Pure Math 30:

TRIGONOMETRY II

$$(\cos x - 1)(\tan x - 1) \equiv 0$$

Lesson Eleven

Identities

Pure Math 30: EXPLAINED!

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TRIGONOMETRY LESSON 11

PART I MULTIPLICATION & DIVISION IDENTITIES

Algebraic proofs of trigonometric identities

In this lesson, we will look at various strategies for proving identities. Try to memorize all the different types, as it will make things much simpler for you when they are mixed together.

Type I: Identities with multiplication & division:

In these proofs, you will need to convert everything to sine and cosine, then use fraction multiplication & division to simplify.

Example 1: Prove: sinxsecx = tanx

$$\begin{aligned}&\sin x \sec x\\ &= \sin x\left(\frac{1}{\cos x}\right)\\ &= \frac{\sin x}{\cos x}\\ &= \tan x\end{aligned}$$

Example 2: Prove: $\frac{\text{tanxcosx}}{\text{sinx}} = 1$

$$\begin{aligned}&\frac{\tan x \cos x}{\sin x}\\ &= \frac{\left(\frac{\sin x}{\cancel{\cos x}}\right)\cancel{\cos x}}{\sin x}\\ &= \frac{\sin x}{\sin x}\\ &= 1\end{aligned}$$

Example 3: Prove: $\frac{\text{cscx}}{\text{cotx}} = \text{secx}$

$$\begin{aligned}&\frac{\csc x}{\cot x}\\ &= \frac{\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}\\ &= \frac{1}{\cancel{\sin x}} \times \frac{\cancel{\sin x}}{\cos x}\\ &= \frac{1}{\cos x}\\ &= \sec x\end{aligned}$$

Fraction Review

Multiplying Fractions:

To multiply fractions, simply multiply the numerators together, and the denominators together.

$$\frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \frac{\sin x}{\cos^2 x}$$

Canceling:

When multiplying fractions you will frequently find factors that can be cancelled. You can cancel something on top with something identical on the bottom.

$$\begin{aligned}&\frac{\cos x}{\sin x} \times \frac{\sin^2 x}{\cos x}\\ &= \frac{\cancel{\cos x}}{\cancel{\sin x}} \times \frac{\sin^{\cancel{2}} x}{\cancel{\cos x}}\\ &= \sin x\end{aligned}$$

Dividing Fractions:

When dividing fractions, rewrite the top fraction, then multiply by the reciprocal of the bottom fraction.

$$\frac{\frac{\sin x}{\cos x}}{\cos x} = \frac{\sin x}{\cos x} \times \frac{1}{\cos x} = \frac{\sin x}{\cos^2 x}$$

TRIGONOMETRY LESSON 11

PART I MULTIPLICATION & DIVISION IDENTITIES

For each of the following, write an algebraic proof.

Identities will always have the following two properties:

1) If you graph the left and right sides, you will obtain exactly the same graph.

2) If you plug in the same angle for x on both sides, you will obtain exactly the same number.

1) Prove: $\cot x \tan x = 1$

2) Prove: $\csc x \cos x = \cot x$

3) Prove: $\dfrac{\sin x}{\tan x} = \cos x$

4) Prove: $\dfrac{1}{\cot x \cos x \tan x} = \sec x$

5) Prove: $\frac{\tan x}{\csc x} = \frac{\sin^2 x}{\cos x}$

6) Prove: $\frac{\tan x}{\sec x} = \sin x$

7) Prove: $\frac{\cos^2 x}{\cot x} = \sin x \cos x$

8) Prove: $\frac{\sec x \csc x}{\cot x} = \sec^2 x$

9) Prove: $\frac{\sec x \csc x}{\csc^2 x} = \tan x$

10) Prove: $\frac{\tan^2 x \cos x}{2 \sec x} = \frac{1}{2}\sin^2 x$

Answers

1) $\cot x \tan x$

$= \left(\frac{\cancel{\cos x}}{\cancel{\sin x}}\right)\left(\frac{\cancel{\sin x}}{\cancel{\cos x}}\right)$

$= 1$

2) $\csc x \cos x$

$= \left(\frac{1}{\sin x}\right)\cos x$

$= \frac{\cos x}{\sin x}$

$= \cot x$

3) $\frac{\sin x}{\tan x}$

$= \frac{\sin x}{\frac{\sin x}{\cos x}}$

$= \cancel{\sin x} \times \frac{\cos x}{\cancel{\sin x}}$

$= \cos x$

4) $\frac{1}{\cot x \cos x \tan x}$

$= \frac{1}{\frac{\cancel{\cos x}}{\cancel{\sin x}} \cos x \frac{\cancel{\sin x}}{\cancel{\cos x}}}$

$= \frac{1}{\cos x}$

$= \sec x$

5) $\frac{\tan x}{\csc x}$

$= \frac{\frac{\sin x}{\cos x}}{\frac{1}{\sin x}}$

$= \frac{\sin x}{\cos x} \times \frac{\sin x}{1}$

$= \frac{\sin^2 x}{\cos x}$

6) $\frac{\tan x}{\sec x}$

$= \frac{\frac{\sin x}{\cos x}}{\frac{1}{\cos x}}$

$= \frac{\sin x}{\cancel{\cos x}} \times \frac{\cancel{\cos x}}{1}$

$= \sin x$

7) $\frac{\cos^2 x}{\cot x}$

$= \frac{\cos^2 x}{\frac{\cos x}{\sin x}}$

$= \cos^{\cancel{2}} x \times \frac{\sin x}{\cancel{\cos x}}$

$= \cos x \sin x$

8) $\frac{\sec x \csc x}{\cot x}$

$= \frac{\frac{1}{\cos x}\frac{1}{\sin x}}{\frac{\cos x}{\sin x}}$

$= \frac{1}{\cos x \cancel{\sin x}} \times \frac{\cancel{\sin x}}{\cos x}$

$= \frac{1}{\cos^2 x}$

$= \sec^2 x$

9) $\frac{\sec x \csc x}{\csc^2 x}$

$= \frac{\sec x}{\csc x}$

$= \frac{\frac{1}{\cos x}}{\frac{1}{\sin x}}$

$= \frac{1}{\cos x} \times \frac{\sin x}{1}$

$= \frac{\sin x}{\cos x}$

$= \tan x$

10) $\frac{\tan^2 x \cos x}{2 \sec x}$

$= \frac{\frac{\sin^2 x}{\cos^2 x}\cos x}{\frac{2}{\cos x}}$

$= \frac{\sin^2 x}{\cancel{\cos x}} \times \frac{\cancel{\cos x}}{2}$

$= \frac{1}{2}\sin^2 x$

TRIGONOMETRY LESSON 11

PART II ADDITION & SUBTRACTION IDENTITIES

We will now look at identities where adding & subtracting is involved. You will first convert everything to sine & cosine, then use a common denominator to simplify the fractions.

Example 1:

Prove: $\sec x + \sin x = \dfrac{1+\sin x\cos x}{\cos x}$

$\sec x + \sin x$

$\dfrac{1}{\cos x} + \dfrac{\sin x}{1}$

$\dfrac{1}{\cos x} + \dfrac{\sin x}{1}\left(\dfrac{\cos x}{\cos x}\right)$

$\dfrac{1}{\cos x} + \dfrac{\sin x\cos x}{\cos x}$

$\dfrac{1+\sin x\cos x}{\cos x}$

Example 2:

Prove $\cot x + \sec x = \dfrac{\cos^2 x + \sin x}{\sin x\cos x}$

$\cot x + \sec x$

$\cot x + \dfrac{1}{\cos x}$

$\dfrac{\cos x}{\sin x} + \dfrac{1}{\cos x}$

$\dfrac{\cos x}{\sin x}\left(\dfrac{\cos x}{\cos x}\right) + \dfrac{1}{\cos x}\left(\dfrac{\sin x}{\sin x}\right)$

$\dfrac{\cos^2 x}{\sin x\cos x} + \dfrac{\sin x}{\sin x\cos x}$

$\dfrac{\cos^2 x + \sin x}{\sin x\cos x}$

Fraction Review

In multiplying & dividing fractions,
you *don't need* a common denominator.

In adding & subtracting fractions,
you *always need* a common denominator.

Example 1:

$\dfrac{1}{\cos x} + \dfrac{1}{\sin x - 1}$

Multiply the first fraction by the denominator of the second fraction.
Multiply the second fraction by the denominator of the first fraction.

$= \dfrac{1}{\cos x}\left(\dfrac{\sin x - 1}{\sin x - 1}\right) + \dfrac{1}{\sin x - 1}\left(\dfrac{\cos x}{\cos x}\right)$

Now multiply the fractions together and simplify

$= \dfrac{\sin x - 1}{\cos x(\sin x - 1)} + \dfrac{\cos x}{\cos x(\sin x - 1)}$

$= \dfrac{\sin x + \cos x - 1}{\cos x(\sin x - 1)}$

Example 2:

$\dfrac{1}{\cos x} + 1$

Multiply the second fraction by the denominator of the first.

We don't need to do anything with the first fraction since we will now have the same denominator.

$= \dfrac{1}{\cos x} + \dfrac{1}{1}\left(\dfrac{\cos x}{\cos x}\right)$

$= \dfrac{1}{\cos x} + \dfrac{\cos x}{\cos x}$

$= \dfrac{1+\cos x}{\cos x}$

TRIGONOMETRY LESSON 11

PART II ADDITION & SUBTRACTION IDENTITIES

Questions: *For each of the following, write an algebraic proof.*

1) $\sec x - \sin x = \dfrac{1 - \sin x \cos x}{\cos x}$

2) $\sin x + \tan x \sin x = \dfrac{\sin x \cos x + \sin^2 x}{\cos x}$

3) $\sec^2 x + \cot x = \dfrac{\sin x + \cos^3 x}{\cos^2 x \sin x}$

4) $\csc^2 x - \tan x = \dfrac{\cos x - \sin^3 x}{\sin^2 x \cos x}$

5) $\csc x - \sec x = \dfrac{\cos x - \sin x}{\sin x \cos x}$

6) $\sec x - \tan x = \dfrac{1 - \sin x}{\cos x}$

7) $\cos x + \tan x = \dfrac{\cos^2 x + \sin x}{\cos x}$

8) $\cot x + \sin x = \dfrac{\cos x + \sin^2 x}{\sin x}$

9) $1 + \tan x = \dfrac{\cos x + \sin x}{\cos x}$

10) $\csc x + 1 = \dfrac{1 + \sin x}{\sin x}$

TRIGONOMETRY LESSON 11

PART II ADDITION & SUBTRACTION IDENTITIES

1)
$$\begin{aligned}&\sec x-\sin x\\&=\frac{1}{\cos x}-\frac{\sin x}{1}\\&=\frac{1}{\cos x}-\left(\frac{\sin x}{1}\right)\frac{\cos x}{\cos x}\\&=\frac{1}{\cos x}-\frac{\sin x\cos x}{\cos x}\\&=\frac{1-\sin x\cos x}{\cos x}\end{aligned}$$

2)
$$\begin{aligned}&\sin x+\tan x\sin x\\&=\sin x+\left(\frac{\sin x}{\cos x}\right)\sin x\\&=\frac{\sin x}{1}+\frac{\sin^2 x}{\cos x}\\&=\left(\frac{\sin x}{1}\right)\frac{\cos x}{\cos x}+\frac{\sin^2 x}{\cos x}\\&=\frac{\sin x\cos x}{\cos x}+\frac{\sin^2 x}{\cos x}\\&=\frac{\sin x\cos x+\sin^2 x}{\cos x}\end{aligned}$$

3)
$$\begin{aligned}&\sec^2 x+\cot x\\&=\frac{1}{\cos^2 x}+\frac{\cos x}{\sin x}\\&=\frac{1}{\cos^2 x}\left(\frac{\sin x}{\sin x}\right)+\frac{\cos x}{\sin x}\left(\frac{\cos^2 x}{\cos^2 x}\right)\\&=\frac{\sin x}{\cos^2 x\sin x}+\frac{\cos^3 x}{\cos^2 x\sin x}\\&=\frac{\sin x+\cos^3 x}{\cos^2 x\sin x}\end{aligned}$$

4)
$$\begin{aligned}&\csc^2 x-\tan x\\&=\frac{1}{\sin^2 x}-\frac{\sin x}{\cos x}\\&=\frac{1}{\sin^2 x}\left(\frac{\cos x}{\cos x}\right)-\frac{\sin x}{\cos x}\left(\frac{\sin^2 x}{\sin^2 x}\right)\\&=\frac{\cos x}{\sin^2 x\cos x}-\frac{\sin^3 x}{\sin^2 x\cos x}\\&=\frac{\cos x-\sin^3 x}{\sin^2 x\cos x}\end{aligned}$$

5)
$$\begin{aligned}&\csc x-\sec x\\&=\frac{1}{\sin x}-\frac{1}{\cos x}\\&=\left(\frac{1}{\sin x}\right)\frac{\cos x}{\cos x}-\left(\frac{1}{\cos x}\right)\frac{\sin x}{\sin x}\\&=\frac{\cos x}{\sin x\cos x}-\frac{\sin x}{\sin x\cos x}\\&=\frac{\cos x-\sin x}{\sin x\cos x}\end{aligned}$$

6)
$$\begin{aligned}&\sec x-\tan x\\&=\frac{1}{\cos x}-\frac{\sin x}{\cos x}\\&=\frac{1-\sin x}{\cos x}\end{aligned}$$

7)
$$\begin{aligned}&\cos x+\tan x\\&=\cos x+\frac{\sin x}{\cos x}\\&=\frac{\cos x}{1}\left(\frac{\cos x}{\cos x}\right)+\frac{\sin x}{\cos x}\\&=\frac{\cos^2 x}{\cos x}+\frac{\sin x}{\cos x}\\&=\frac{\cos^2 x+\sin x}{\cos x}\end{aligned}$$

8)
$$\begin{aligned}&\cot x+\sin x\\&=\frac{\cos x}{\sin x}+\sin x\\&=\frac{\cos x}{\sin x}+\left(\frac{\sin x}{1}\right)\frac{\sin x}{\sin x}\\&=\frac{\cos x}{\sin x}+\left(\frac{\sin x}{1}\right)\frac{\sin x}{\sin x}\\&=\frac{\cos x}{\sin x}+\frac{\sin^2 x}{\sin x}\\&=\frac{\cos x+\sin^2 x}{\sin x}\end{aligned}$$

9)
$$\begin{aligned}&1+\tan x\\&=1+\frac{\sin x}{\cos x}\\&=\left(\frac{1}{1}\right)\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}\\&=\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}\\&=\frac{\cos x+\sin x}{\cos x}\end{aligned}$$

10)
$$\begin{aligned}&\csc x+1\\&=\frac{1}{\sin x}+\frac{\sin x}{\sin x}\\&=\frac{1+\sin x}{\sin x}\end{aligned}$$

TRIGONOMETRY LESSON 11

PART III THREE SPECIAL IDENTITIES

The three special identities (below) are critical when simplifying trigonometric expressions. The basic idea is that when you come across one of these special identities during simplification, you should immediately replace it with whatever that identity is equal to. Watch out for manipulations of these identities, as you will be expected to recognize them as well.

Example 1:

Prove: $1 - \cos^2 x \tan^2 x = \cos^2 x$

$$1 - \cos^2 x \tan^2 x$$

$$= 1 - \cancel{\cos^2 x}\left(\frac{\sin^2 x}{\cancel{\cos^2 x}}\right)$$

$$= 1 - \sin^2 x$$

Use Special Identity Here.

$$= \cos^2 x$$

Example 2:

Prove: $\sin x - \csc x = -\cot x \cos x$

$$\sin x - \csc x$$

$$= \sin x - \frac{1}{\sin x}$$

$$= \left(\frac{\sin x}{1}\right)\left(\frac{\sin x}{\sin x}\right) - \frac{1}{\sin x}$$

$$= \frac{\sin^2 x}{\sin x} - \frac{1}{\sin x}$$

$$= \frac{\sin^2 x - 1}{\sin x}$$

Use Special Identity Here.

$$= \frac{-\cos^2 x}{\sin x}$$

$$= -\left(\frac{\cos x}{\sin x}\right)\cos x$$

Spread out the two cosines so you can form cot *x*.

$$= -\cot x \cos x$$

$$\sin^2 x + \cos^2 x = 1$$

$\sin^2 x = 1 - \cos^2 x$ $\quad$ $\cos^2 x = 1 - \sin^2 x$

$-\sin^2 x = \cos^2 x - 1$ $\quad$ $-\cos^2 x = \sin^2 x - 1$

$$\tan^2 x + 1 = \sec^2 x$$

$\tan^2 x = \sec^2 x - 1$

$-\tan^2 x = 1 - \sec^2 x$

$$\cot^2 x + 1 = \csc^2 x$$

$\cot^2 x = \csc^2 x - 1$

$-\cot^2 x = 1 - \csc^2 x$

PART III THREE SPECIAL IDENTITIES

Questions: *Use the special identities to do each of the following proofs.*

1) $\sec x - \tan x \sin x = \cos x$

2) $\cos x + \tan x \sin x = \sec x$

3) $\tan x + \cot x = \sec x \csc x$

4) $1 + \tan^2 x = \sec^2 x$

5) $\sec x - \cos x = \tan x \sin x$

6) $\sin x + \cot x \cos x = \csc x$

7) $\sec^2 x - 1 = \sin^2 x \sec^2 x$

8) $1 - \csc^2 x = -\cot^2 x$

9) $\csc x - \sin x = \cos x \cot x$

10) $1 - \sec^2 x = -\tan^2 x$

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PART III THREE SPECIAL IDENTITIES

1)

$$\begin{aligned}&\sec x - \tan x \sin x\\ &= \frac{1}{\cos x} - \left(\frac{\sin x}{\cos x}\right)\sin x\\ &= \frac{1}{\cos x} - \frac{\sin^2 x}{\cos x}\\ &= \frac{1-\sin^2 x}{\cos x}\\ &= \frac{\cos^2 x}{\cancel{\cos x}}\\ &= \cos x\end{aligned}$$

2)

$$\begin{aligned}&\cos x + \tan x \sin x\\ &= \cos x + \left(\frac{\sin x}{\cos x}\right)\sin x\\ &= \frac{\cos x}{1} + \frac{\sin^2 x}{\cos x}\\ &= \left(\frac{\cos x}{1}\right)\frac{\cos x}{\cos x} + \frac{\sin^2 x}{\cos x}\\ &= \frac{\cos^2 x}{\cos x} + \frac{\sin^2 x}{\cos x}\\ &= \frac{1}{\cos x}\\ &= \sec x\end{aligned}$$

3)

$$\begin{aligned}&\tan x + \cot x\\ &= \frac{\sin x}{\cos x} + \frac{\cos x}{\sin x}\\ &= \left(\frac{\sin x}{\cos x}\right)\frac{\sin x}{\sin x} + \left(\frac{\cos x}{\sin x}\right)\frac{\cos x}{\cos x}\\ &= \frac{\sin^2 x}{\sin x \cos x} + \frac{\cos^2 x}{\sin x \cos x}\\ &= \frac{1}{\sin x \cos x}\\ &= \sec x \csc x\end{aligned}$$

4)

$$\begin{aligned}&1 + \tan^2 x\\ &= 1 + \frac{\sin^2 x}{\cos^2 x}\\ &= \left(\frac{1}{1}\right)\frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\\ &= \frac{\cos^2 x}{\cos^2 x} + \frac{\sin^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x}\\ &= \sec^2 x\end{aligned}$$

5)

$$\begin{aligned}&\sec x - \cos x\\ &= \frac{1}{\cos x} - \cos x\\ &= \frac{1}{\cos x} - \left(\frac{\cos x}{1}\right)\frac{\cos x}{\cos x}\\ &= \frac{1}{\cos x} - \frac{\cos^2 x}{\cos x}\\ &= \frac{1-\cos^2 x}{\cos x}\\ &= \frac{\sin^2 x}{\cos x}\\ &= \frac{\sin x}{\cos x}\sin x\\ &= \tan x \sin x\end{aligned}$$

6)

$$\begin{aligned}&\sin x + \cot x \cos x\\ &= \sin x + \left(\frac{\cos x}{\sin x}\right)\cos x\\ &= \sin x + \frac{\cos^2 x}{\sin x}\\ &= \left(\frac{\sin x}{1}\right)\frac{\sin x}{\sin x} + \frac{\cos^2 x}{\sin x}\\ &= \frac{\sin^2 x}{\sin x} + \frac{\cos^2 x}{\sin x}\\ &= \frac{1}{\sin x}\\ &= \csc x\end{aligned}$$

7)

$$\begin{aligned}&\sec^2 x - 1\\ &= \frac{1}{\cos^2 x} - 1\\ &= \frac{1}{\cos^2 x} - \left(\frac{1}{1}\right)\frac{\cos^2 x}{\cos^2 x}\\ &= \frac{1}{\cos^2 x} - \frac{\cos^2 x}{\cos^2 x}\\ &= \frac{1-\cos^2 x}{\cos^2 x}\\ &= \frac{\sin^2 x}{\cos^2 x}\\ &= \sin^2 x\left(\frac{1}{\cos^2 x}\right)\\ &= \sin^2 x \sec^2 x\end{aligned}$$

8)

$$\begin{aligned}&1 - \csc^2 x\\ &= 1 - \frac{1}{\sin^2 x}\\ &= \left(\frac{1}{1}\right)\frac{\sin^2 x}{\sin^2 x} - \frac{1}{\sin^2 x}\\ &= \frac{\sin^2 x}{\sin^2 x} - \frac{1}{\sin^2 x}\\ &= \frac{\sin^2 x - 1}{\sin^2 x}\\ &= \frac{-\cos^2 x}{\sin^2 x}\\ &= -\cot^2 x\end{aligned}$$

9)

$$\begin{aligned}&\csc x - \sin x\\ &= \frac{1}{\sin x} - \sin x\\ &= \frac{1}{\sin x} - \left(\frac{\sin x}{1}\right)\frac{\sin x}{\sin x}\\ &= \frac{1}{\sin x} - \frac{\sin^2 x}{\sin x}\\ &= \frac{1-\sin^2 x}{\sin x}\\ &= \frac{\cos^2 x}{\sin x}\\ &= \left(\frac{\cos x}{\sin x}\right)\cos x\\ &= \cot x \cos x\end{aligned}$$

10)

$$\begin{aligned}&1 - \sec^2 x\\ &= 1 - \frac{1}{\cos^2 x}\\ &= \left(\frac{1}{1}\right)\frac{\cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}\\ &= \frac{\cos^2 x}{\cos^2 x} - \frac{1}{\cos^2 x}\\ &= \frac{\cos^2 x - 1}{\cos^2 x}\\ &= \frac{-\sin^2 x}{\cos^2 x}\\ &= -\tan^2 x\end{aligned}$$

PART IV COMPOUND FRACTIONS & SPECIAL IDENTITIES

Next we'll look at compound fractions. Everything here is basically the same as in the previous section, just be sure to follow your rules for dividing fractions & watch for special identities.

Example 1: Prove: $\dfrac{\tan^2 x+1}{\csc^2 x-1}=\sec^2 x\tan^2 x$

Without using the special identities

$$\frac{\tan^2 x+1}{\csc^2 x-1}$$

$$=\frac{\dfrac{\sin^2 x}{\cos^2 x}+1}{\dfrac{1}{\sin^2 x}-1}$$

$$=\frac{\dfrac{\sin^2 x}{\cos^2 x}+\left(\dfrac{1}{1}\right)\dfrac{\cos^2 x}{\cos^2 x}}{\dfrac{1}{\sin^2 x}-\left(\dfrac{1}{1}\right)\dfrac{\sin^2 x}{\sin^2 x}}$$

$$=\frac{\dfrac{\sin^2 x}{\cos^2 x}+\dfrac{\cos^2 x}{\cos^2 x}}{\dfrac{1}{\sin^2 x}-\dfrac{\sin^2 x}{\sin^2 x}}$$

$$=\frac{\dfrac{\sin^2 x+\cos^2 x}{\cos^2 x}}{\dfrac{1-\sin^2 x}{\sin^2 x}}$$

$$=\frac{\dfrac{1}{\cos^2 x}}{\dfrac{\cos^2 x}{\sin^2 x}}$$

$$=\frac{1}{\cos^2 x}\times\frac{\sin^2 x}{\cos^2 x}$$

$$=\sec^2 x\tan^2 x$$

With the special identities

$$\frac{\tan^2 x+1}{\csc^2 x-1}$$

$$=\frac{\sec^2 x}{\cot^2 x}$$

$$=\sec^2 x\tan^2 x$$

Questions: *Prove each of the following:*

1) $\dfrac{\sec x}{\cot x + \tan x} = \sin x$

2) $\dfrac{\sin x + \tan x}{\cos x + 1} = \tan x$

3) $\dfrac{\cos x - \csc x}{\sin x - \sec x} = \cot x$

4) $\dfrac{\sin x + \cos x}{\sec x + \csc x} = \sin x \cos x$

5) $\dfrac{\tan x - \sin x}{\tan x \sin x} = \dfrac{1 - \cos x}{\sin x}$

6) $\dfrac{1 + \cos x}{\tan x + \sin x} = \cot x$

7) $\dfrac{1+\tan^2 x}{1+\cot^2 x} = \tan^2 x$

8) $\dfrac{1}{\sec^2 x} + \dfrac{1}{\csc^2 x} = 1$

9) $\dfrac{1+\tan x}{1+\cot x} = \tan x$

10) $\dfrac{\cos x}{\sec x - 1} + \dfrac{\cos x}{\sec x + 1} = 2\cot^2 x$

11) $\dfrac{\tan x}{1+\tan x} = \dfrac{\sin x}{\sin x + \cos x}$

12) $\dfrac{\sin^2 x}{1-\sin x} + \dfrac{\sin^2 x}{1+\sin x} = 2\tan^2 x$

TRIGONOMETRY LESSON 11

PART IV COMPOUND FRACTIONS & SPECIAL IDENTITIES

1) $\dfrac{\sec x}{\cot x+\tan x}$

$$=\frac{\frac{1}{\cos x}}{\frac{\cos x}{\sin x}+\frac{\sin x}{\cos x}}$$

$$=\frac{\frac{1}{\cos x}}{\left(\frac{\cos x}{\sin x}\right)\frac{\cos x}{\cos x}+\left(\frac{\sin x}{\cos x}\right)\frac{\sin x}{\sin x}}$$

$$=\frac{\frac{1}{\cos x}}{\frac{\cos^2 x}{\sin x\cos x}+\frac{\sin^2 x}{\sin x\cos x}}$$

$$=\frac{\frac{1}{\cos x}}{\frac{\cos^2 x+\sin^2 x}{\sin x\cos x}}$$

$$=\frac{\frac{1}{\cos x}}{\frac{1}{\sin x\cos x}}$$

$$=\frac{1}{\cancel{\cos x}}\times\frac{\sin x\cancel{\cos x}}{1}$$

$$=\sin x$$

2) $\dfrac{\sin x+\tan x}{\cos x+1}$

$$=\frac{\sin x+\frac{\sin x}{\cos x}}{\cos x+1}$$

$$=\frac{\left(\frac{\sin x}{1}\right)\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}{\cos x+1}$$

$$=\frac{\frac{\sin x\cos x}{\cos x}+\frac{\sin x}{\cos x}}{\cos x+1}$$

$$=\frac{\frac{\sin x\cos x+\sin x}{\cos x}}{\cos x+1}$$

$$=\frac{\sin x\cos x+\sin x}{\cos x}\times\frac{1}{\cos x+1}$$

$$=\frac{\sin x\cancel{(\cos x+1)}}{\cos x}\times\frac{1}{\cancel{\cos x+1}}$$

$$=\frac{\sin x}{\cos x}$$

$$=\tan x$$

3) $\dfrac{\cos x-\csc x}{\sin x-\sec x}$

$$=\frac{\cos x-\frac{1}{\sin x}}{\sin x-\frac{1}{\cos x}}$$

$$=\frac{\left(\frac{\cos x}{1}\right)\frac{\sin x}{\sin x}-\frac{1}{\sin x}}{\left(\frac{\sin x}{1}\right)\frac{\cos x}{\cos x}-\frac{1}{\cos x}}$$

$$=\frac{\frac{\cos x\sin x}{\sin x}-\frac{1}{\sin x}}{\frac{\sin x\cos x}{\cos x}-\frac{1}{\cos x}}$$

$$=\frac{\frac{\cos x\sin x-1}{\sin x}}{\frac{\cos x\sin x-1}{\cos x}}$$

$$=\frac{\cancel{\cos x\sin x-1}}{\sin x}\times\frac{\cos x}{\cancel{\cos x\sin x-1}}$$

$$=\frac{\cos x}{\sin x}$$

$$=\cot x$$

4) $\dfrac{\sin x+\cos x}{\sec x+\csc x}$

$$=\frac{\sin x+\cos x}{\frac{1}{\cos x}+\frac{1}{\sin x}}$$

$$=\frac{\sin x+\cos x}{\left(\frac{1}{\cos x}\right)\frac{\sin x}{\sin x}+\left(\frac{1}{\sin x}\right)\frac{\cos x}{\cos x}}$$

$$=\frac{\sin x+\cos x}{\frac{\sin x}{\cos x\sin x}+\frac{\cos x}{\cos x\sin x}}$$

$$=\frac{\sin x+\cos x}{\frac{\sin x+\cos x}{\cos x\sin x}}$$

$$=\frac{\cancel{\sin x+\cos x}}{1}\times\frac{\cos x\sin x}{\cancel{\sin x+\cos x}}$$

$$=\cos x\sin x$$

5) $\dfrac{\tan x-\sin x}{\tan x\sin x}$

$$=\frac{\frac{\sin x}{\cos x}-\sin x}{\left(\frac{\sin x}{\cos x}\right)\sin x}$$

$$=\frac{\frac{\sin x}{\cos x}-\left(\frac{\sin x}{1}\right)\frac{\cos x}{\cos x}}{\frac{\sin^2 x}{\cos x}}$$

$$=\frac{\frac{\sin x}{\cos x}-\frac{\sin x\cos x}{\cos x}}{\frac{\sin^2 x}{\cos x}}$$

$$=\frac{\frac{\sin x-\sin x\cos x}{\cos x}}{\frac{\sin^2 x}{\cos x}}$$

$$=\frac{\sin x-\sin x\cos x}{\cos x}\times\frac{\cos x}{\sin^2 x}$$

$$=\frac{\cancel{\sin x}(1-\cos x)}{\cancel{\cos x}}\times\frac{\cancel{\cos x}}{\cancel{\sin^2} x}$$

$$=\frac{1-\cos x}{\sin x}$$

6) $\dfrac{1+\cos x}{\tan x+\sin x}$

$$=\frac{1+\cos x}{\frac{\sin x}{\cos x}+\sin x}$$

$$=\frac{1+\cos x}{\frac{\sin x}{\cos x}+\left(\frac{\sin x}{1}\right)\frac{\cos x}{\cos x}}$$

$$=\frac{1+\cos x}{\frac{\sin x}{\cos x}+\frac{\sin x\cos x}{\cos x}}$$

$$=\frac{1+\cos x}{\frac{\sin x+\sin x\cos x}{\cos x}}$$

$$=\frac{1+\cos x}{1}\times\frac{\cos x}{\sin x+\sin x\cos x}$$

$$=\frac{\cancel{1+\cos x}}{1}\times\frac{\cos x}{\sin x\cancel{(1+\cos x)}}$$

$$=\frac{\cos x}{\sin x}$$

$$=\cot x$$

7) $\dfrac{1+\tan^2 x}{1+\cot^2 x}$

$= \dfrac{\sec^2 x}{\csc^2 x}$

$= \dfrac{\frac{1}{\cos^2 x}}{\frac{1}{\sin^2 x}}$

$= \dfrac{1}{\cos^2 x} \bullet \sin^2 x$

$= \dfrac{\sin^2 x}{\cos^2 x}$

$= \tan^2 x$

8)

$\dfrac{1}{\sec^2 x} + \dfrac{1}{\csc^2 x}$

$= \cos^2 x + \sin^2 x$

$= 1$

9) $\dfrac{1+\tan x}{1+\cot x}$

$= \dfrac{1+\frac{\sin x}{\cos x}}{1+\frac{\cos x}{\sin x}}$

$= \dfrac{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}{\frac{\sin x}{\sin x}+\frac{\cos x}{\sin x}}$

$= \dfrac{\frac{\cos x+\sin x}{\cos x}}{\frac{\sin x+\cos x}{\sin x}}$

$= \dfrac{\cancel{\cos x+\sin x}}{\cos x} \times \dfrac{\sin x}{\cancel{\sin x+\cos x}}$

$= \dfrac{\sin x}{\cos x}$

$= \tan x$

10) $\dfrac{\cos x}{\sec x - 1} + \dfrac{\cos x}{\sec x + 1}$

$= \dfrac{\cos x}{\sec x - 1}\left(\dfrac{\sec x + 1}{\sec x + 1}\right) + \dfrac{\cos x}{\sec x + 1}\left(\dfrac{\sec x - 1}{\sec x - 1}\right)$

$= \dfrac{\cos x \sec x + \cos x}{(\sec x - 1)(\sec x + 1)} + \dfrac{\cos x \sec x - \cos x}{(\sec x + 1)(\sec x - 1)}$

$= \dfrac{\cos x \sec x + \cos x + \cos x \sec x - \cos x}{(\sec x - 1)(\sec x + 1)}$

$= \dfrac{2\cos x \sec x}{(\sec x - 1)(\sec x + 1)}$

$= \dfrac{2\cos x \sec x}{(\sec^2 x - 1)}$

$= \dfrac{2\cos x \sec x}{\tan^2 x}$

$= \dfrac{2\cancel{\cos x}\frac{1}{\cancel{\cos x}}}{\tan^2 x}$

$= \dfrac{2}{\tan^2 x}$

$= 2\cot^2 x$

11)

$\dfrac{\tan x}{1+\tan x}$

$= \dfrac{\frac{\sin x}{\cos x}}{1+\frac{\sin x}{\cos x}}$

$= \dfrac{\frac{\sin x}{\cos x}}{\frac{\cos x}{\cos x}+\frac{\sin x}{\cos x}}$

$= \dfrac{\frac{\sin x}{\cos x}}{\frac{\cos x+\sin x}{\cos x}}$

$= \dfrac{\sin x}{\cancel{\cos x}} \times \dfrac{\cancel{\cos x}}{\cos x+\sin x}$

$= \dfrac{\sin x}{\cos x+\sin x}$

12)

$\dfrac{\sin^2 x}{1-\sin x} + \dfrac{\sin^2 x}{1+\sin x}$

$= \dfrac{\sin^2 x}{1-\sin x}\left(\dfrac{1+\sin x}{1+\sin x}\right) + \dfrac{\sin^2 x}{1+\sin x}\left(\dfrac{1-\sin x}{1-\sin x}\right)$

$= \dfrac{\sin^2 x+\sin^3 x}{(1-\sin x)(1+\sin x)} + \dfrac{\sin^2 x-\sin^3 x}{(1+\sin x)(1-\sin x)}$

$= \dfrac{\sin^2 x+\sin^3 x+\sin^2 x-\sin^3 x}{(1-\sin x)(1+\sin x)}$

$= \dfrac{2\sin^2 x}{(1-\sin x)(1+\sin x)}$

$= \dfrac{2\sin^2 x}{(1-\sin^2 x)}$

$= \dfrac{2\sin^2 x}{\cos^2 x}$

$= 2\tan^2 x$

Difference of Squares: See the review on the side of the page, then study the example.

Example 1: Prove: $\sec^4 x - 1 = \dfrac{\sin^2 x + \sin^2 x \cos^2 x}{\cos^4 x}$

$\sec^4 x - 1$

$= (\sec^2 x - 1)(\sec^2 x + 1)$

$= \tan^2 x(\sec^2 x + 1)$

$= \dfrac{\sin^2 x}{\cos^2 x}\left(\dfrac{1}{\cos^2 x} + 1\right)$

$= \dfrac{\sin^2 x}{\cos^2 x}\left(\dfrac{1}{\cos^2 x} + \dfrac{\cos^2 x}{\cos^2 x}\right)$

$= \dfrac{\sin^2 x}{\cos^2 x}\left(\dfrac{1+\cos^2 x}{\cos^2 x}\right)$

$= \dfrac{\sin^2 x + \sin^2 x \cos^2 x}{\cos^4 x}$

Rearranging an expression to make an identity:

Example 2:

Prove that $(\cot x - 1)^2 = \csc^2 x - 2\cot x$

$(\cot x - 1)^2$

$= (\cot x - 1)(\cot x - 1)$

$= \cot^2 x - 2\cot x + 1$

$= \cot^2 x + 1 - 2\cot x$ *Rearranging to allow use of special identity

$= \csc^2 x - 2\cot x$

Factoring out a negative to make an identity:

Example 3:

Prove: $4 - \sec^2 x + 1 = 4 - \tan^2 x$

$4 - \sec^2 x + 1$

$= 4 - (\sec^2 x - 1)$

$= 4 - \tan^2 x$

Difference of Squares

You can recognize a difference of squares by the following:

- *It is a binomial, with a minus in the middle. (Watch out for binomials with a plus, this will **not** be a difference of squares.)*
- *The first & last terms are perfect squares.*

Example 1: Factor $4\sin^2 x - 9\cos^2 x$

$4\sin^2 x - 9\cos^2 x$

Square root first term and put first in each bracket.

Square root second term and put last in each bracket.

$= (2\sin x - 3\cos x)(2\sin x + 3\cos x)$

In one bracket put a plus, and a minus in the other.

Example 2: Factor $x^8 - y^8$

(Watch for multiple difference of squares)

$x^8 - y^8$

$= (x^4 - y^4)(x^4 + y^4)$

$= (x^2 - y^2)(x^2 + y^2)(x^4 + y^4)$

$= (x - y)(x + y)(x^2 + y^2)(x^4 + y^4)$

TRIGONOMETRY LESSON 11

PART V OTHER PROOFS

1) $\dfrac{3\tan x}{1+\tan^2 x} = 3\sin x \cos x$

2) $\dfrac{1}{1+\cot^2 x} = \sin^2 x$

3) $\sec^2 x - \cos^2 x - \sin^2 x = \tan^2 x$

4) $(\sin x + \cos x)^2 + (\sin x - \cos x)^2 = 2$

5) $(1+\sin x)^2 + \cos^2 x = 2(1+\sin x)$

6) $\sin^4 x - \cos^4 x = 2\sin^2 x - 1$

PART V OTHER PROOFS

7) $(\tan x - 1)^2 = \sec^2 x - 2\tan x$

8) $(1 - \sec^2 x)(1 - \sin^2 x) = -\sin^2 x$

9) $\csc x - \csc^3 x = \dfrac{-\cos^2 x}{\sin^3 x}$

10) $\csc^4 x - 1 = \dfrac{\cos^2 x(1 + \sin^2 x)}{\sin^4 x}$

11) $\tan^2 x - \cot^2 x = \dfrac{\sin^2 x - \cos^2 x}{\sin^2 x \cos^2 x}$

TRIGONOMETRY LESSON 11

PART V OTHER PROOFS

1)
$$\frac{3\tan x}{1+\tan^2 x}$$
$$=\frac{3\tan x}{\sec^2 x}$$
$$=\frac{3\dfrac{\sin x}{\cos x}}{\dfrac{1}{\cos^2 x}}$$
$$=3\frac{\sin x}{\cancel{\cos x}}\times\frac{\cancel{\cos^2} x}{1}$$
$$=3\sin x\cos x$$

2)
$$\frac{1}{1+\cot^2 x}$$
$$=\frac{1}{\csc^2 x}$$
$$=\frac{1}{\dfrac{1}{\sin^2 x}}$$
$$=1\times\frac{\sin^2 x}{1}$$
$$=\sin^2 x$$

3)
$$\sec^2 x-\cos^2 x-\sin^2 x$$
$$\sec^2 x-(\cos^2 x+\sin^2 x)$$
$$\sec^2 x-1$$
$$\tan^2 x$$

4)
$$(\sin x+\cos x)^2+(\sin x-\cos x)^2$$
$$=\sin^2 x+2\sin x\cos x+\cos^2 x+\sin^2 x-2\sin x\cos x+\cos^2 x$$
$$=2\sin^2 x+2\cos^2 x$$
$$=2(\sin^2 x+\cos^2 x)$$
$$=2$$

5)
$$(1+\sin x)^2+\cos^2 x$$
$$=1+2\sin x+\sin^2 x+\cos^2 x$$
$$=1+2\sin x+1$$
$$=2+2\sin x$$
$$=2(1+\sin x)$$

6)
$$\sin^4 x-\cos^4 x$$
$$=(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)$$
$$=(\sin^2 x-\cos^2 x)$$
$$=\sin^2 x-(1-\sin^2 x)$$
$$=\sin^2 x-1+\sin^2 x$$
$$=2\sin^2 x-1$$

7)
$$(\tan x-1)^2$$
$$=(\tan x-1)(\tan x-1)$$
$$=\tan^2 x-2\tan x+1$$
$$=\tan^2 x+1-2\tan x$$
$$=\sec^2 x-2\tan x$$

8)
$$(1-\sec^2 x)(1-\sin^2 x)$$
$$=(-\tan^2 x)(\cos^2 x)$$
$$=-\frac{\sin^2 x}{\cos^2 x}\cos^2 x$$
$$=-\sin^2 x$$

9)
$$\csc x-\csc^3 x$$
$$=\csc x(1-\csc^2 x)$$
$$=\csc x(-\cot^2 x)$$
$$=\frac{1}{\sin x}\left(-\frac{\cos^2 x}{\sin^2 x}\right)$$
$$=-\frac{\cos^2 x}{\sin^3 x}$$

10)
$$\csc^4 x-1$$
$$=(\csc^2 x-1)(\csc^2 x+1)$$
$$=\cot^2 x(\csc^2 x+1)$$
$$=\frac{\cos^2 x}{\sin^2 x}\left(\frac{1}{\sin^2 x}+1\right)$$
$$=\frac{\cos^2 x}{\sin^4 x}+\frac{\cos^2 x}{\sin^2 x}$$
$$=\frac{\cos^2 x}{\sin^4 x}+\frac{\cos^2 x}{\sin^2 x}\left(\frac{\sin^2 x}{\sin^2 x}\right)$$
$$=\frac{\cos^2 x}{\sin^4 x}+\frac{\cos^2 x\sin^2 x}{\sin^4 x}$$
$$=\frac{\cos^2 x+\cos^2 x\sin^2 x}{\sin^4 x}$$
$$=\frac{\cos^2 x(1+\sin^2 x)}{\sin^4 x}$$

11)
$$\tan^2 x-\cot^2 x$$
$$=\frac{\sin^2 x}{\cos^2 x}-\frac{\cos^2 x}{\sin^2 x}$$
$$=\frac{\sin^2 x}{\cos^2 x}\left(\frac{\sin^2 x}{\sin^2 x}\right)-\frac{\cos^2 x}{\sin^2 x}\left(\frac{\cos^2 x}{\cos^2 x}\right)$$
$$=\frac{\sin^4 x}{\cos^2 x\sin^2 x}-\frac{\cos^4 x}{\sin^2 x\cos^2 x}$$
$$=\frac{\sin^4 x-\cos^4 x}{\cos^2 x\sin^2 x}$$
$$=\frac{(\sin^2 x-\cos^2 x)(\sin^2 x+\cos^2 x)}{\cos^2 x\sin^2 x}$$
$$=\frac{(\sin^2 x-\cos^2 x)}{\cos^2 x\sin^2 x}$$

TRIGONOMETRY LESSON 11

PART VI CONJUGATES

Sometimes you will get identities that can't be broken down any further. In these cases, you can multiply numerator & denominator by the conjugate. This will convert the fraction into something that will give you identities to work with.

The conjugate is obtained by taking a binomial from the original expression and changing the sign in the middle.

Example 1: Prove $\frac{1+\cos x}{\sin x} = \frac{\sin x}{1-\cos x}$

$\frac{1+\cos x}{\sin x}$ has the conjugate: **1 - cos x**

$$\frac{1+\cos x}{\sin x}$$

$$= \frac{1+\cos x}{\sin x}\left(\frac{1-\cos x}{1-\cos x}\right)$$

$$= \frac{1-\cos^2 x}{\sin x(1-\cos x)}$$

$$= \frac{\sin^2 x}{\sin x(1-\cos x)}$$

$$= \frac{\sin x}{1-\cos x}$$

Prove each of the following identities:

1) $\frac{\cos x}{1-\sin x} = \frac{1+\sin x}{\cos x}$ **2)** $\frac{1}{1-\sin x} = \frac{1+\sin x}{\cos^2 x}$ **3)** $\frac{1-\cos x}{\sin x} = \frac{\sin x}{1+\cos x}$ **4)** $\frac{1-\sin x}{\cos x} = \frac{\cos x}{1+\sin x}$

TRIGONOMETRY LESSON 11

PART VI CONJUGATES

1) $\dfrac{\cos x}{1-\sin x}$

$$= \frac{\cos x}{1-\sin x}\left(\frac{1+\sin x}{1+\sin x}\right)$$

$$= \frac{\cos x(1+\sin x)}{1-\sin^2 x}$$

$$= \frac{\cancel{\cos x}(1+\sin x)}{\cos^{\cancel{2}} x}$$

$$= \frac{(1+\sin x)}{\cos x}$$

2) $\dfrac{1}{1-\sin x}$

$$= \frac{1}{1-\sin x}\left(\frac{1+\sin x}{1+\sin x}\right)$$

$$= \frac{1+\sin x}{1-\sin^2 x}$$

$$= \frac{1+\sin x}{\cos^2 x}$$

3) $\dfrac{1-\cos x}{\sin x}$

$$= \frac{1-\cos x}{\sin x}\left(\frac{1+\cos x}{1+\cos x}\right)$$

$$= \frac{1-\cos^2 x}{\sin x(1+\cos x)}$$

$$= \frac{\sin^2 x}{\sin x(1+\cos x)}$$

$$= \frac{\sin x}{1+\cos x}$$

4) $\dfrac{1-\sin x}{\cos x}$

$$= \frac{1-\sin x}{\cos x}\left(\frac{1+\sin x}{1+\sin x}\right)$$

$$= \frac{1-\sin^2 x}{\cos x(1+\sin x)}$$

$$= \frac{\cos^{\cancel{2}} x}{\cancel{\cos x}(1+\sin x)}$$

$$= \frac{\cos x}{1+\sin x}$$