Principles of Mathematics 12

Probability

Lesson 1

Basics of Probability

Principles of Math 12 EXPLAINED!

By Barry Mabillard

Part I: Basic Elements of Probability

Sample Spaces:

Consider the following situation: *A six – sided die is rolled*

- The **sample space** refers to the complete set of all possible outcomes. In the case of rolling a die, the sample space is 1, 2, 3, 4, 5, or 6.

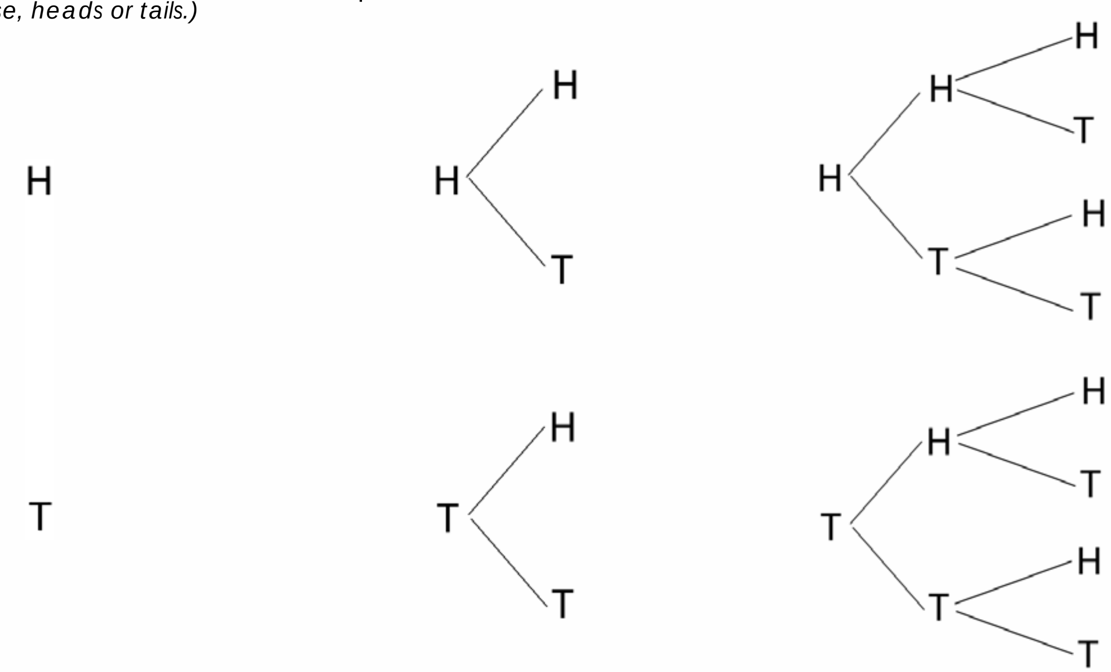
Sample spaces are often represented using *tree diagrams* or *charts*.

Example 1: A coin is flipped three times. Illustrate the sample space using a tree diagram.

Step 1: Start the tree diagram by vertically indicating the two possible results from the first flip.
(In this case, heads or tails.)

Step 2: Continue the tree diagram by connecting all possibilities for the second flip.

Step 3: Finish the tree diagram by connecting all possibilities for the third flip.



The sample space for this experiment is:

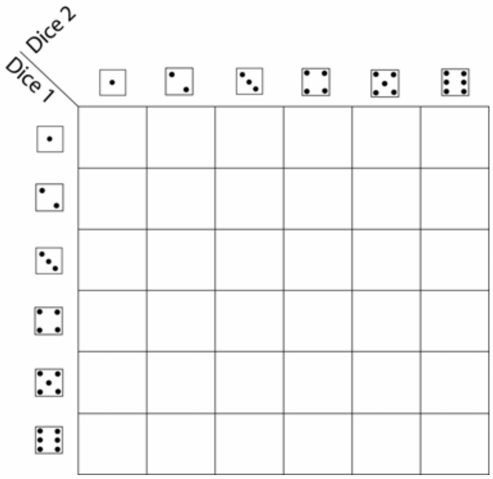
HHH
HHT
HTH
HTT
THH
THT
TTH
TTT

There are 8 possible outcomes

Part I: Basic Elements of Probability

Example 2: Two dice are rolled, and the sum is recorded. Illustrate the sample space in a chart.

Step 1: Start the chart as shown below:



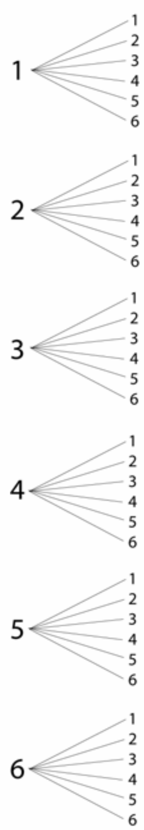
Step 2: Fill in the sums.

Dice 1 \ Dice 2	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

The sample space for the sum of two dice is {2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12}

The chart form is useful for large amounts of data since it allows for easy reading.

The situation above *could* be expressed as a tree diagram, but it would be more cumbersome.



Sample spaces are useful in quickly determining the number of particular outcomes in an experiment.

Example: If you roll two dice and want to know how many possible outcomes have a sum of seven, the chart above can be used very quickly to see there are six possible outcomes.

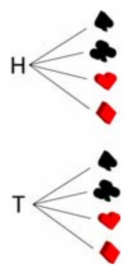
Part I: Basic Elements of Probability

Basic Probability: The formula for basic probability is

$$P(\text{event}) = \frac{\text{Favorable Outcomes}}{\text{Total Number of Possible Outcomes}}$$

Example 3: A coin is flipped, then a card is randomly drawn from a deck of 52 cards.
a) Determine the probability of flipping a head, then pulling a diamond.
b) Determine the probability of flipping a head, then pulling a diamond or a heart.

a) **Step 1:** Start by drawing the sample space:



Step 2: Now apply the above formula. There is only one favorable outcome out of 8 possible outcomes.

$$P(Head,\ Diamond) = \frac{1}{8}$$

b) Using the sample space above, there are two favorable outcomes out of eight.

$$P(Head, Diamond\ \ \textbf{or}\ \ Head,\ Heart) = \frac{Favorable\ Cases}{Total\ Cases} = \frac{2}{8} = \frac{1}{4}$$

Adding Probabilities: The probability of A **or** B, if the events are mutually exclusive, may be found by adding the individual probabilities.

$$P(A \text{ or } B) = P(A) + P(B)$$

Example 4: Two fair six-sided die are rolled and the sum is recorded. If the probability of rolling a sum greater than 10 is 0.083, and the probability of rolling a sum less than 5 is 0.167, what is the probability of rolling a sum greater than 10 or a sum less than 5?

The probability of rolling a sum greater than 10 is 0.083
The probability of rolling a sum less than 5 is 0.167

The probability of getting the first case **or** the second case is 0.083 + 0.167 = 0.25

> The phrase **mutually exclusive** means that one event taking place prevents the other event from taking place.
>
> **Example 1:** Flipping a coin gives mutually exclusive outcomes since you can't get heads and tails at the same time.
>
> **Example 2:** The outcomes of tossing a six-sided die are mutually exclusive since no two results can occur at the same time.

Multiplying Probabilities: The probability of A & B, if the events are independent, may be found by multiplying the individual probabilities.

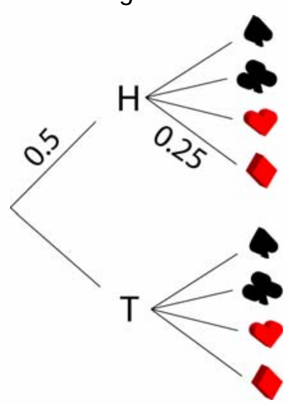
$$P(A \text{ and } B) = P(A) \times P(B)$$

Two events are **independent** if the outcome of one event does not change the probability of the second event occurring.

Example: A student tosses a six-sided die and gets a 3, then re-tosses and gets a 5. The two events do not influence each other in any way, so they are independent events.

Example 5: A coin is flipped, then a card is randomly drawn from a deck of 52 cards. If the probability of flipping a head is 0.5, and the probability of drawing a diamond is 0.25, then determine the probability of obtaining a head and a diamond.

Use a tree diagram and state the given probabilities.



$$P(Head \text{ and } Diamond) = P(Head) \times P(Diamond)$$

$$P(Head \text{ and } Diamond) = 0.5 \times 0.25$$

$$P(Head \text{ and } Diamond) = 0.125 = \frac{1}{8}$$

The Complement:

Probabilities of mutually exclusive events add up to one.

Consider a coin:

The probability of getting a head is 0.5, and getting a tail is 0.5. There is a 100% chance of getting a head or a tail when you flip a coin, so the total probability is 1.

Suppose you now have a weighted (trick) coin, such that the probability of getting a head is 0.43

To calculate the probability of getting a tail, just subtract the given probability from 1.
0.57 is the *compliment probability*

$$P(\bar{A})$$

The expression above is another way of writing the complement. The dash above the A tells you to calculate the probability it's NOT A

Example: If $P(A) = 0.65$, calculate $P(\bar{A})$

$$P(\bar{A}) = 1 - P(A)$$

$$P(\bar{A}) = 1 - 0.65$$

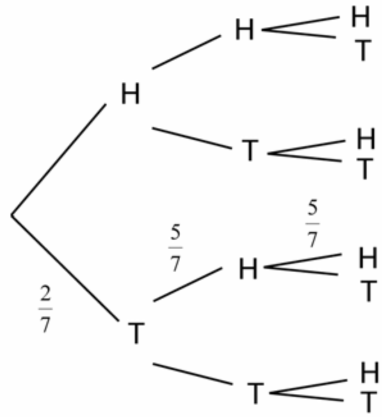
$$P(\bar{A}) = 0.35$$

Example 6: **A trick (weighted) coin is altered so the probability of it landing on a head for each flip is 5/7.**

a) The trick coin is flipped 3 times. What is the probability of getting a tail on the first flip and heads on the next two flips?

Probability of getting a head = $\frac{5}{7}$

Probability of getting a tail = $1-\frac{5}{7}=\frac{2}{7}$



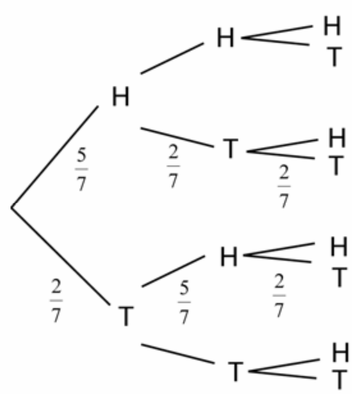
$$\frac{2}{7}\times\frac{5}{7}\times\frac{5}{7}=\frac{50}{343}$$

b) If the trick coin is flipped until two tails appear, what is the probability that the second tail will appear in three flips?

Think of this question as follows: ***What elements in the sample space will have two tails, one of which must be on the third flip?***

The answer is that you can have HTT or THT, since both of these will have the second tail come up on the third toss.

Once you find the probability of each event occurring, add the results to get the total probability.



$$\frac{5}{7}\times\frac{2}{7}\times\frac{2}{7}=\frac{20}{343}$$

$$\frac{2}{7}\times\frac{5}{7}\times\frac{2}{7}=\frac{20}{343}$$

$$\frac{20}{343}+\frac{20}{343}=\frac{40}{343}$$

Example 7: **A fair six-sided die is tossed twice. What is the probability the first toss will be a number greater than (or equal to) 3, and the second toss will a number less than 3?**

The probability of getting a 3, 4, 5, or 6 is $\frac{4}{6} \rightarrow \frac{2}{3}$

The probability of getting a 1 or 2 is $\frac{2}{6} \rightarrow \frac{1}{3}$

Multiply the probabilities to get $\frac{2}{3}\times\frac{1}{3}=\frac{2}{9}$

Questions:

1) Two six-sided dice are rolled and the sum is recorded. Determine the probability of obtaining:

a) a sum greater than 8

b) a sum less than 6

c) a sum greater than 8 **or** less than 6

d) a sum greater than 8 **and** less than 6

Dice 1 \ Dice 2	⚀	⚁	⚂	⚃	⚄	⚅
⚀	2	3	4	5	6	7
⚁	3	4	5	6	7	8
⚂	4	5	6	7	8	9
⚃	5	6	7	8	9	10
⚄	6	7	8	9	10	11
⚅	7	8	9	10	11	12

2) A card is drawn from a deck of 52 cards. Determine the probability of drawing:

a) the six of clubs or the eight of hearts.

b) the six of clubs or a heart.

c) a black card or a diamond

d) a nine of any suit

e) a black or red card

Quick Card Facts:

There are 52 cards in a deck with the jokers removed.

There are 26 black cards and 26 red cards.

There are 4 suits: Spades, Clubs, Hearts, and Diamonds.

Each suit has 13 cards of different rank.

Face cards are Jacks, Queens, and Kings.

3) There are 3 red & 2 yellow balls in one bag, and 6 blue & 7 green balls in a second bag. What is the probability of pulling a yellow ball from the first bag, and then a green ball from the second bag?

4) A fair coin is flipped three times. What is the probability of getting two heads and then a tail?

5) A trick coin (where the probability of getting a head is 3/4), is flipped three times. What is the probability of getting exactly two heads such that the second head will appear on the third flip?

6) A fair six-sided die is tossed twice. What is the probability the first toss will be a number less than 3, and the second toss will a number greater than 3?

Part I: Basic Elements of Probability

Answers:

1.

a) $\frac{10}{36} \rightarrow \frac{5}{18}$

b) $\frac{10}{36} \rightarrow \frac{5}{18}$

c) $\frac{5}{18}+\frac{5}{18}=\frac{10}{18} \rightarrow \frac{5}{9}$

d) Zero → You can't have two different sums at the same time!

2.

a) $\frac{1}{52}+\frac{1}{52}=\frac{2}{52} \rightarrow \frac{1}{26}$

b) $\frac{1}{52}+\frac{13}{52}=\frac{14}{52} \rightarrow \frac{7}{26}$

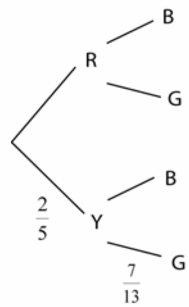
c) $\frac{26}{52}+\frac{13}{52}=\frac{39}{52} \rightarrow \frac{3}{4}$

d) $\frac{4}{52} \rightarrow \frac{1}{13}$

e) $\frac{26}{52}+\frac{26}{52}=\frac{52}{52} \rightarrow 1$

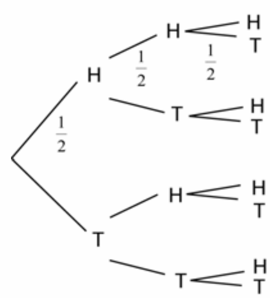
3. The probability of pulling a yellow ball from the first bag is $\frac{\text{favorable cases}}{\text{total cases}}=\frac{2}{5}$

The probability of pulling a green ball from the second bag is $\frac{\text{favorable cases}}{\text{total cases}}=\frac{7}{13}$



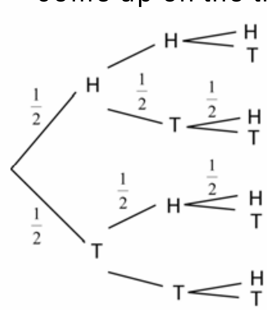
$\frac{2}{5} \times \frac{7}{13} = \frac{14}{65}$

4.



$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

5. What this question is basically saying is that you can have HTH or THH, since both of these will have the second tail come up on the third toss.



$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

$\frac{1}{8}+\frac{1}{8}=\frac{2}{8}=\frac{1}{4}$

$\frac{1}{2}\times\frac{1}{2}\times\frac{1}{2}=\frac{1}{8}$

6.

The probability of getting a 1or 2 is $\frac{2}{6} \rightarrow \frac{1}{3}$

The probability of getting a 4, 5, or 6 is $\frac{3}{6} \rightarrow \frac{1}{2}$

Multiply the probabilities to get $\frac{1}{3}\times\frac{1}{2}=\frac{1}{6}$

Probability Lesson 1

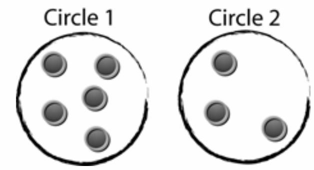
Part II: Venn Diagrams

<u>Venn Diagrams:</u> Outcomes of events are frequently organized in Venn Diagrams as a visual aid to the underlying mathematics.

<u>Venn Diagrams of Mutually Exclusive Events:</u>

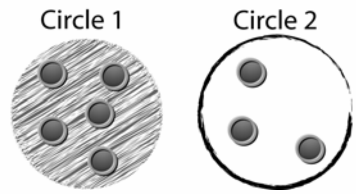
If two events are mutually exclusive, the Venn diagrams must illustrate that there is no overlap between the two sets of outcomes.

<u>Example 1:</u> In a game, discs are thrown into two circles on the other side of the room, as shown in the diagram.



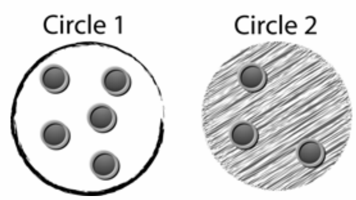
a) Calculate the probability of a disc being in Circle 1

$$P(\text{Circle 1}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{5}{8}$$



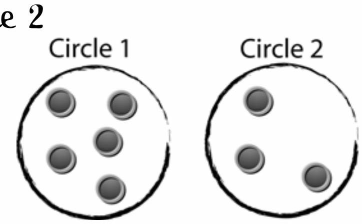
b) Calculate the probability of a disc being in Circle 2

$$P(\text{Circle 2}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{3}{8}$$



c) Calculate the probability of a disc being in Circle 1 and Circle 2

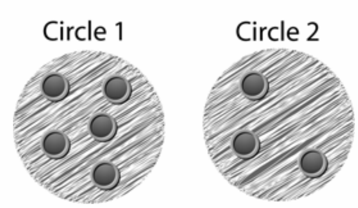
$P(\text{Circle 1 and 2}) = 0$ since a disc can't be in both circles at once!



d) Calculate the probability of a disc being in Circle 1 or Circle 2

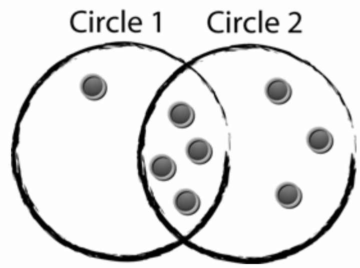
Since we have an **or** probability, simply add the probability of the disc being in Circle 1 to the probability of the disc being in Circle 2.

$$P(\text{Circle 1 or 2}) = \frac{5}{8} + \frac{3}{8} = \frac{8}{8} \rightarrow 1$$



Venn Diagrams of Non - Mutually Exclusive Events:

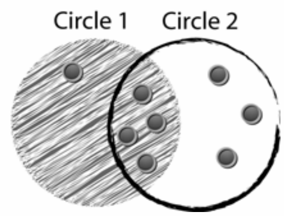
If two events are non - mutually exclusive, there is overlap between the two sets of outcomes.



Example 2: **In a game, discs are thrown into two circles on the other side of the room, as shown in the diagram.**

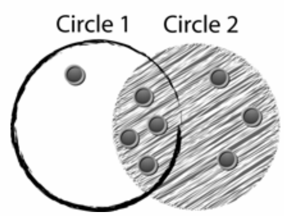
a) Calculate the probability of a disc being in Circle 1

$$P(\text{Circle 1}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{5}{8}$$



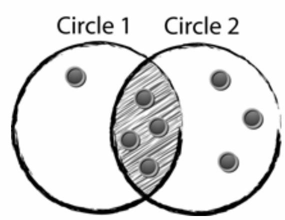
b) Calculate the probability of a disc being in Circle 2

$$P(\text{Circle 2}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{7}{8}$$



c) Calculate the probability of a disc being in Circle 1 and Circle 2

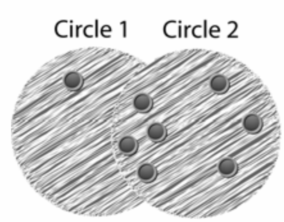
$$P(\text{Circle 1 and 2}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{4}{8} \rightarrow \frac{1}{2}$$



d) Calculate the probability of a disc being in Circle 1 or Circle 2

Since we have an **or** probability, simply add the probability of the disc being in Circle 1 to the probability of the disc being in Circle 2.

$$P(\text{Circle 1 or 2}) = \frac{5}{8} + \frac{7}{8} = \frac{12}{8} \rightarrow \frac{3}{2} \rightarrow \textbf{INCORRECT}$$



? Probability can't be bigger than one!

What Happened?

Adding the probabilities for Circle 1 and Circle 2 led to overlap in the middle since the 4 discs are counted in each case! Subtract them out to get the correct answer.

$$\mathbf{P(A \text{ or } B) = P(A) + P(B) - P(A \text{ and } B)}$$

Using the new formula to calculate the probability for Circle 1 OR 2 gives:

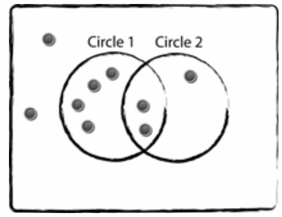
$$\frac{5}{8} + \frac{7}{8} - \frac{4}{8} = 1$$

This is the correct result.

Probability Lesson 1

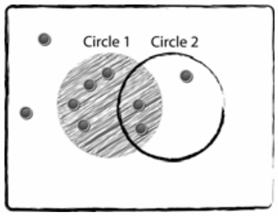
Part II: Venn Diagrams

Example 3: In a game, discs are thrown into two circles on the other side of the room, as shown in the diagram.



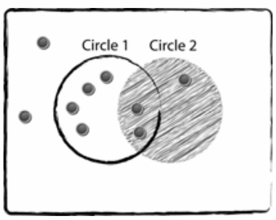
a) Calculate the probability of a disc being in Circle 1

$$P(\text{Circle 1}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{6}{9} \rightarrow \frac{2}{3}$$



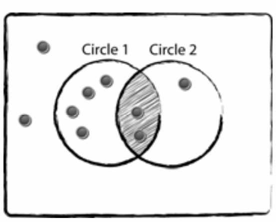
b) Calculate the probability of a disc being in Circle 2

$$P(\text{Circle 2}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{3}{9} \rightarrow \frac{1}{3}$$



c) Calculate the probability of a disc being in Circle 1 and Circle 2

$$P(\text{Circle 1 and 2}) = \frac{\text{favorable cases}}{\text{total cases}} = \frac{2}{9}$$



You may have noticed that the probabilities can be read off the diagram without using the formula.

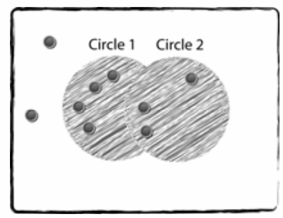
The formula method is included to show you how to do these calculations when there is no diagram to look at. Some questions will just give you numbers for the probabilities, and you will have to work it out from the formulas only.

d) Calculate the probability of a disc being in Circle 1 or Circle 2

$$P(\text{Circle 1 or 2}) = P(\text{Circle 1}) + P(\text{Circle 2}) - P(\text{Circle 1 and 2})$$

$$P(\text{Circle 1 or 2}) = \frac{6}{9} + \frac{3}{9} - \frac{2}{9}$$

$$P(\text{Circle 1 or 2}) = \frac{7}{9}$$

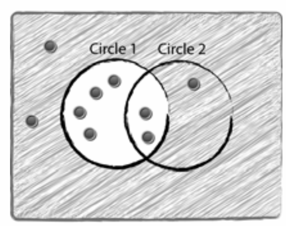


e) Calculate the probability of a disc not being in circle 1

$$P(\textbf{NOT}\ \text{Circle 1}) = 1 - P(\text{Circle 1})$$

$$P(\textbf{NOT}\ \text{Circle 1}) = 1 - \frac{2}{3}$$

$$P(\textbf{NOT}\ \text{Circle 1}) = \frac{1}{3}$$

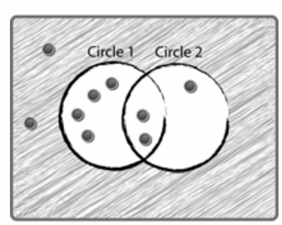


f) Calculate the probability of a disc not being in Circle 1 or Circle 2

$$P(\textbf{NOT}\ \text{Circle 1}) = 1 - P(\text{Circle 1 or 2})$$

$$P(\textbf{NOT}\ \text{Circle 1}) = 1 - \frac{7}{9}$$

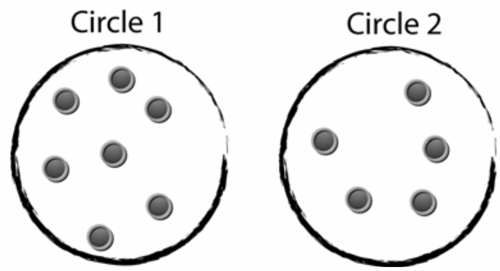
$$P(\textbf{NOT}\ \text{Circle 1}) = \frac{2}{9}$$



Part II: Venn Diagrams

Questions:

1) In a game, discs are thrown into two circles as shown in the diagram.



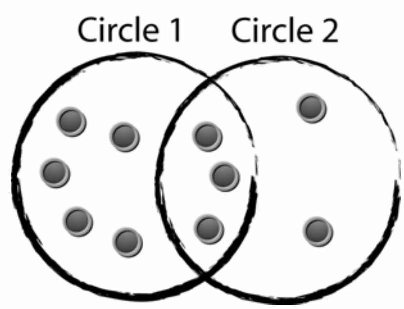
a) Calculate the probability of a disc being in Circle 1

b) Calculate the probability of a disc being in Circle 2

c) Calculate the probability of a disc being in Circle 1 **and** Circle 2

d) Calculate the probability of a disc being in Circle 1 **or** Circle 2

2) In a game, discs are thrown into two circles as shown in the diagram.



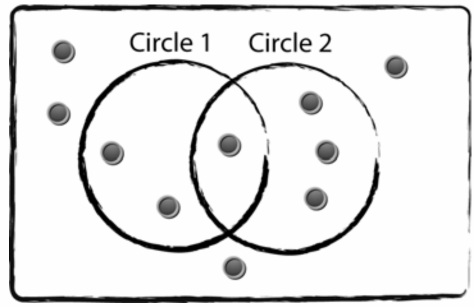
a) Calculate the probability of a disc being in Circle 1

b) Calculate the probability of a disc being in Circle 2

c) Calculate the probability of a disc being in Circle 1 **and** Circle 2

d) Calculate the probability of a disc being in Circle 1 **or** Circle 2

3) In a game, discs are thrown into two circles as shown in the diagram.



a) Calculate the probability of a disc being in Circle 1

b) Calculate the probability of a disc being in Circle 2

c) Calculate the probability of a disc being in Circle 1 **and** Circle 2

d) Calculate the probability of a disc being in Circle 1 **or** Circle 2

e) Calculate the probability of a disc **not** being in circle 1

f) Calculate the probability of a disc **not** being in Circle 1 **or** Circle 2

4) Calculate the unknown quantity in each of the following:

a) P(A) = 0.611, P(B) = 0.833, P(A and B) = 0.444.
Calculate P(A or B)

b) P(A) = 0.737, P(B) = 0.789, P(A or B) = 1
Calculate P(A and B)

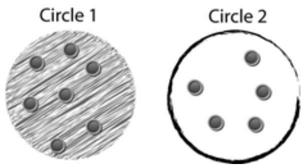
c) P(A) = 0.5, P(A or B) = 1, P(A and B) = 0.0625
Calculate P(B)

Probability Lesson 1

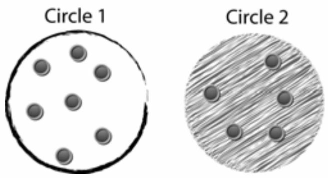
Part II: Venn Diagrams

Answers:

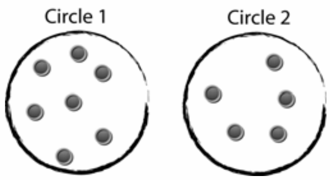
1.



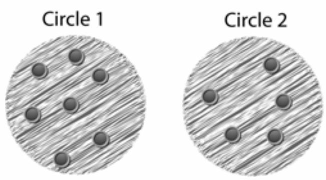
P(Circle 1) = 0.583



P(Circle 2) = 0.417

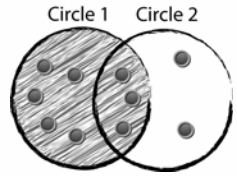


P(Circle 1 & 2) = 0

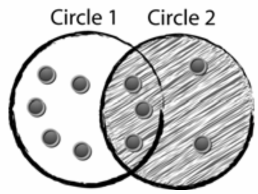


P(Circle 1 or 2) = 1

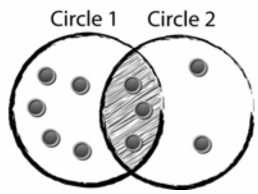
2.



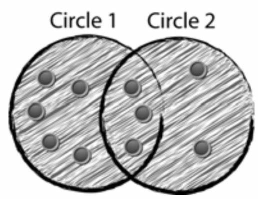
P(Circle 1) = 0.8



P(Circle 2) = 0.5

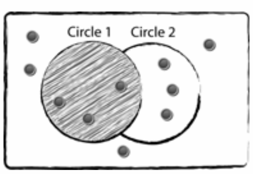


P(Circle 1 & 2) = 0.3

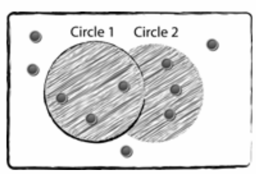


P(Circle 1 or 2) = 1

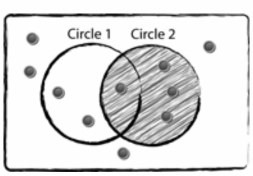
3.



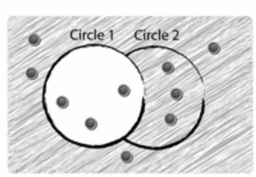
P(Circle 1) = 0.3



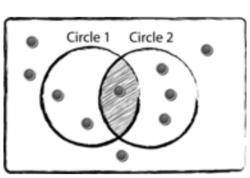
P(Circle 1 or 2) = 0.6



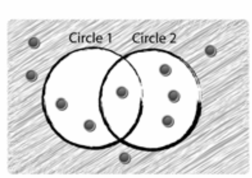
P(Circle 2) = 0.4



P(Not Circle 1) = 0.7



P(Circle 1 & 2) = 0.1



P(Not Circle 1 or 2) = 0.4

4.

a) P(A or B) = P(A) + P(B) – P(A and B)
P(A or B) = 0.611 + 0.833 – 0.444
P(A or B) = **1**

b) P(A or B) = P(A) + P(B) – P(A and B)

Solve for P(A and B)

P(A and B) = P(A) + P(B) – P(A or B)
P(A and B) = 0.737 + 0.789 – 1
P(A and B) = **0.526**

c) P(A or B) = P(A) + P(B) – P(A and B)

Solve for P(B)

P(B) = P(A or B) + P(A and B) - P(A)
P(B) = 1 – 0.0625 – 0.5
P(B) = **0.4375**