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PERMUTATION PATTERNS, ORDERED TREES AND CONTINUED FRACTIONS

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ABSTRACT. We enumerate permutations which have exactly r 123-patterns and s 132-patterns where $r + s \leq 2$.

We also give a new bijection between the ordered trees on $n+1$ vertices and 123-avoiding permutations of length n. We define the weight of ordered trees so that the bijection becomes weight-preserving, and find the generating function, in the form of a continued fraction, of 123-avoiding permutations of length n that have exactly r 132-patterns.

1. INTRODUCTION

It is known that the number of permutations which avoid both patterns 123 and 132 of length n is 2^{n-1}. Robertson shows in [R] that the number of permutations of length n which avoid 132-pattern but contain exactly one 123-pattern equals $(n-2)2^{n-3}$. He also defines a natural bijection between these permutations and the permutations of length n which avoid 123-pattern but contain exactly one 132-pattern. And then he shows that the number of permutations of length n which contain exactly one 132-pattern and one 123-pattern is $(n-3)(n-4)2^{n-5}$.

In this paper, we will find useful bijections between previous permutations and some special sequences. Using these bijections, we confirm Robertson's results and also find some new results.

It is shown in [BJS] that the number of 132-avoiding (or 123-avoiding) permutations of length n is the n-th Catalan number,

$$C_n = \frac{1}{n+1}\binom{2n}{n}.$$

In section 3, we will give a bijection between 132-avoiding permutations and the ordered trees on $n+1$ vertices. Moreover, in section 5, we define a weight of ordered trees so that this bijection becomes weight-preserving, where the weight of permutations is defined to be the number of 123-patterns. And then we will find, in the form of a continued fraction, the generating function for the number of 132-avoiding permutations that have a given number of 123-patterns. This generating function has been given in [RWZ, JR]. But the proof in [RWZ] is complicated and is difficult to interpret combinatorially. The proof in [JR] is combinatorial and the weight-preserving bijection between ordered trees and 132-avoiding

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permutations in [JR] is identical with the bijection in this paper, but their method to build the bijection is different from ours. This difference allows us to make a weight-preserving bijection between ordered trees and 123-avoiding permutations.

In section 6, we also find the generating function for the number of 123-avoiding permutations that have a given number of 132-patterns.

2. Preliminaries

2.1. **Permutations**

Let S_n be the set of all permutations of length n. A 123-pattern (*resp.* a 132-pattern) in a permutation $\pi = \pi_1\pi_2\cdots\pi_n$ is a triple (π_i, π_j, π_k) such that $1 \le i < j < k \le n$ and $\pi_i < \pi_j < \pi_k$ (*resp.* $\pi_i < \pi_k < \pi_j$). Define $p(\pi)$ to be the number of p-patterns in π, where p denotes 123 or 132. If $p(\pi) = 0$, π is called *p-avoiding* (or *p-pattern avoiding*). Let $A_n(r,s)$ denote the set of all permutations of length n which have exactly r 123-patterns and s 132-patterns, i.e.

$$A_n(r,s) = \{\pi \in S_n |\ 123(\pi) = r \text{ and } 132(\pi) = s\}$$

and A_n^p the set of all p-avoiding permutations of length n, i.e.

$$A_n^{132} = \bigcup_{i=0}^{\infty} A_n(i,0), \quad A_n^{123} = \bigcup_{i=0}^{\infty} A_n(0,i).$$

2.2. **Sequences**

For a natural number n, a sequence $(a_1, a_2, \cdots, a_n)$ of integers is called a *position sequence* of length n, if $0 \le a_i \le n - i$ for $i = 1, 2, \cdots, n$. A sequence $(a_1, a_2, \cdots, a_n)$ is called *good* if it is a position sequence and $(a_i + i)_{i=1}^n$ is non-decreasing i.e. $a_{i-1} \le a_i + 1$ for $i = 2, 3, \cdots, n$.

From now on, let P_n denote the set of all position sequences of length n and G_n the set of all good sequences of length n. P_n and S_n are in 1–1 correspondence and G_n is a subset of P_n with

$$|G_n| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

2.3. **Ordered Trees**

An *ordered tree T* is a *direct graph* with the following properties :

a. There is one vertex, called the *root*, that has no predecessors and from which there is a path to every vertex.
b. Each vertex other than the root has exactly one predecessor.
c. The successors of each vertex are ordered *left-to-right*.

A vertex with no successors is called a *leaf*. We shall draw trees with the root at top and all arcs pointing downward. The arrows on the arcs are therefore not needed to indicate direction, and they will not be shown. The successors of each vertex will be drawn in left-to-right order.

$$\begin{tikzcd}
R_n \arrow[d, "\phi"'] & \\
G_n \arrow[r, "id"] \arrow[d, "Lemma\ 5.1"', "?"] & P_n \arrow[d, shift left, "\psi"] \\
A_n^{132} \arrow[r, "id"] & S_n \arrow[u, shift left, "\rho"]
\end{tikzcd}$$

FIGURE 1. A diagram

Let R_n be the set of all ordered trees on $n+1$ vertices. We know that

$$|R_n| = C_n = \frac{1}{n+1}\binom{2n}{n}.$$

3. BIJECTIONS

3.1. Bijection ψ between P_n and S_n

The bijection ρ will send permutations to position sequences. For $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$, define $\pi_j^{-1} = k$ if $\pi_k = j$ and define

$$\rho(\pi) = (a_1, a_2, \cdots, a_n)$$

where

$$a_i = \left|\left\{j \mid i < j \text{ and } \pi_i^{-1} < \pi_j^{-1}\right\}\right|.$$

Note that ρ actually counts non-inversions, and

$$\rho(\pi) = (n-1-b_1, n-2-b_2, \cdots, n-n-b_n)$$

if $(b_1, b_2, \cdots, b_n)$ is the *inversion sequence* of π where $b_i = \left|\left\{j \mid i < j \text{ and } \pi_i^{-1} > \pi_j^{-1}\right\}\right|$

Clearly, ρ is injective. We define $\psi = \rho^{-1}$. In fact, we can think of the insertion algorithm for accomplishing ψ that naturally produces a permutation from a position sequence. Given such a sequence $a = (a_1, a_2, \cdots, a_n)$, assume that $1, 2, \cdots, i-1$ have been inserted into n cells, expressed as a *word*. Then insert i so that it has a_i vacant cells to its right.

For example, if $a = (5, 2, 1, 3, 1, 0, 0)$ then $*1*****\rightarrow *1**2** \rightarrow *1**23* \rightarrow 41**23* \rightarrow 41*523* \rightarrow 41*5236 \rightarrow \psi(a) = 4175236$. By a construction, a_i is the number of entries j of $\psi(a)$ to the right of i satisfying $j > i$. Now, we can identify a permutation $a = (a_1, a_2, \cdots, a_n)$ with its position sequence.

3.2. Bijection ϕ between R_n and G_n

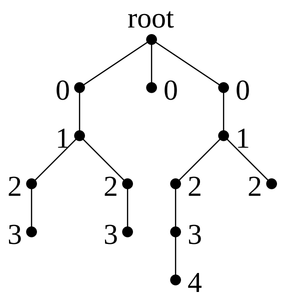
We now define a bijection $\phi : R_n \longrightarrow G_n$ from ordered trees to good sequences. Let T be an ordered tree. We label each vertex v of T with the number, $d(v) - 1$ where $d(v)$, called the *depth* of vertex v, is the *distance* from vertex v to the root of T. We now construct the sequence $(c_1, c_2, \cdots, c_n)$ written as an n-tuple by *reading* the vertices of the labeled tree T except root in *(leftmost-child-first) postorder traversal.* Then $c_{i-1} \leq c_i + 1$ for $i = 2, 3, \cdots, n$

and $c_n = 0$. So $0 \leq c_i \leq n - i$ for all i. Therefore, the sequence $(c_1, c_2, \cdots, c_n)$ is good. We define

$$\phi(T) = (c_1, c_2, \cdots, c_n).$$

Since ϕ is clearly an injection and $|R_n| = |G_n| = C_n$, ϕ is a bijection.

For example, let T be the following ordered tree.



Then $\phi(T) = (3, 2, 3, 2, 1, 0, 0, 4, 3, 2, 2, 1, 0)$

4. Numbers

Lemma 4.1. *The sum of the number of 123-patterns and that of 132-patterns in a permutation π with a position sequence $(a_1, a_2, \cdots, a_n)$ is*

$$\sum_{i=1}^{n} \binom{a_i}{2}.$$

Proof. For a fixed i, the number of 123-patterns or 132-patterns which are of the form $(i, \cdot, \cdot)$ equals $\binom{a_i}{2}$. Therefore,

$$123(\pi) + 132(\pi) = \sum_{i=1}^{n} \binom{a_i}{2}.$$

$\square$

Lemma 4.2. *For any position sequence $a = (a_1, a_2, \cdots, a_n)$ and $\pi = \psi(a)$,*

(1) If $a_k \leq a_{k+1},\ a_k \leq a_{k+2},\ \cdots,\ a_k \leq a_{k+i-1}$ and $a_k - a_{k+i} \geq 2$ for some k and i, then the permutation π has a 132-pattern of the form $(k, \cdot, \cdot)$.

(2) If $a_k = a_{k+1}+1 = a_{k+2}+2 = \cdots = a_{k+a_k-1}+(a_k-1)$ for some k, then the permutation π has no 132-pattern of the form $(k, \cdot, \cdot)$.

(3) If the permutation π has a 132-pattern, then $a_k - a_{k+1} \geq 2$ for some k.

(4) Therefore, a position sequence $a = (a_1, a_2, \cdots, a_n)$ is good if and only if the permutation π is 132-avoiding

Proof. (1) Let $B_k = \left\{ j \mid k < j \text{ and } \pi_k^{-1} < \pi_j^{-1} \right\}$ and $J_{k,k+i} = B_k \backslash \left(B_{k+i} \cup \{k+i\}\right)$. Since $|J_{k,k+i}| \geq |B_k| - |B_{k+i}| - |\{k+i\}| = a_k - a_{k+i} - 1 \geq 1$, $J_{k,k+i}$ is non-empty. Then the permutation $\psi(a)$ has the 132-pattern $(k, j, k+i)$ for some $j \in J_{k,k+i}$.

(2) For all $i = 1, 2, \cdots, a_k - 1$, $a_{k+i-1} = a_{k+i} + 1$ and $B_{k+i-1} = B_{k+i} \cup \{k+i-1\}$. Then $B_k = \{k+1, k+2, \cdots, k+a_i-1, \alpha\}$ assuming $B_{k+a_i-1} = \{\alpha\}$ where $\alpha > k + a_i - 1$. Because that any element of B_{k+i} is on the right of $k+i-1$,

$$\pi_k^{-1} < \pi_{k+1}^{-1} < \cdots < \pi_{k+a_k-1}^{-1} < \pi_\alpha^{-1}.$$

So any $(k, \cdot, \cdot)$ is not 132-pattern.

(3) Let (x, j, y) be a 132-pattern in the permutation π. Then $\pi_x^{-1} < \pi_j^{-1} < \pi_y^{-1}$. For some $x \leq k < y$,

$$\pi_x^{-1} < \pi_j^{-1}, \quad \pi_{x+1}^{-1} < \pi_j^{-1}, \quad \cdots, \quad \pi_k^{-1} < \pi_j^{-1} \quad \text{and} \quad \pi_{k+1}^{-1} > \pi_j^{-1}.$$

So $(k, j, k+1)$ is a 132-pattern in the permutation π. Then

$$j \in J_{k,k+1}, \quad \text{i.e.} \quad |J_{k,k+1}| \geq 1.$$

Hence $a_k - a_{k+1} \geq 2$.

(4) It is the composition of contrapositions of (1) and (3).

□

In the following theorem 4.3, the results (1)–(4) have been proved indirectly by Robertson in [R]. We prove them directly here. The results (5)–(6) are new.

Theorem 4.3 ($|A_n(r,s)|$ for $r + s \leq 2$)**.**

(1) $|A_n(0,0)| = 2^{n-1}$ *for* $n \geq 1$*. (No 123-pattern and no 132-pattern)*

(2) $|A_n(1,0)| = (n-2)2^{n-3}$ *for* $n \geq 3$ *(One 123-pattern but no 132-pattern)*

(3) $|A_n(0,1)| = (n-2)2^{n-3}$ *for* $n \geq 3$ *(One 132-pattern but no 123-pattern)*

(4) $|A_n(1,1)| = (n-3)(n-4)2^{n-5}$ *for* $n \geq 5$*. (One 123-pattern and one 132-pattern)*

(5) $|A_n(2,0)| = (n-3)(n-4)2^{n-5} + (n-3)2^{n-4}$ *for* $n \geq 4$*. (Two 123-patterns but no 132-pattern)*

(6) $|A_n(0,2)| = (n-3)(n-4)2^{n-5} + (n-3)2^{n-4}$ *for* $n \geq 4$*. (Two 132-patterns but no 123-pattern)*

Proof. By definition, a_n is zero for any position sequence $(a_1, \cdots, a_n)$.

(1) Consider the position sequence $(a_1, \cdots, a_{n-1}, 0)$ of any permutation which avoids both 123- and 132-patterns. By lemma 4.1, $\sum_{i=1}^n \binom{a_i}{2} = 0$. Then $a_i = 0$ or 1 for all $i = 1, 2, \cdots, n-1$. There exist 2^{n-1} possibilities. Hence $|A_n(0,0)| = 2^{n-1}$ for $n \geq 1$.

(2) Consider the position sequence $(a_1, \cdots, a_{n-1}, 0)$ of any 132-avoiding permutation which has exactly one 123-pattern. $\sum_{i=1}^n \binom{a_i}{2} = 1$ by lemma 4.1. Then $a_k = 2$ for some $1 \leq k \leq n-2$ and $a_i = 0$ or 1 for all $i \neq k$. Since $a_{k+1} = 1$ by lemma 4.2, there exist $(n-2)2^{n-3}$ possibilities. Therefore, $|A_n(1,0)| = (n-2)2^{n-3}$ for $n \geq 3$.

(3) Similarly, $a_k = 2$ for some $1 \leq k \leq n-2$ and $a_i = 0$ or 1 for all $i \neq k$. Since $a_{k+1} = 0$ by lemma 4.2, there exist $(n-2)2^{n-3}$ possibilities. Therefore, $|A_n(0,1)| = (n-2)2^{n-3}$ for $n \geq 3$.

(4) Similarly, by lemma 4.1, $\sum_{i=1}^n \binom{a_i}{2} = 2$. Since $\binom{3}{2} = 3 > 2$, there are only two 2's among a_i and the others are 0 or 1. By lemma 4.2, the position sequence has exactly one substring $(2,0)$ and one substring $(2,1)$. Since a_n is zero, there exist $(n-3)(n-4)2^{n-5}$ possibilities. Therefore, $|A_n(1,1)| = (n-3)(n-4)2^{n-5}$ for $n \geq 5$.

(5) By lemmas 4.1 and 4.2, the position sequence has exactly two substring $(2,1)$ or one substring $(2,2,1)$ and the other elements are 0 or 1. Then there exist $(n-3)(n-4)2^{n-5}+(n-3)2^{n-4}$ possibilities. Therefore, $|A_n(2,0)| = (n-3)(n-4)2^{n-5}+(n-3)2^{n-4}$ for $n \geq 4$.
(6) By lemmas 4.1 and 4.2, the position sequence has exactly two $(2,0)$ or one $(2,2,0)$ and the other elements are 0 or 1. Then there exist $(n-3)(n-4)2^{n-5}+(n-3)2^{n-4}$ possibilities. Therefore, $|A_n(0,2)| = (n-3)(n-4)2^{n-5}+(n-3)2^{n-4}$ for $n \geq 4$.

□

5. 132-avoiding Permutations

5.1. Bijection $\psi|_{G_n}$ between G_n and A_n^{132}

Lemma 5.1. *The position sequence $a = (a_1, a_2, \cdots, a_n)$ is good if and only if the permutation $\psi(a)$ is 132-avoiding.*

Proof. This easily follows from lemma 4.2. □

By lemma 5.1, the restriction $\psi|_{G_n}$ is a well-defined bijection between G_n and A_n^{132}. We can define $\psi \circ \phi : R_n \longrightarrow A_n^{132}$, naturally.

5.2. Weights of ordered trees corresponding to the number of 123-patterns

Define a weight of 132-avoiding permutations of length n for the number of 123-patterns, i.e.

$$w(\pi) = z^n q^{123(\pi)}$$

for all π in A_n^{132}. We now want to define the weight of ordered trees in R_n to make $\psi \circ \phi$ weight-preserving. By lemma 4.1, the number of 123-patterns of 132-avoiding permutation $\psi \circ \phi(T)$ is

$$\sum_{i=1}^{n} \binom{a_i}{2}$$

where $(a_1, a_2, \cdots, a_n)$ is its position sequence $\phi(T)$. So each vertex v of ordered tree T except the root contributes $\binom{d(v)-1}{2}$ 123-patterns in the permutation $\psi \circ \phi(T)$, where $d(v)$ is the depth of v.

We define weights of ordered tree T and vertex v in T by

$$w(T) = \prod_{v \in T} w(v) \quad \text{and} \quad w(v) = \begin{cases} 1 & \text{if } v \text{ is a root,} \\ zq^{\binom{d(v)-1}{2}} & \text{if } v \text{ is not a root.} \end{cases} \tag{1}$$

Then clearly $\psi \circ \phi$ becomes a weight-preserving map, i.e.

$$w(T) = w(\pi) \quad \text{if} \quad \psi \circ \phi(T) = \pi.$$

5.3. Generating function for ordered trees

Lemma 5.2. *Let w be the weight function on T defined by $\prod_{v\in T} x_{d(v)}$, where $d(v)$ denotes the depth of v, and $R = \bigcup_{n=0}^{\infty} R_n$. Then*

$$\sum_{T\in R} w(T) = \cfrac{x_0}{1-\cfrac{x_1}{1-\cfrac{x_2}{1-\cfrac{x_3}{1-\cfrac{x_4}{1-\cfrac{x_5}{\cdots}}}}}}. \qquad (2)$$

Proof. For each i, define a weight function $w_i(T)$ by the product of $w_i(v) = x_{d(v)+i}$ for each vertex v in T and $F_i = \sum_{T\in R} w_i(T)$. Any ordered tree is decomposed into subtrees by deleting the root. Each of such subtrees contributes a factor of F_{i+1} because of the level change. Thus, the trees with k subtrees are enumerated by $x_i F_{i+1}^k$ in which x_i is the weight of the root, so that the generating function satisfies

$$F_i = x_i + x_i F_{i+1} + x_i F_{i+1}^2 + \cdots = \frac{x_i}{1 - F_{i+1}}.$$

Then

$$\sum_{T\in R} w(T) = F_0 = \frac{x_0}{1-F_1} = \cfrac{x_0}{1-\cfrac{x_1}{1-F_2}} = \cdots$$

$$= \cfrac{x_0}{1-\cfrac{x_1}{1-\cfrac{x_2}{1-\cfrac{x_3}{1-\cfrac{x_4}{1-\cfrac{x_5}{\cdots}}}}}}.$$

□

5.4. First Continued Fraction

Let $f(n,r)$ be the number of 132-avoiding permutations of length n that have exactly r 123-patterns. We have the following expression of the generating function for $f(n,r)$.

Theorem 5.3. *The generating function for the $f(n,r)$ is*

$$\sum_{n,r\geq 0} f(n,r)z^n q^r = \cfrac{1}{1-\cfrac{z}{1-\cfrac{z}{1-\cfrac{zq}{1-\cfrac{zq^3}{1-\cfrac{zq^6}{\cdots}}}}}}$$

in which the n-th numerator is $zq^{\binom{n-1}{2}}$.

Proof. Let $A^{132} = \bigcup_{n=0}^{\infty} A_n^{132}$. Since $\psi\circ\phi$ is the weight-preserving map from A^{132} to R with weight defined in (1), we obtain

$$\sum_{n,r\geq 0} f(n,r)z^n q^r = \sum_{\pi\in A^{132}} w(\pi) = \sum_{T\in R} w(T).$$

By lemma 5.2, we find the generating function by substituting

$$x_n = \begin{cases} 1 & \text{if } n=0, \\ zq^{\binom{n-1}{2}} & \text{if } n>0 \end{cases}$$

in (2). □

In fact, theorem 5.3 was already shown by Robertson, Wilf and Zeilberger in [RWZ], but they can not interpret the n-th numerator combinatorially.

Remark. We can generalize theorem 5.3 as follows. $\tilde{f}(n,r)$ be the number of 132-avoiding permutations of length n that have exactly r $(12\cdots k)$-patterns. Note that the generating function for $\tilde{f}(n,r)$ is

$$\sum_{n,r\geq 0} \tilde{f}(n,r)z^n q^r = \cfrac{1}{1-\cfrac{zq^{\binom{0}{k-1}}}{1-\cfrac{zq^{\binom{1}{k-1}}}{1-\cfrac{zq^{\binom{2}{k-1}}}{1-\cfrac{zq^{\binom{3}{k-1}}}{1-\cfrac{zq^{\binom{4}{k-1}}}{\cdots}}}}}}$$

in which the n-th numerator is $zq^{\binom{n-1}{k-1}}$.

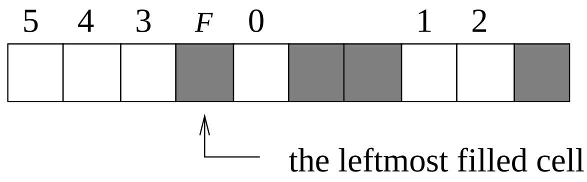


FIGURE 2. An example of labeled cell

6. 123-AVOIDING PERMUTATIONS

6.1. Bijection $\tilde{\psi}$ between G_n and A_n^{123}

We now construct a bijection $\tilde{\psi}$ from G_n to A_n^{123}. The insertion algorithm which generates the 123-avoiding permutation from a good sequence

$$a = (a_1, a_2, \cdots, a_n) \in G_n$$

runs as follows :

Step 1. Draw n cells.
Step 2. Insert 1 in the $(a_1 + 1)$-th cell from right.
Step 3. Assuming that $i - 1$ has been inserted, insert i as follows :
(1) Find the leftmost filled cell, and name it F.
(2) Label the vacant cells to the right of F, from left to right, and then the vacant cells to the left of F, from right to left, with the integers $0, 1, 2, \cdots$. (as Figure 2)
(3) Insert i in the cell label a_i.
Step 4. Repeat **Step 3** until all cells are filled.
Step 5. Define $\tilde{\psi}(a)$ to be the permutation obtained by reading numbers in the cells from left to right.

For example, if $a = (2, 3, 2, 1, 0, 1, 0)$ then $* * * * 1 * * \to * * 2 * 1 * * \to * * 2 * 1 * 3 \to * * 2 * 143 \to * * 25143 \to 6 * 25143 \to \tilde{\psi}(a) = 6725143$. By the lemma 6.1, $\tilde{\psi}$ is a well-defined bijection between G_n and A_n^{123}.

Lemma 6.1. *The position sequence $a = (a_1, a_2, \cdots, a_n)$ is good if and only if the permutation $\tilde{\psi}(a)$ is 123-avoiding.*

Proof. Let $a = (a_1, a_2, \cdots, a_n)$ be good, i.e. $a_{i-1} \le a_i + 1$ for all $i = 2, 3, \cdots, n$. Since $i (> 1)$ is inserted to the left of the leftmost filled cell or in the rightmost vacant cell, there is no 123-pattern in the form of $(\cdot, i, \cdot)$ for all $i = 2, 3, \cdots, n$. Hence $\tilde{\psi}(a)$ is 123-avoiding permutation.

Conversely, let k be the smallest integer satisfying $a_{k-1} > a_k + 1$. When k is inserted, there exists a vacant cell, called C, on the right of the inserted cell, which is located on the right of the leftmost filled cell, called F. Assume that i was inserted in the leftmost filled cell F and j will be inserted in the vacant cell C. Then

$$i < k < j, \quad \pi_i^{-1} < \pi_k^{-1} < \pi_j^{-1}.$$

and so, (i,k,j) is a 123-pattern. □

Remark. The mapping $\tilde{\psi} \circ \rho = \tilde{\psi} \circ \psi^{-1} : A_n^{132} \longrightarrow A_n^{123}$ is a bijection.

$$A_n^{132} \xrightarrow{\rho=\psi^{-1}} G_n \xrightarrow{\tilde{\psi}} A_n^{123}$$

6.2. **Weights of ordered trees corresponding to the number of 132-patterns**

Define a weight function w on A_n^{123} by

$$w(\pi) = z^n q^{132(\pi)}$$

for π in A_n^{123}. We now want to define the weight of ordered trees in R_n so that $\tilde{\psi} \circ \phi$ is weight-preserving. Let T be an ordered tree where $\phi(T) = (a_1, a_2, \cdots, a_n)$. Consider the algorithm for the bijection $\tilde{\psi}$ in section 6.1. If $a_{i-1} = a_i + 1$, then i is inserted in the rightmost vacant cell, and there is no 132-pattern of the form $(i,\cdot,\cdot)$ in $\tilde{\psi}\circ\phi(T)$. If $a_{i-1} < a_i + 1$, then i is inserted on the left of the leftmost filled cell, so that there exist a_i vacant cells on the right of the cell containing i. Hence there exist exactly $\binom{a_i}{2}$ 132-patterns of the form $(i,\cdot,\cdot)$ in $\tilde{\psi}\circ\phi(T)$. So each leaf v of ordered tree T contributes $\binom{d(v)-1}{2}$ 132-patterns, and the other vertices do not contribute any 132-patterns. Define a weight function w on an ordered tree T as follows :

$$w(T) = \prod_{v \in T} w(v), \quad w(v) = \begin{cases} 1 & \text{if } v \text{ is a root,} \\ z & \text{if } v \text{ is neither a root nor a leaf,} \\ zq^{\binom{d(v)-1}{2}} & \text{if } v \text{ is a leaf.} \end{cases} \tag{3}$$

Then $\tilde{\psi} \circ \phi$ becomes weight-preserving , i.e.

$$w(T) = w(\pi) \quad \text{if} \quad \tilde{\psi}\circ\phi(T) = \pi.$$

6.3. **When weights of leaves and non-leaves are different**

Lemma 6.2. *Assume that weights of vertices in T are defined by*

$$w(v) = \begin{cases} x_{d(v)} & \textit{if } v \textit{ is a non-leaf,} \\ y_{d(v)} & \textit{if } v \textit{ is a leaf.} \end{cases}$$

Let $R = \bigcup_{n=0}^{\infty} R_n$. Then

$$\sum_{T \in R} w(T) = (y_0 - x_0) + \cfrac{x_0}{1 - (y_1 - x_1) - \cfrac{x_1}{1 - (y_2 - x_2) - \cfrac{x_2}{\cdots}}}. \tag{4}$$

Proof. Define $F_i = \sum_{T\in R} w_i(T)$ where $w_i(v) = y_{d(v)+i}$ for any leaf v and $w_i(\tilde{v}) = x_{d(\tilde{v})+i}$ for any non-leaf $\tilde{v}$. Since the root is a leaf in the trees without subtrees and the converse of that is also true, the generating function satisfies

$$F_i = y_i + x_iF_{i+1} + x_iF_{i+1}^2 + \cdots = (y_i - x_i) + \frac{x_i}{1 - F_{i+1}} \ .$$

Then

$$\begin{aligned}
\sum_{T\in R} w(T) &= F_0 \\
&= (y_0 - x_0) + \frac{x_0}{1 - F_1} \\
&= (y_0 - x_0) + \cfrac{x_0}{1 - (y_1 - x_1) - \cfrac{x_1}{1 - F_2}} = \cdots \\
&= (y_0 - x_0) + \cfrac{x_0}{1 - (y_1 - x_1) - \cfrac{x_1}{1 - (y_2 - x_2) - \cfrac{x_2}{\cdots}}} \ .
\end{aligned}$$

□

6.4. **Second Continued Fraction**

Let $g(n, r)$ be the number of 123-avoiding permutations of length n that have exactly r 132-patterns.

Theorem 6.3. *The generating function for the $g(n, r)$ is*

$$\sum_{n,r\geq 0} g(n, r)z^nq^r = \cfrac{1}{1 - \cfrac{z}{1 - \cfrac{z}{1 - (zq - z) - \cfrac{z}{1 - (zq^3 - z) - \cfrac{z}{\cdots}}}}}$$

in which the n-th denominator is $1 - (zq^{\binom{n-1}{2}} - z)$.

Proof. Let $A^{123} = \bigcup_{n=0}^{\infty} A_n^{123}$. Since $\tilde{\psi} \circ \phi$ is the weight-preserving map from A^{123} to R with weight defined in (3), we obtain

$$\sum_{n,r\geq 0} g(n, r)z^nq^r = \sum_{\pi\in A^{123}} w(\pi) = \sum_{T\in R} w(T).$$

By lemma 6.2, we find the generating function by substituting

$$x_n = \begin{cases} 1 & \text{if } n = 0, \\ z & \text{if } n > 0 \end{cases} \quad \text{and} \quad y_n = \begin{cases} 1 & \text{if } n = 0, \\ zq^{\binom{n-1}{2}} & \text{if } n > 0 \end{cases}$$

in (4). $\square$

Remark. This theorem also has the following generalization. Let $\tilde{g}(n,r)$ be the number of 123-avoiding permutations of length n that have exactly r $(1k\cdots32)$-patterns. Note that the generating function for $\tilde{g}(n,r)$ is

$$\sum_{n,r\geq 0} \tilde{g}(n,r) z^n q^r = \cfrac{1}{1 - (zq^{\binom{0}{k-1}} - z) - \cfrac{z}{1 - (zq^{\binom{1}{k-1}} - z) - \cfrac{z}{\cdots}}}$$

in which the n-th denominator is $1 - (zq^{\binom{n-1}{k-1}} - z)$.

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