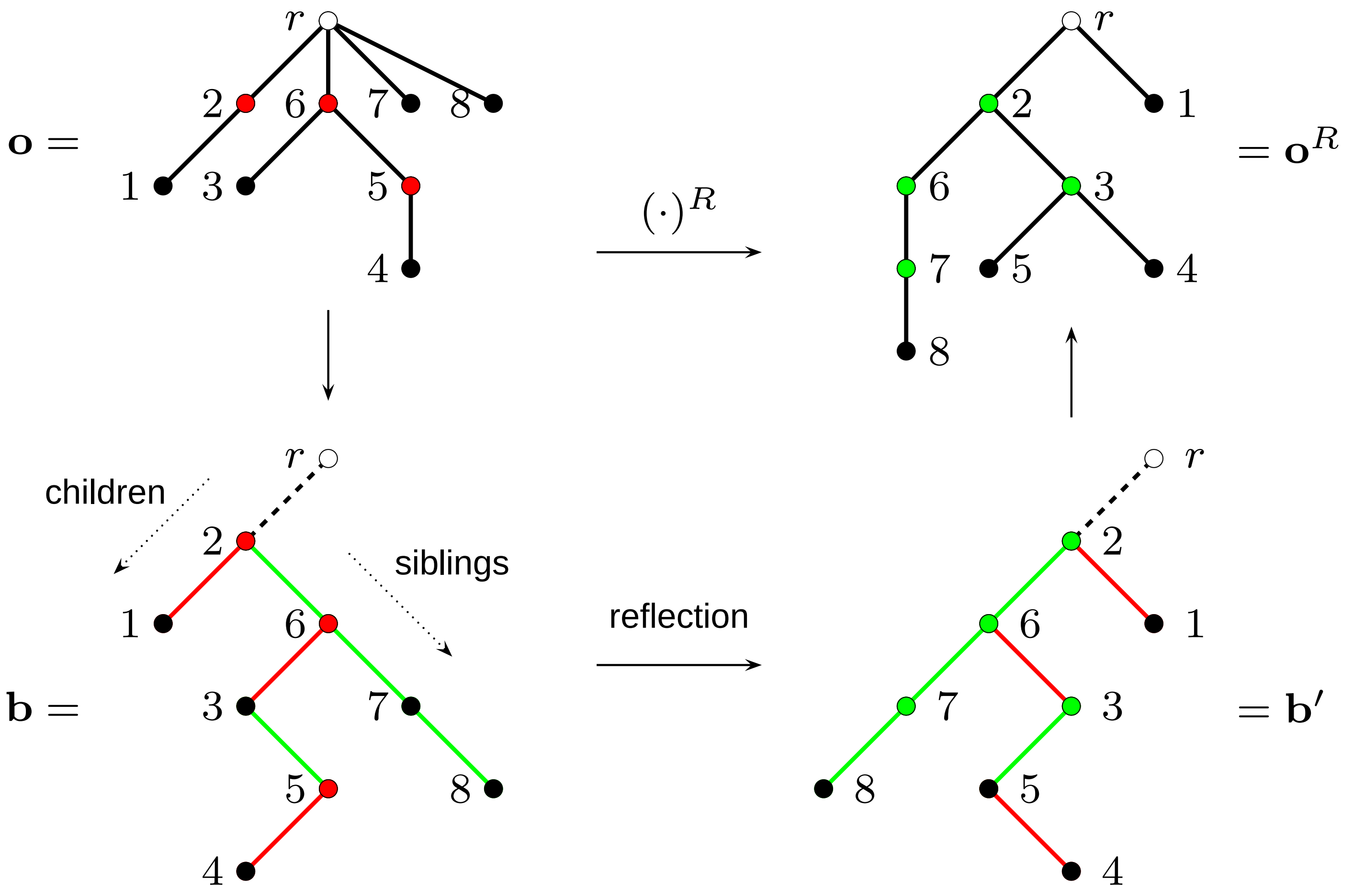


Figure 2: $|I(\mathbf{o})| + |I(\mathbf{o}^R)| = n$, so $|I(\mathbf{o})| = |T(\mathbf{o}^R)|$.

Proof of Therem 1 using Dyck paths (Seo, 2001).

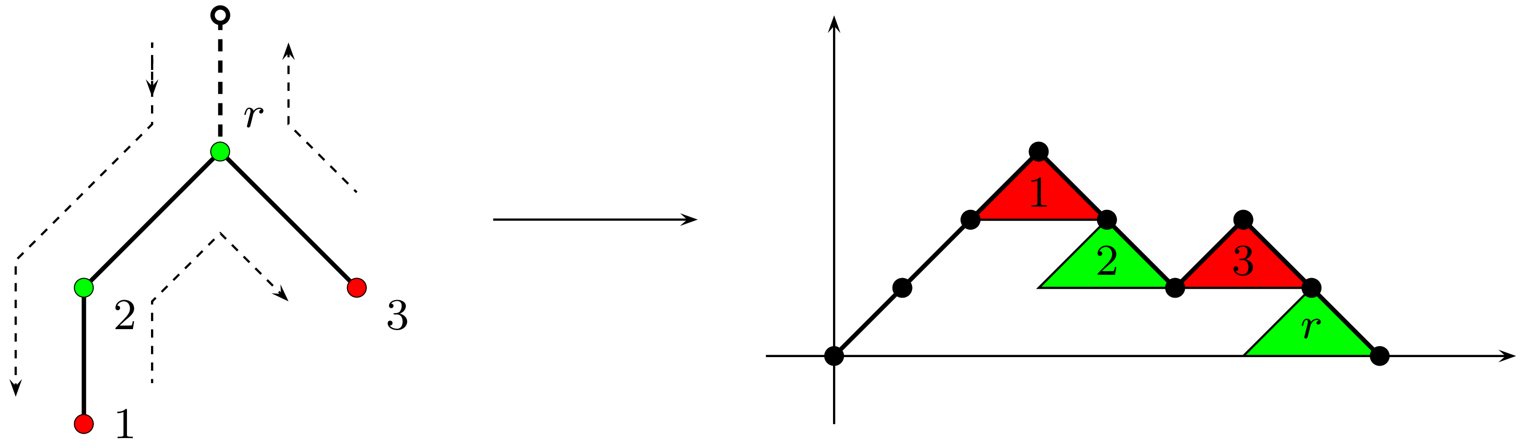


Figure 3: The bijection between ordered trees and Dyck paths

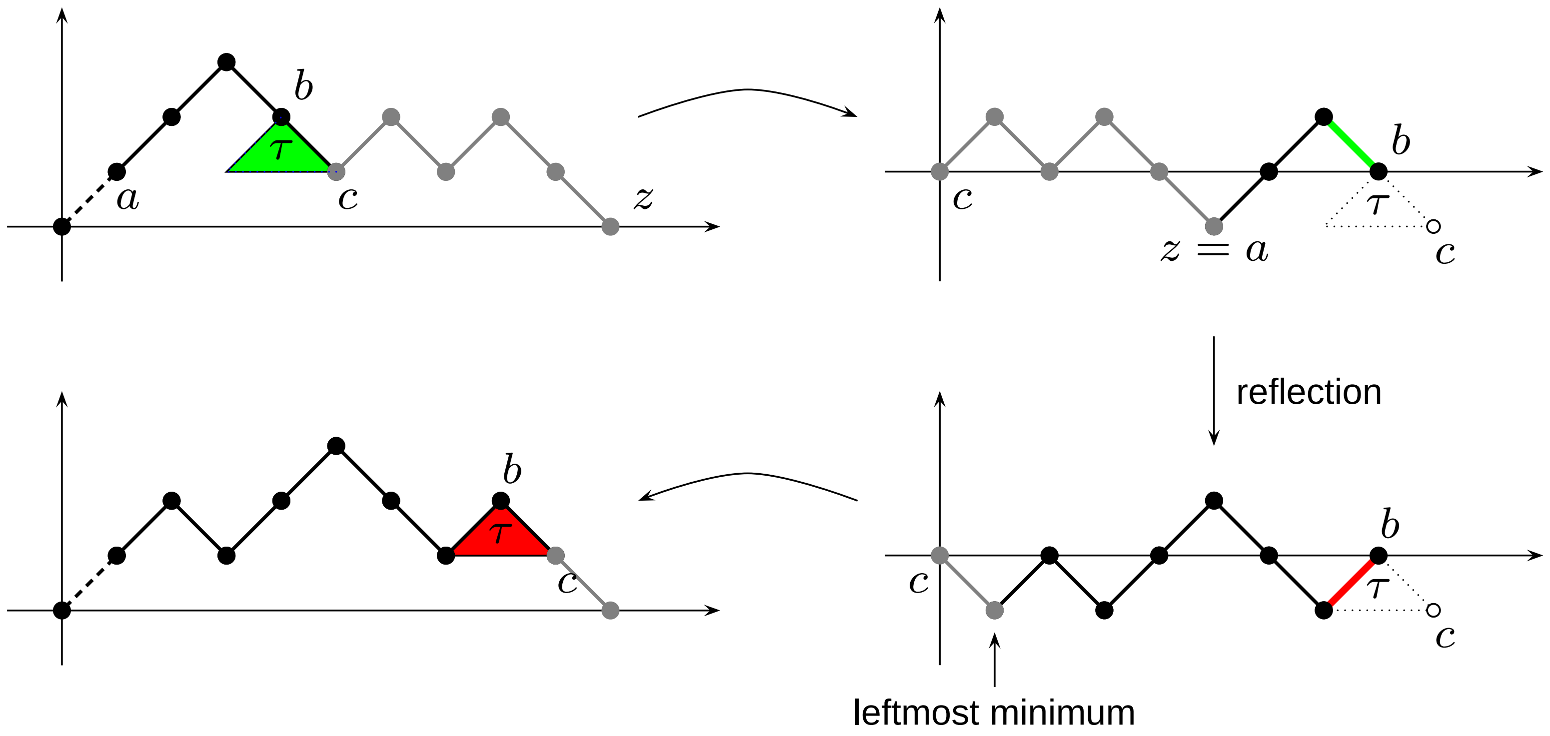


Figure 4: The relation between triangles and final steps

Pointed Ordered Trees

For an ordered tree $\mathbf{o}$ and a vertex v in $\mathbf{o}$, let $(\mathbf{o}, v)$ (*abbre.* $\mathbf{o}^v$) denote the *pointed ordered tree* with v pointed.

- $\mathcal{O}_n^{\bullet} = \{\mathbf{o}^v \ : \ \mathbf{o} \in \mathcal{O}_n \, , \ v \in V(\mathbf{o})\}$.
- $\mathcal{O}_n^{+} = \{\mathbf{o}^v \ : \ \mathbf{o} \in \mathcal{O}_n \, , \ v \in I(\mathbf{o})\}$.
- $\mathcal{O}_n^{-} = \{\mathbf{o}^v \ : \ \mathbf{o} \in \mathcal{O}_n \, , \ v \in T(\mathbf{o})\}$.

To verify the Theorem 1, it is enough to show that there exists a *bijection* between $\mathcal{O}_n^{+}$ and $\mathcal{O}_n^{-}$, where $n > 0$.

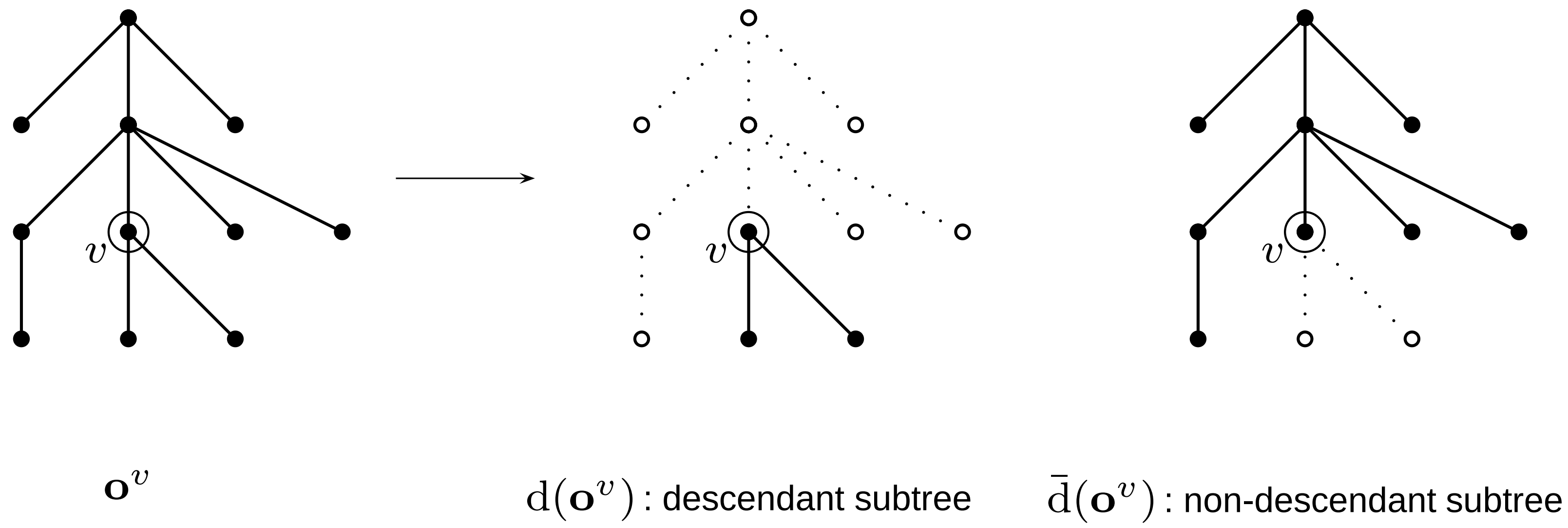


Figure 5: General decomposition of $\mathbf{o}^v$

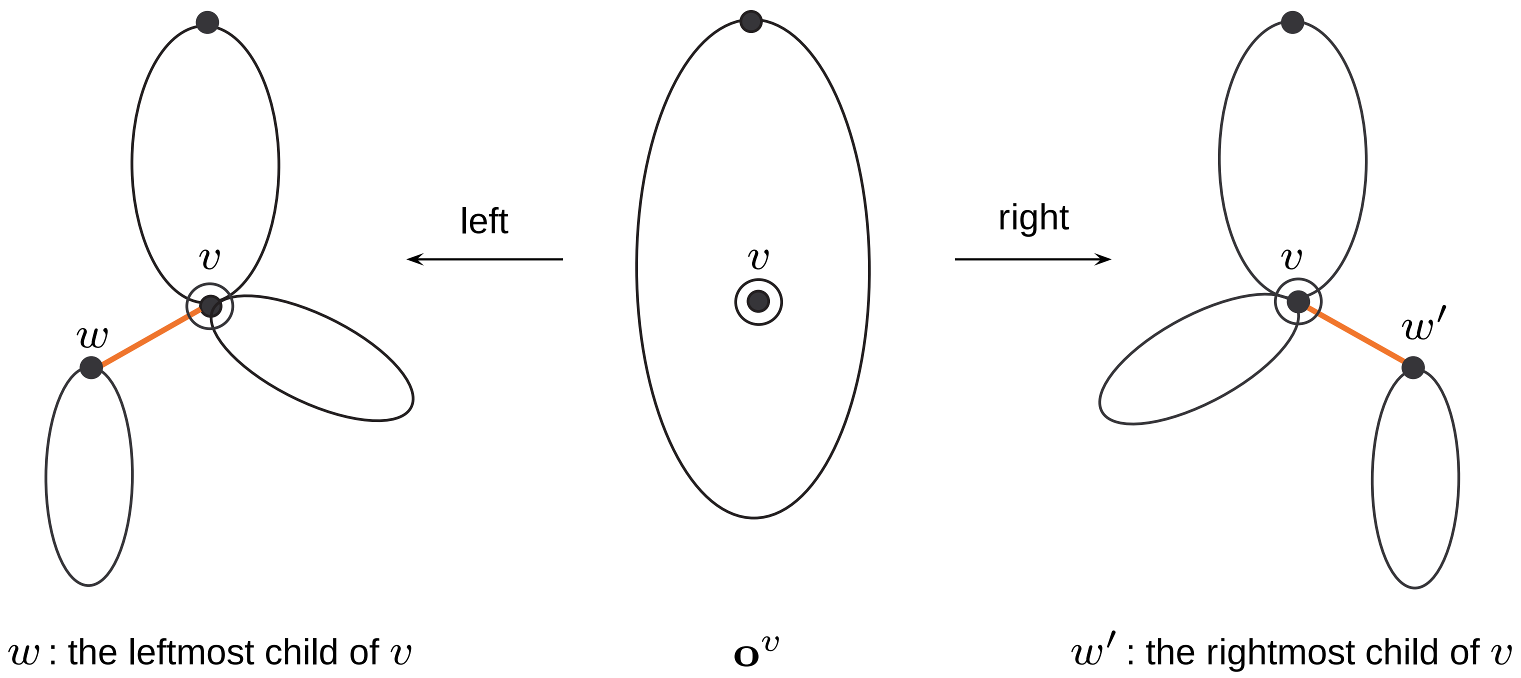


Figure 6: Left and right decompositions of internal $\mathbf{o}^v$

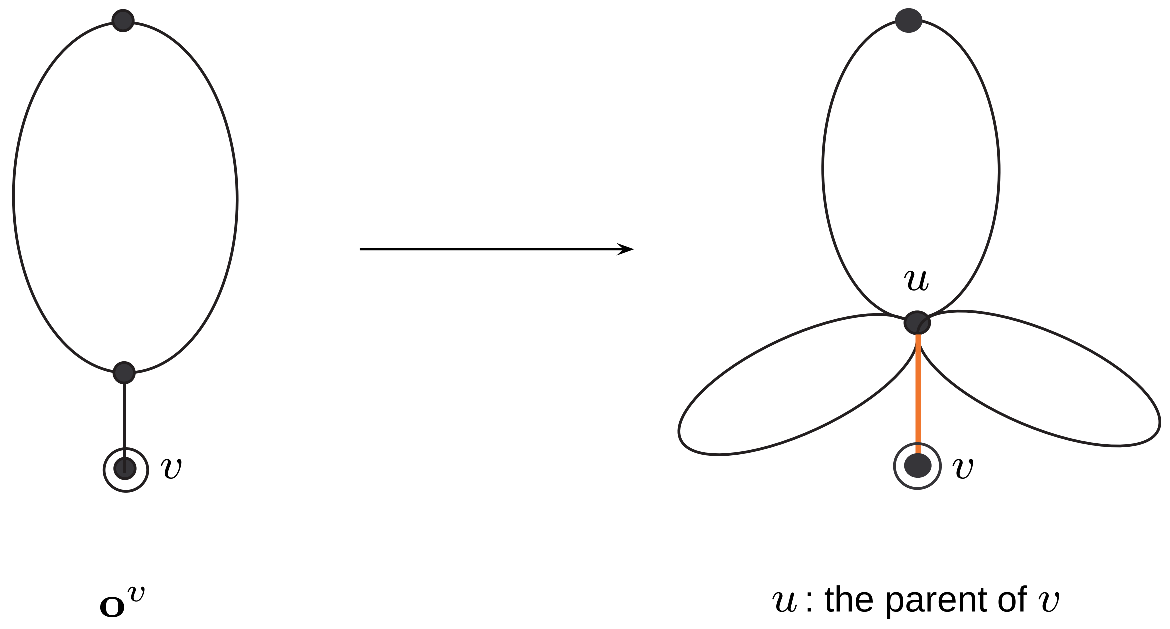


Figure 7: Terminal decomposition of terminal $\mathbf{o}^v$

Involutions on Pointed Ordered Trees

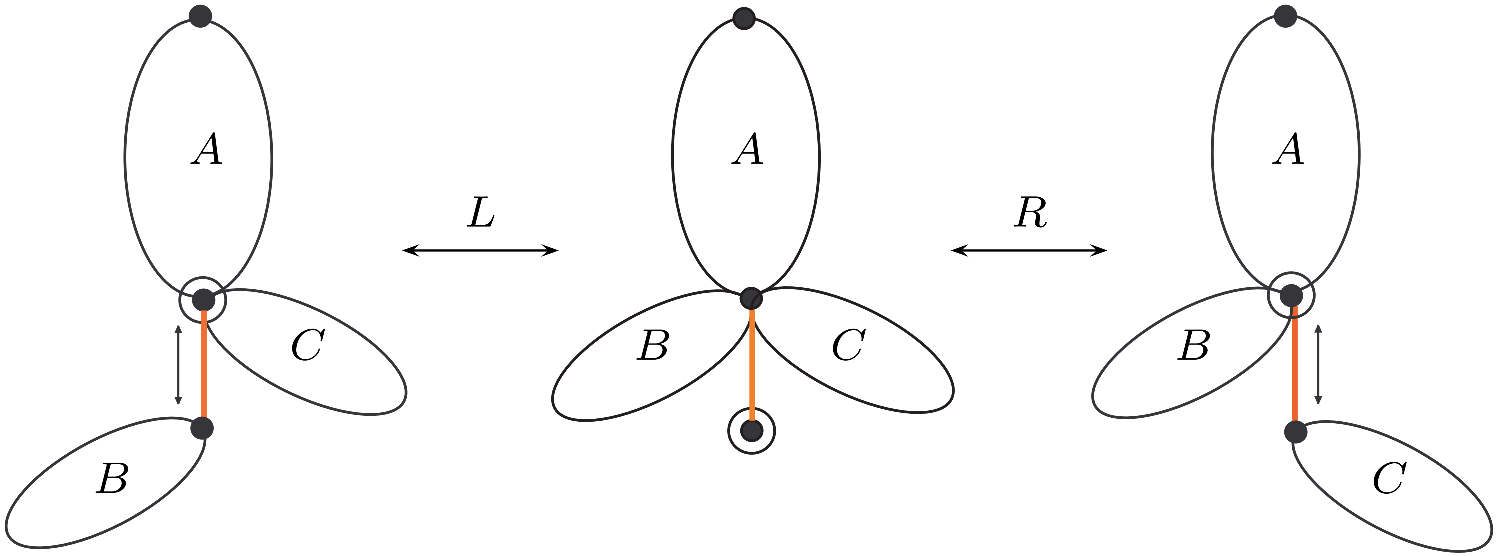


Figure 8: Two maps L and R

Theorem 2 *For all* $n > 0$*, the maps* L *and* R *are* **involutions** *in* $\mathcal{O}_n^\bullet$ *with* $L(\mathcal{O}_n^-) = \mathcal{O}_n^+$ *and* $R(\mathcal{O}_n^-) = \mathcal{O}_n^+$*. So* L *and* R *are* **bijections** *between* $\mathcal{O}_n^-$ *and* $\mathcal{O}_n^+$*.*

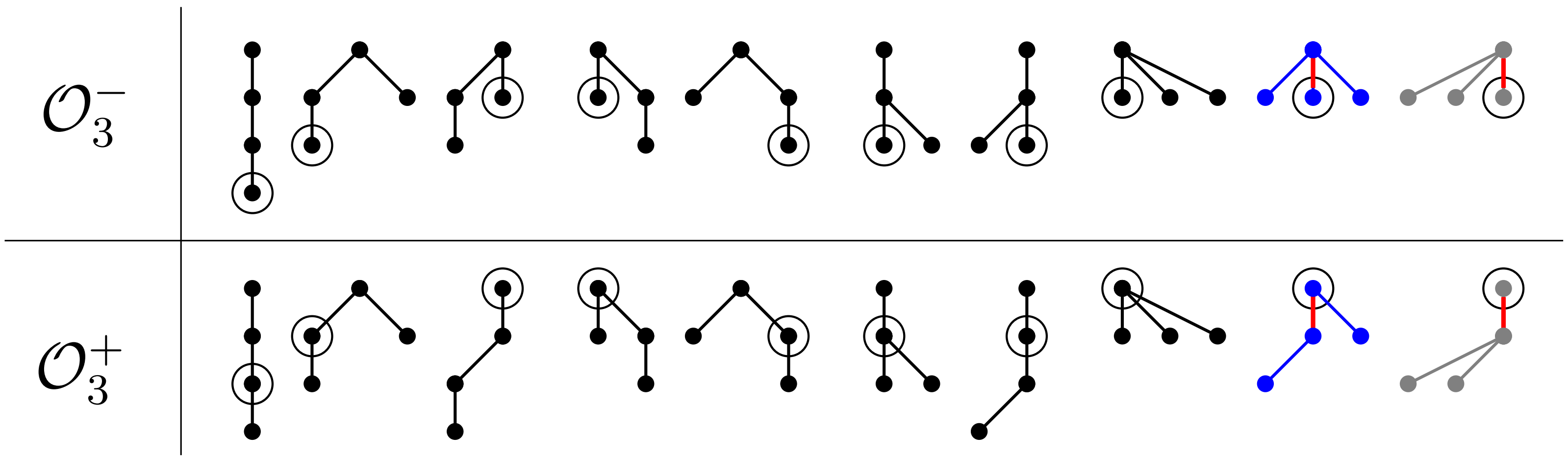


Figure 9: Correspondence in $\mathcal{O}_3^{\bullet}$ by the map L

Let $\rho\big(\mathbf{o}^v\big)$ be the length of the path from v to the root of $\mathbf{o}$. We call it the *level* of $\mathbf{o}^v$.

Corollary 3 *For a* $\mathbf{o}^v \in \mathcal{O}_n^{\bullet}$*,*

$$\rho\big(L(\mathbf{o}^v)\,\big) = \rho\big(R(\mathbf{o}^v)\,\big) = \begin{cases} \rho\big(\mathbf{o}^v\big) + 1, & \textit{if } \mathbf{o}^v \textit{ is internal,} \\ \rho\big(\mathbf{o}^v\big) - 1, & \textit{if } \mathbf{o}^v \textit{ is terminal.} \end{cases}$$

Consequently,

$$\big[\,\textit{average level of } \mathcal{O}_n^- \,\big] = \big[\,\textit{average level of } \mathcal{O}_n^+ \,\big] + 1\,,$$

which has been proved by Dershowitz and Zaks (1981).

A Group Action on $\mathcal{O}_n^\bullet$

Let $\mathbf{G}$ be the group generated by L and R with composition as the operation. Since L and R are involutions, $\mathbf{G}$ has the following presentation.

$$\mathbf{G} = \langle L, R : L^2 = 1,\ R^2 = 1 \rangle.$$

The group $\mathbf{G}$ acts on $\mathcal{O}_n^\bullet$ by

$$G \cdot \mathbf{o}^v = G(\mathbf{o}^v) \quad \text{for all } \mathbf{o}^v \in \mathcal{O}_n^\bullet \text{ and all } G \in \mathbf{G}.$$

By this $\mathbf{G}$-action, $\mathcal{O}_n^\bullet$ is partitioned into $\mathbf{G}$-orbits.

Given $\mathbf{o}^v$, let $[\mathbf{o}^v]$ denote the $\mathbf{G}$-orbit of $\mathbf{o}^v$ and $\mathcal{O}_n^\bullet/\mathbf{G}$ the set of all distinct orbits in $\mathcal{O}_n^\bullet$, i.e.,

$$\begin{aligned} [\mathbf{o}^v] &= \{G\mathbf{o}^v : G \in \mathbf{G}\}, \\ \mathcal{O}_n^\bullet/\mathbf{G} &= \{[\mathbf{o}^v] : \mathbf{o}^v \in \mathcal{O}_n^\bullet\}. \end{aligned}$$

Then we can raise two natural questions:

Question 1. How many distinct orbits are there, i.e., $|\mathcal{O}_n^\bullet/\mathbf{G}| = ?$

Question 2. Given $\mathbf{o}^v \in \mathcal{O}_n^\bullet$, what is the cardinality of $[\mathbf{o}^v]$?

Let $\mathbf{H} = \langle RL \rangle$ and $[\mathbf{o}^v]^+ = [\mathbf{o}^v] \cap \mathcal{O}_n^+$.

Clearly, $\mathbf{H}$ acts on $\mathcal{O}_n^+$ and

$$[\mathbf{o}^v]^+ = \{H\mathbf{o}^v \;:\; H \in \mathbf{H}\}, \quad \text{where } \mathbf{o}^v \in \mathcal{O}_n^+.$$

Observation 1. $|\mathcal{O}_n^\bullet/\mathbf{G}| = |\mathcal{O}_n^+/\mathbf{H}|$.

Observation 2. $\big|[\mathbf{o}^v]\big| = 2\,\big|[\mathbf{o}^v]^+\big|$, where $\mathbf{o}^v \in \mathcal{O}_n^+$.

With these two observations, instead of the $\mathbf{G}$-action on $\mathcal{O}_n^\bullet$, we will discuss the $\mathbf{H}$-*action on* $\mathcal{O}_n^+$ to answer the previous two questions.

Enumeration of Distinct $\mathbb{H}$-orbits

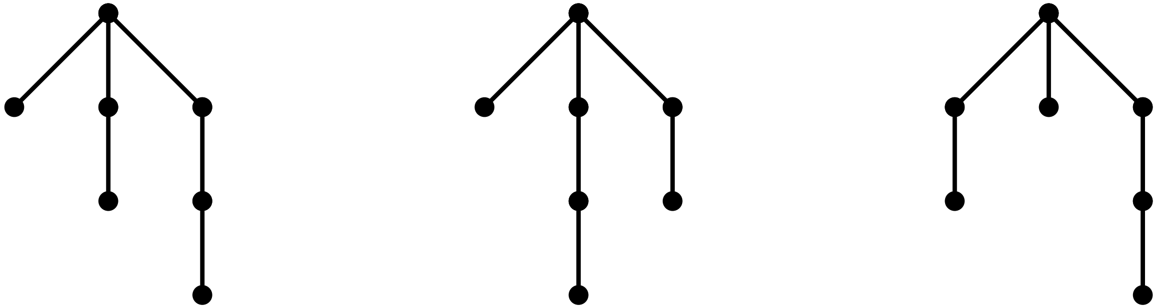


Figure 10: Trees and plane trees

- As *trees*, $\mathbf{t}_1 = \mathbf{t}_2 = \mathbf{t}_3$.
- As *plane trees*, $\mathbf{t}_2 = \mathbf{t}_3$, but $\mathbf{t}_1 \neq \mathbf{t}_2$.

$\mathcal{P}_n$: the set of plane trees with n edges p_n : the cardinality of $\mathcal{P}_n$

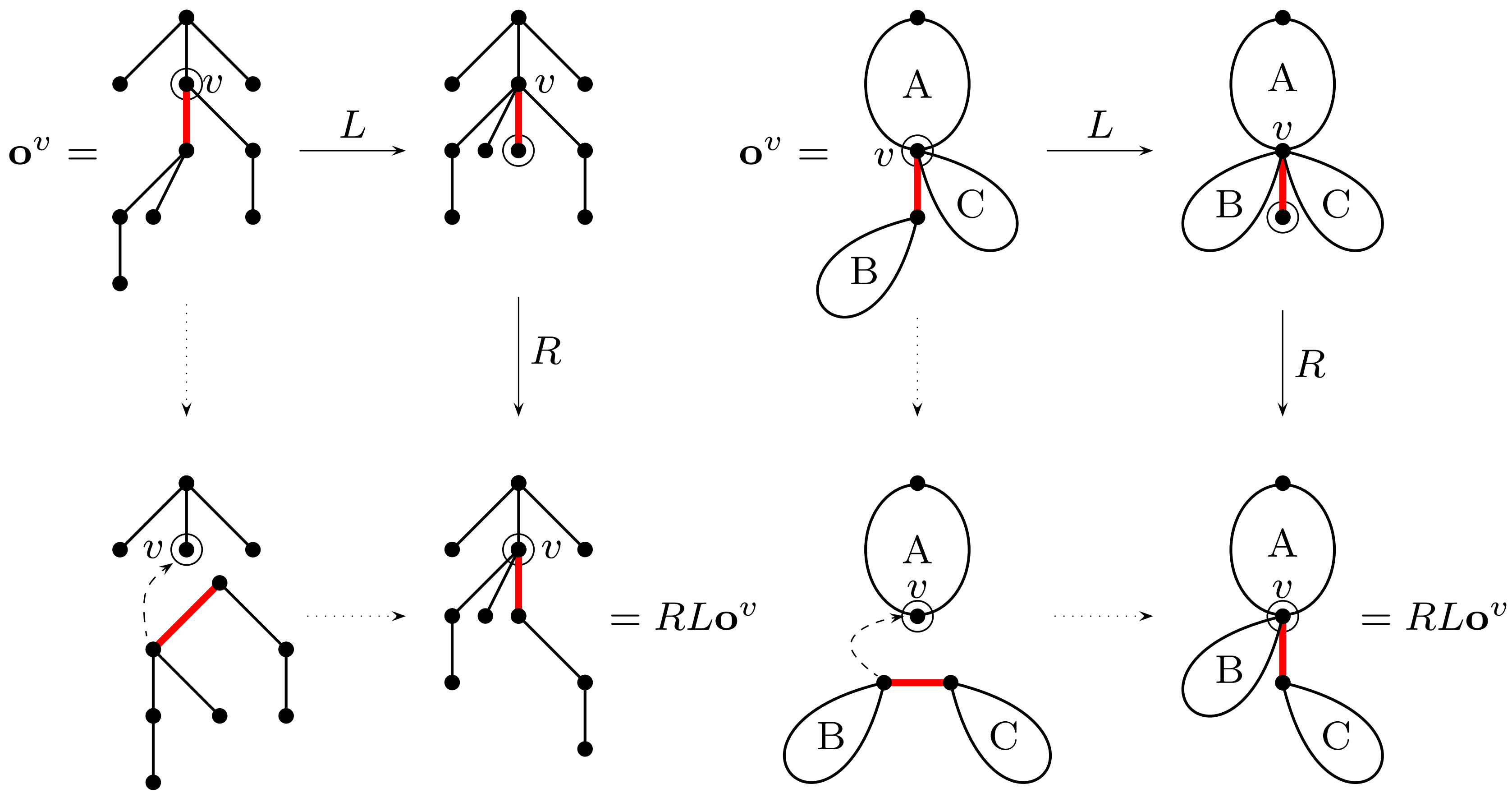


Figure 11: How RL acts on an internal pointed ordered tree $\mathbf{o}^v$

Define $P : \mathcal{O}_n \longrightarrow \mathcal{P}_n$ by *forgetting* the root and the distinguished edge of each ordered tree. By Figure 11,

$$\bar{\mathrm{d}}(RL\mathbf{o}^v) = \bar{\mathrm{d}}(\mathbf{o}^v) \quad \text{and} \quad P(\mathrm{d}(RL\mathbf{o}^v)\,) = P(\mathrm{d}(\mathbf{o}^v)\,)\,.$$

Hence

$$[\mathbf{o}_1^v]^+ = [\mathbf{o}_2^w]^+ \Longleftrightarrow$$
$$\bar{\mathrm{d}}(\mathbf{o}_1^v) = \bar{\mathrm{d}}(\mathbf{o}_2^w) \quad \text{and} \quad P(\mathrm{d}(\mathbf{o}_1^v)\,) = P(\mathrm{d}(\mathbf{o}_2^w)\,)\,.$$

Theorem 4 *Let* $orb_n = |\mathcal{O}_n^+/\mathbf{H}|$*. Then*

$$orb_n = \sum_{k=0}^{n-1} |\mathcal{O}_k^-|\cdot|\mathcal{P}_{n-k}| \;=\; p_n + \sum_{k=1}^{n-1} \frac{1}{2}\binom{2k}{k} \cdot p_{n-k}. \qquad (1)$$

Let $\mathcal{P}(x)$ denote the ordinary generating function for p_n, and $\mathcal{O}(x)$ for Catalan number c_n. Then by *dissymmetry Theorem for trees* (Bergeron, Labelle and Leroux, 1998),

$$\mathcal{P}(x) = 1 + \sum_{n\geq 1} \frac{\phi(n)}{n} \log \frac{1}{1 - x^n\mathcal{O}(x^n)} + \frac{x}{2}\left(\mathcal{O}(x^2) - \mathcal{O}^2(x)\right)$$

and

$$p_n = \frac{1}{2n}\sum_{d|n} \phi\left(\frac{n}{d}\right)\binom{2d}{d} - \frac{1}{2}c_n + \frac{1}{2}\chi_{\mathrm{odd}}(n)\, c_{\frac{n-1}{2}}\,. \qquad (2)$$

From (1) and (2), we can have the summation form of orb_n, but we cannot find a simple formula. The sequence $\{orb_n\}_{n=0}^{\infty}$ starts with $1, 1, 2, 6, 18, 60, 210, 754, 2766, 10280, 38568, \ldots$, and it does not appear in On-Line Encyclopedia of Integer Sequences.

Counting the Cardinality of an orbit

For $\mathbf{p} \in \mathcal{P}_n$, define the *center* of $\mathbf{p}$ by the center of the longest path in $\mathbf{p}$ (Knuth, 1973; Bergeron, Labelle and Leroux, 1998). Let $c(\mathbf{p})$ denote the *center* of $\mathbf{p}$.

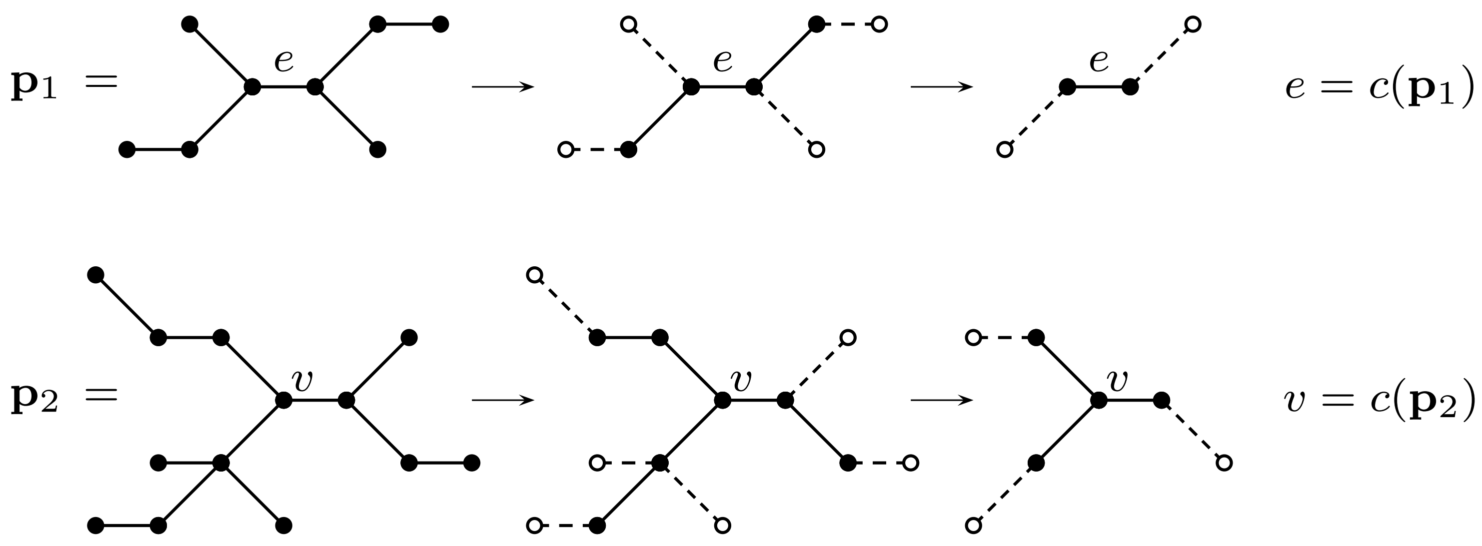


Figure 12: The process to find center of given plane tree

symmetry index : Define the *symmetry index* of $\mathbf{p}$ by the number of periods of components around the center of $\mathbf{p}$.

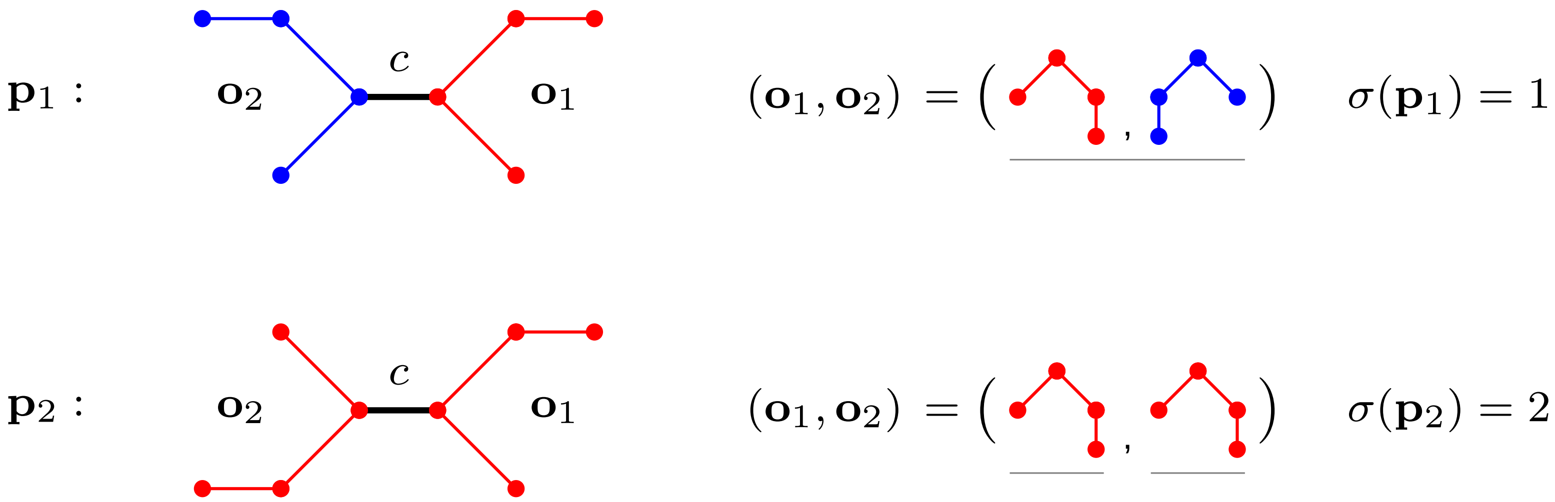


Figure 13: $\sigma(\mathbf{p})$: when $c(\mathbf{p})$ is an edge.

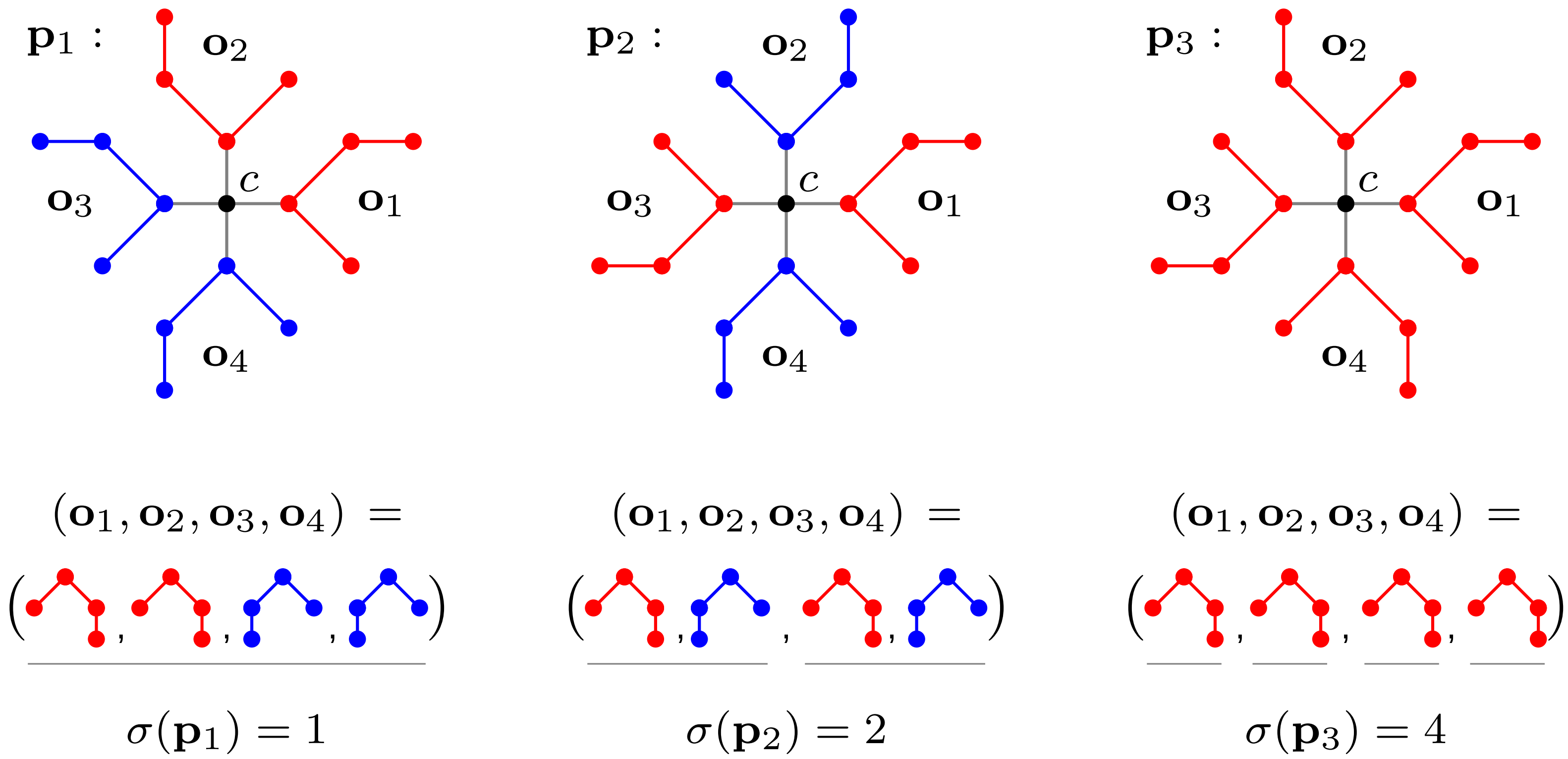


Figure 14: $\sigma(\mathbf{p})$: when $c(\mathbf{p})$ is a vertex.

The symmetry index plays an important role in obtaining the size of an orbit as follows:

Theorem 5 *Given* $\mathbf{o}^v \in \mathcal{O}_n^+$*, the cardinality of* $[\mathbf{o}^v]^+$ *is*

$$\left|[\mathbf{o}^v]^+\right| = \frac{2\,\epsilon(\mathbf{p})}{\sigma(\mathbf{p})}, \tag{3}$$

where $\mathbf{p} = P(\mathrm{d}(\mathbf{o}^v)\,)$*, and* $\epsilon(\mathbf{p})$ *is the number of* edges *in* $\mathbf{p}$*.*

Sketch of the proof: The size of $[\mathbf{o}^v]^+$ equals the number of ways of identifying a vertex w in $\mathbf{p}$ with $v \in \bar{\mathrm{d}}(\mathbf{o}^v)$. For each vertex w in $\mathbf{p}$, we have $\deg(w)$ distinct ways of identifying w with $v \in \mathrm{d}(\mathbf{o}^v)$. So, if we allow repetition, the number of all possible ways of attaching $\mathbf{p}$ to $\bar{\mathrm{d}}(\mathbf{o}^v)$ is $\sum_{w\in\mathbf{p}} \deg(w) = 2\epsilon(\mathbf{p})$. But by the definition of the symmetry number, each pattern occurs exactly $\sigma(\mathbf{p})$ times. This yields (3).

p							
$\varepsilon(p)$	1	2	3	3	4	4	4
$\sigma(p)$	2	2	3	2	2	1	4
$\frac{2\varepsilon(p)}{\sigma(p)}$	1	2	2	3	4	8	2

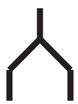
p						
$\varepsilon(p)$	5	5	5	5	5	5
$\sigma(p)$	2	1	1	1	2	5
$\frac{2\varepsilon(p)}{\sigma(p)}$	5	10	10	10	5	2

Figure 15: symmetry index of all plane trees having 5 or less edges

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