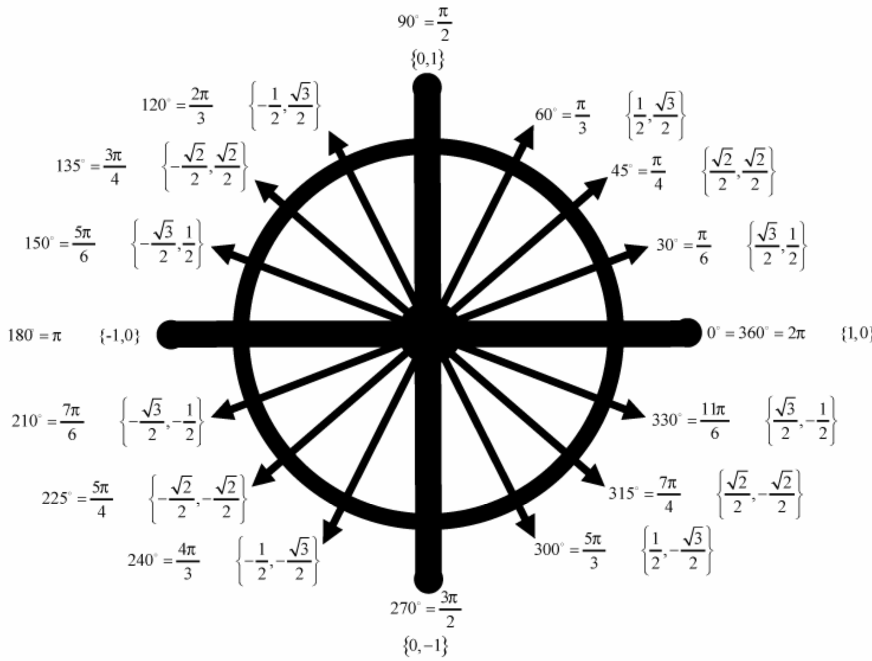
Principles of Mathematics 12

TRIGONOMETRY 1



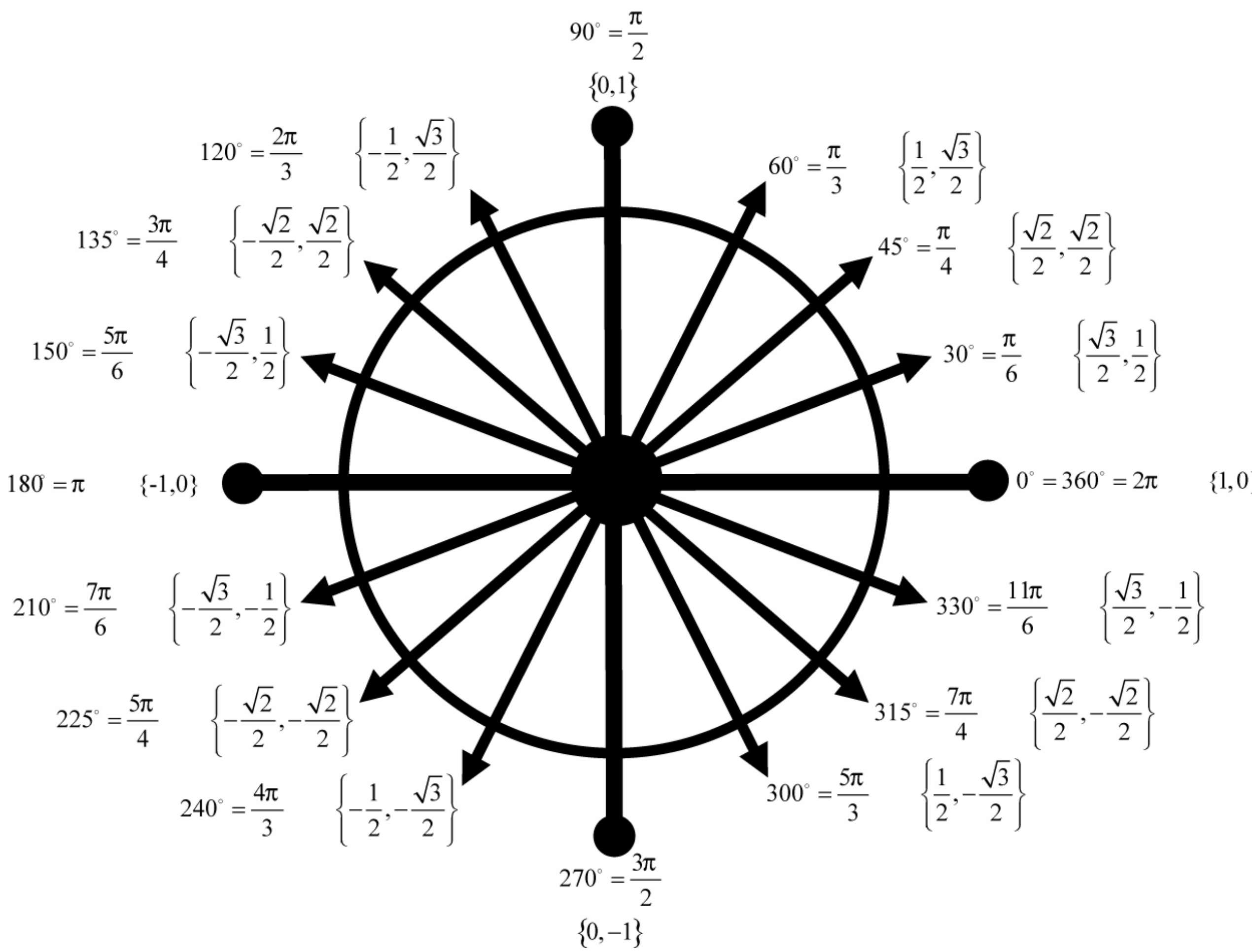
LESSON TWO

The Unit Circle

Principles of Math 12 EXPLAINED!
By Barry Mabillard

Trigonometry Lesson 2
Part One – The Unit Circle

The Unit Circle



What you see here is the unit circle. This is a useful tool in:

a) Comparing angles in degrees & radians.
b) Finding exact values of the six trigonometric ratios.

It is very important you memorize the unit circle as it will not be provided on the diploma.

The x-coordinate is the ***cosine*** of the angle
The y-coordinate is the ***sine*** of the angle.

Trigonometry Lesson 2
Part One – The Unit Circle

Example 1: Find the exact value of cos 135°

From the unit circle, we can see that the x-coordinate of 135° is $-\frac{\sqrt{2}}{2}$

Example 2: Find the exact value of sin 750°

First convert the given angle to a principal angle, then use the unit circle to find the exact value.

First, find the principal angle for 750°.
750° - 360° = 390°
390° - 360° = 30°

From the unit circle, sin 30° = $\frac{1}{2}$

Example 3: Find the exact value of cos(–1020°)

First, find the principal angle for cos(-1020°)

It is considered proper form to keep negative angles inside brackets.

-1020° + 360° + 360° + 360° = 60°

From the unit circle, cos 60° = $\frac{1}{2}$

Example 4: Find the exact value of $\sin^2 \frac{5\pi}{6}$

$\sin^2\theta$ is the same thing as $(\sin\theta)^2$
Evaluate what is inside the brackets first, then square the result to get the answer.

First convert the radian fraction to degrees.

$$\frac{5\pi}{6} \times \frac{180^\circ}{\pi} = 150^\circ$$

From the unit circle, sin 150° = $\frac{1}{2}$

Now square that answer = $\frac{1}{4}$

Example 5: Find the exact value of $\cos \frac{16\pi}{3}$

$$\frac{16\pi}{3} \times \frac{180^\circ}{\pi} = 960^\circ$$

960° - 360° - 360° = 240°

From the unit circle, cos 240° = $-\frac{1}{2}$

Example 6: Find the exact value of $-\sin\left(-\frac{9\pi}{2}\right)$

$$-\frac{9\pi}{2} \times \frac{180^\circ}{\pi} = -810^\circ$$

-810° + 360° + 360° + 360° = 270°

Finally, -sin 270° = - (-1) = 1

Trigonometry Lesson 2
Part One – The Unit Circle

The other four trigonometric ratios can be found from the unit circle using the following formulas:

$$\csc\theta = \frac{1}{\sin\theta} \qquad \sec\theta = \frac{1}{\cos\theta}$$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{\cos\theta}{\sin\theta}$$

Example 7: Find the exact value of $\sin\left(-\frac{\pi}{3}\right)$

$$-\frac{\pi}{3} \times \frac{180^\circ}{\pi} = -60^\circ$$

$$-60^\circ + 360^\circ = 300^\circ$$

$$\sin 300^\circ = -\frac{\sqrt{3}}{2}$$

Example 8: Find the exact value of $\cot\left(\frac{20\pi}{3}\right)$

$$\frac{20\pi}{3} \times \frac{180^\circ}{\pi} = 1200^\circ$$

$$1200^\circ - 360^\circ - 360^\circ - 360^\circ = 120^\circ$$

$$\cot 120^\circ = \frac{\cos 120^\circ}{\sin 120^\circ}$$

$$= \frac{-\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= -\frac{1}{2} \times \frac{2}{\sqrt{3}}$$

$$= -\frac{1}{\sqrt{3}}$$ Now Rationalize The Denominator

$$= -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= -\frac{\sqrt{3}}{3}$$

Example 9: Find the exact value of $\tan\left(-\frac{23\pi}{6}\right)$

$$-\frac{23\pi}{6} \times \frac{180^\circ}{\pi} = -690^\circ$$

$$-690^\circ + 360^\circ + 360^\circ = 30^\circ$$

$$\tan 30^\circ = \frac{\sin 30^\circ}{\cos 30^\circ}$$

$$= \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}$$

$$= \frac{1}{2} \times \frac{2}{\sqrt{3}}$$

$$= \frac{1}{\sqrt{3}}$$ Now Rationalize The Denominator

$$= \frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{\sqrt{3}}{3}$$

Example 10: Find the exact value of $\sec\left(\frac{3\pi}{2}\right)$

$$\frac{3\pi}{2} = 270^\circ$$

$$\sec 270^\circ$$

$$= \frac{1}{\cos 270^\circ}$$

$$= \frac{1}{0} = \text{undefined}$$

Trigonometry Lesson 2
Part One – The Unit Circle

Evaluate each of the following:

Express all fractions with rationalized denominators.

1) $\sin\frac{5\pi}{4}$

2) $\sin 180^\circ$

3) $\cos(-240^\circ)$

4) $\sec^2(-420^\circ)$

5) $\csc\left(\frac{-7\pi}{6}\right)$

6) $\cot\left(\frac{-10\pi}{3}\right)$

7) $-\tan\left(\frac{13\pi}{6}\right)$

8) $\tan\frac{\pi}{2}$

9) $\csc\left(\frac{-3\pi}{4}\right)$

10) $\tan\frac{7\pi}{6}$

11) $\sec\left(\frac{-11\pi}{2}\right)$

12) $-\sec 180^\circ$

13) $\cot 240^\circ$

14) $\csc\left(\frac{-5\pi}{4}\right)$

15) $\sec\frac{\pi}{6}$

16) $\cot^2\frac{20\pi}{3}$

17) $\sec\left(-\frac{35\pi}{6}\right)$

18) $\tan\frac{29\pi}{4}$

19) $\sec 0$

20) $-\csc 1380^\circ$

21) $\cot(-10\pi)$

22) $\tan\frac{13\pi}{4}$

23) $-\sec\left(-\frac{4\pi}{3}\right)$

24) $-\tan 0$

25) $\cos\frac{11\pi}{6}$

26) $\sec\frac{5\pi}{3}$

27) $-\csc 180^\circ$

28) $\cot\frac{\pi}{2}$

29) $\sin\frac{\pi}{3}$

30) $-\tan\frac{\pi}{3}$

Trigonometry Lesson Two

Part I - The Unit Circle

1) $-\frac{\sqrt{2}}{2}$

2) 0

3) $-\frac{1}{2}$

4) 4

5) 2

6) $-\frac{\sqrt{3}}{3}$

7) $-\frac{\sqrt{3}}{3}$

8) undefined

9) $-\sqrt{2}$

10) $\frac{\sqrt{3}}{3}$

11) undefined

12) 1

13) $\frac{\sqrt{3}}{3}$

14) $\sqrt{2}$

15) $\frac{2\sqrt{3}}{3}$

16) $\left(-\frac{\sqrt{3}}{3}\right)^2 = \frac{3}{9} = \frac{1}{3}$

17) $\frac{2\sqrt{3}}{3}$

18) 1

19) 1

20) $\frac{2\sqrt{3}}{3}$

21) undefined

22) 1

23) 2

24) 0

25) $\frac{\sqrt{3}}{2}$

26) 2

27) undefined

28) 0

29) $\frac{\sqrt{3}}{2}$

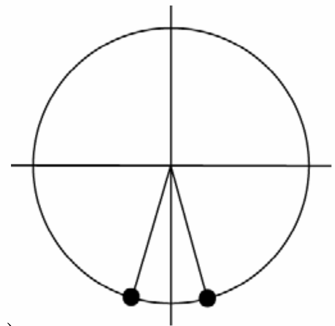
30) $-\sqrt{3}$

Trigonometry Lesson Two

Part II – Solving Basic Equations

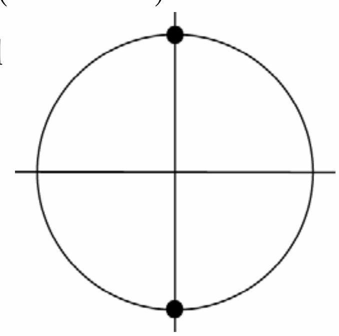
In the last section, we were looking for the exact value when given an angle.
In this section, we'll look for the angle(s) given the exact value.

Example 1: What is the solution to: $\sin\theta = -\frac{\sqrt{3}}{2}$ $(0 \le \theta \le 2\pi)$



To solve this, look for y-coordinates equal to $-\frac{\sqrt{3}}{2}$ on the unit circle and see what angles correspond to it.
The solution is $\theta = \frac{4\pi}{3}$ and $\frac{5\pi}{3}$

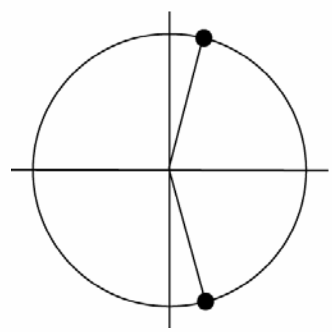
Example 2: Where is $\sec\theta$ undefined? $(0 \le \theta \le 2\pi)$



We know that $\sec\theta = \frac{1}{\cos\theta}$, so $\sec\theta$ is undefined whenever the denominator becomes zero.
$\cos\theta = 0$ when $\theta = \frac{\pi}{2}, \frac{3\pi}{2}$

In the previous two questions, we were given $(0 \le \theta \le 2\pi)$, which means we only need to go around the circle once. Most of the time, you will want the general solution, which includes all possible co-terminal angles.

Example 3: Find the general solution to $\cos\theta = \frac{1}{2}$



From the unit circle, we have an x-coordinate of $\frac{1}{2}$ when $\theta = \frac{\pi}{3}$ and $\frac{5\pi}{3}$

The general solution for $\frac{\pi}{3}$ is $\frac{\pi}{3} \pm n(2\pi)$

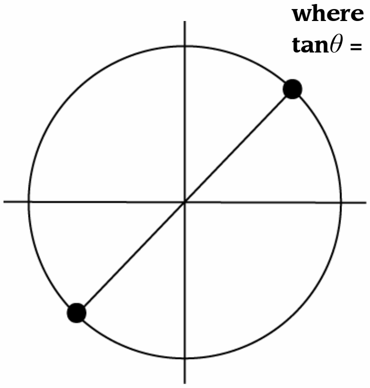
The general solution for $\frac{5\pi}{3}$ is $\frac{5\pi}{3} \pm n(2\pi)$

In Lesson 1, we used 360° for general solutions. If you are dealing with radians, use 2π instead.

Trigonometry Lesson Two

Part II – Solving Basic Equations

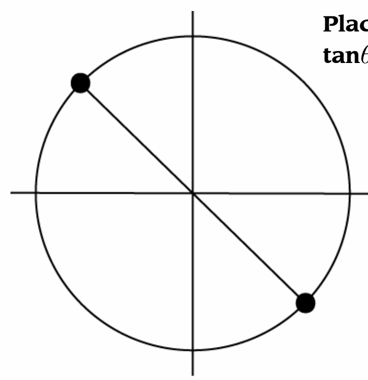
Watch out for questions involving tan θ. It has special rules.



Places where $\tan\theta = 1$

$\tan\theta = 1$ at $\frac{\pi}{4}$ and $\frac{5\pi}{4}$, since $\sin\theta$ and $\cos\theta$ are both $\frac{\sqrt{2}}{2}$

$$\textbf{\textit{example :}}\quad \tan\frac{5\pi}{4} = \frac{\sin\frac{5\pi}{4}}{\cos\frac{5\pi}{4}} = \frac{-\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}} = 1$$



Places where $\tan\theta = -1$

$\tan\theta = -1$ at $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$, since both $\sin\theta$ and $\cos\theta$ have a magnitude of $\frac{\sqrt{2}}{2}$ but differ by a negative.

$$\textbf{\textit{example :}}\quad \tan\frac{7\pi}{4} = \frac{\sin\frac{7\pi}{4}}{\cos\frac{7\pi}{4}} = \frac{-\frac{\sqrt{2}}{2}}{\frac{\sqrt{2}}{2}} = -1$$

In the special case of $\tan\theta = 1$ or $\tan\theta = -1$, we can combine all answers into one general solution. The reason for this is that each solution is exactly π units away from the other solution.

<u>Example 4:</u> Find the general solution to $\tan\theta = -1$

We know $\tan\theta = -1$ at $\frac{3\pi}{4}$ and $\frac{7\pi}{4}$. The angles are drawn on the right.

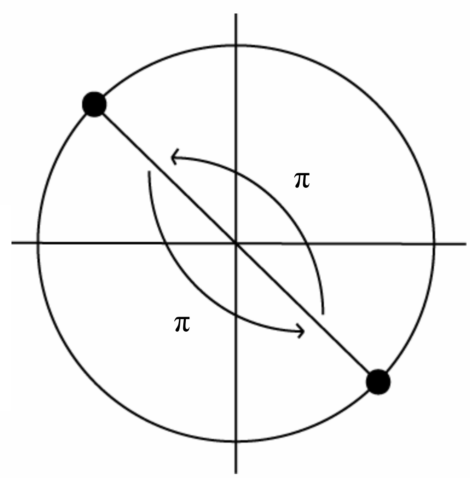
In this case, simply write the general solution as: $\frac{3\pi}{4} \pm n\pi$

When n = 0 , we will have $\frac{3\pi}{4}$

When n = 1 , we will have $\frac{7\pi}{4}$

<u>Common Denominator:</u>

$$\frac{3\pi}{4} + \pi = \frac{3\pi}{4} + \frac{\pi}{1} = \frac{3\pi}{4} + \frac{4\pi}{4} = \frac{7\pi}{4}$$



As you can see by trying out a few values of n, we will always obtain a difference of π . Thus, we can account for all co-terminal angles using $\frac{3\pi}{4} \pm n\pi$ without having to write two separate general solutions.

Trigonometry Lesson Two

Part II – Solving Basic Equations

For each of the following, find the solution for $(0 \le \theta \le 2\pi)$

1) $\sin\theta = \frac{\sqrt{3}}{2}$

2) $\sin\theta = 0$

3) $\cos\theta = -1$

4) $\sin\theta = -\frac{\sqrt{2}}{2}$

5) $\tan\theta = 0$

6) $\csc\theta = \textit{undefined}$

7) $\tan\theta = -1$

Memorize these general solutions:

$\tan\theta = 1$ or $\cot\theta = 1 \rightarrow \frac{\pi}{4} \pm n\pi$

$\tan\theta = -1$ or $\cot\theta = -1 \rightarrow \frac{3\pi}{4} \pm n\pi$

For each of the following, find the general solution. (Watch for solutions that are separated by π*)*

8) $\cos\theta = \frac{1}{2}$

9) $\cos\theta = 0$

10) $\sin\theta = -\frac{1}{2}$

11) $\cos\theta = \frac{\sqrt{3}}{2}$

12) $\cos\theta = \frac{\sqrt{2}}{2}$

13) $\tan\theta = \textit{undefined}$

14) $\cot\theta = 0$

15) $\tan\theta = 1$

16) $\cot\theta = -1$

Trigonometry Lesson Two

Part II – Solving Basic Equations

1) $\theta = \frac{\pi}{3}, \frac{2\pi}{3}$

2) $\theta = 0, \pi, 2\pi$ *Notice that $(0 \le \theta \le 2\pi)$ has a less than/equals sign for 2π, so include the 2π.

3) $\theta = \pi$

4) $\theta = \frac{5\pi}{4}, \frac{7\pi}{4}$

5) $\theta = 0, \pi, 2\pi$

6) $\csc\theta = undefined$ whenever $\frac{1}{\sin\theta}$ is undefined. $\sin\theta = 0$ when $\theta = 0, \pi, 2\pi$

7) $\theta = \frac{3\pi}{4}, \frac{7\pi}{4}$

8) $\theta = \frac{\pi}{3} \pm n(2\pi)$

$\theta = \frac{5\pi}{3} \pm n(2\pi)$

9) $\theta = \frac{\pi}{2} \pm n\pi$ *Note the combined general solution, since both angles where $\cos\theta = 0$ are separated by π

10) $\theta = \frac{7\pi}{6} \pm n(2\pi)$

$\theta = \frac{11\pi}{6} \pm n(2\pi)$

11) $\theta = \frac{\pi}{6} \pm n(2\pi)$

$\theta = \frac{11\pi}{6} \pm n(2\pi)$

12) $\theta = \frac{\pi}{4} \pm n(2\pi)$

$\theta = \frac{7\pi}{4} \pm n(2\pi)$

13) $\tan\theta = undefined$ when $\frac{\sin\theta}{\cos\theta}$ is undefined, and this happens when the denominator is zero

$\theta = \frac{\pi}{2} \pm n\pi$ is the solution to $\cos\theta = 0$

14) $\cot\theta = 0$ whenever $\frac{\cos\theta}{\sin\theta}$ is zero, and this happens when the numerator is zero.

$\theta = \frac{\pi}{2} \pm n\pi$ is the solution to $\cos\theta = 0$

15) $\theta = \frac{\pi}{4} \pm n\pi$

16) $\theta = \frac{3\pi}{4} \pm n\pi$

Part III – Graphically Solving Equations

In the last section, all the solutions could be found on the unit circle.
Now we will look at equations involving solutions not on the unit circle.

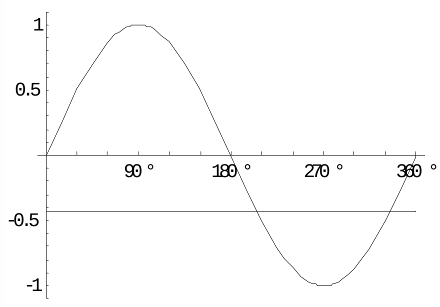
<u>Example 1:</u> Find the angles where $\sin\theta = -0.4235,\ \left(0 \le \theta \le 360°\right)$

To solve this, we graph in the TI-83 *(use **degree** mode):*

$y_1 = \sin\theta$

$y_2 = -0.4235$

X min = 0
X max = 360
X scl = 90
Y min = -1
Y max = 1
Y scl = 1



When the domain is in degrees, the answer should be in degrees also.

When the domain is in radians, the answer should be in radians also.

Use 2nd → Trace → Intersect to find the points of intersection of the two lines.

The *x*-coordinates will give you the angles that solve the equation.

θ = 205° and 335°

<u>Example 2:</u> Find the angles where $\cos\theta = \dfrac{\sqrt{5}}{7},\ \left(0 \le \theta \le 2\pi\right)$

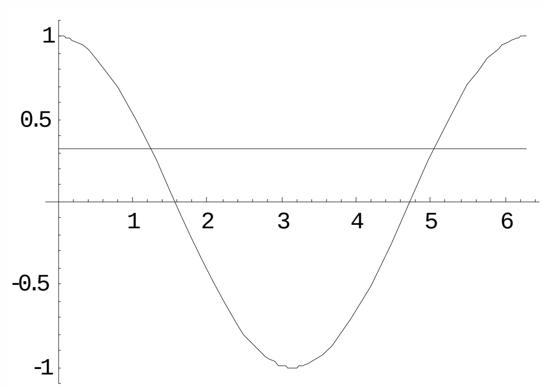
When the domain is given in radians, the answer should be in radians too.
To solve this, we graph in the TI-83 *(use **radian** mode):*

$y_1 = \cos\theta$

$y_2 = \dfrac{\sqrt{5}}{7}$

Window for radian mode

X min = 0
X max = 2π
X scl = $\dfrac{\pi}{2}$
Y min = -1
Y max = 1
Y scl = 1



Use 2nd → Trace → Intersect to find the points of intersection of the two lines.

The *x*-coordinates will give you the angles that solve the equation.

θ = 1.25 and 5.04 radians

Trigonometry Lesson Two

Part III – Graphically Solving Equations

Solve the following equations for the domain $0^o \leq x \leq 360^o$

*In some of the graphs, you will see vertical lines. These represent asymptotes, and do not contribute to the solution. Don't try to obtain intersection points at asymptotes.

1) $\cos x = -0.1288$

2) $\sin x = 0.5$

3) $\tan x = -1$

4) $\sec x = -1.3342$

5) $\cot x = 2$

6) $\csc x = 3.4219$

7) $\tan x = -0.5397$

8) $\cos x = 0.3994$

9) $\sec x = 1$

10) $\sin x = -0.4398$

Solve the following equations for the domain $0 \leq x \leq 2\pi$

11) $\cos x = \frac{\sqrt{3}}{2}$

12) $\tan x = 0$

13) $\sin x = 2$

14) $\cot x = 0.1123$

15) $\csc x = 0.5$

16) $\sec x = -1$

17) $\sin x = -0.5$

18) $\tan x = 1.7321$

19) $\cos x = -0.7071$

20) $\cot x = -1$

Answers:

The following answers are in degrees.

1) 97.4 & 262.6
2) 30 & 150
3) 135 & 315
4) 138.5 & 221.5
5) 26.6 & 206.6
6) 17 & 163
7) 151.6 & 331.6
8) 66.5 & 293.5
9) 0 & 360
10) 206.1 & 333.9

The following answers are in radians.

11) $\frac{\pi}{6}, \frac{11\pi}{6}$
12) $0, \pi, 2\pi$
13) Undefined
14) 1.46 & 4.60
15) Undefined
16) π
17) $\frac{7\pi}{6}, \frac{11\pi}{6}$
18) $\frac{\pi}{3}, \frac{4\pi}{3}$
19) $\frac{3\pi}{4}, \frac{5\pi}{4}$
20) $\frac{3\pi}{4}, \frac{7\pi}{4}$