Pure Math 30:

TRIGONOMETRY II

$$(\cos x - 1)(\tan x - 1) = 0$$

LESSON TWELVE

Sum & Difference Identities

Pure Math 30: EXPLAINED!

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Sum & Difference Identities: ***The following formulas are used to expand trigonometric functions that have addition & subtraction in brackets.***

$\sin(A + B) \neq \sin A + \sin B$, *so we must use these rules whenever we want to expand.*

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$

$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$

$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

Example 1: Expand $\sin(60° - 45°)$

$$\sin(60^\circ - 45^\circ)$$
$$= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ$$
$$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$$
$$= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 2: Expand $\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

$$\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$$
$$= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6}$$
$$= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$$
$$= \left(\frac{\sqrt{6}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right)$$
$$= \frac{\sqrt{6} - \sqrt{2}}{4}$$

Trigonometry Lesson 12
Part I Expanding Sum & Difference

Find the exact value by expanding each of the following:

1) $\sin(45^\circ + 60^\circ)$

5) $\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

2) $\cos(45^\circ - 30^\circ)$

6) $\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

3) $\sin(60^\circ - 135^\circ)$

7) $\cos\left(0 - \frac{3\pi}{4}\right)$

4) $\cos(150^\circ + 45^\circ)$

8) $\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$

1. $\sin(45^\circ + 60^\circ)$

$=\sin45^\circ\cos60^\circ+\cos45^\circ\sin60^\circ$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$

$=\frac{\sqrt{2}+\sqrt{6}}{4}$

2. $\cos(45^\circ - 30^\circ)$

$=\cos45^\circ\cos30^\circ+\sin45^\circ\sin30^\circ$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$

$=\frac{\sqrt{6}+\sqrt{2}}{4}$

3. $\sin(60^\circ - 135^\circ)$

$=\sin60^\circ\cos135^\circ - \cos60^\circ\sin135^\circ$

$=\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\frac{-\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$

$=\frac{-\sqrt{6}-\sqrt{2}}{4}$

4. $\cos(150^\circ + 45^\circ)$

$=\cos150^\circ\cos45^\circ - \sin150^\circ\sin45^\circ$

$=\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\frac{-\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$

$=\frac{-\sqrt{6}-\sqrt{2}}{4}$

5. $\sin(\frac{\pi}{2} - \frac{\pi}{3})$

$=\sin\frac{\pi}{2}\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\sin\frac{\pi}{3}$

$=(1)\left(\frac{1}{2}\right)-(0)\left(\frac{\sqrt{3}}{2}\right)$

$=\frac{1}{2}$

6. $\sin(\frac{\pi}{4} + \frac{\pi}{6})$

$=\sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$

$=\frac{\sqrt{6}+\sqrt{2}}{4}$

7. $\cos(0-\frac{3\pi}{4})$

$=\cos0\cos\frac{3\pi}{4}+\sin0\sin\frac{3\pi}{4}$

$=(1)\left(-\frac{\sqrt{2}}{2}\right)+(0)\left(\frac{\sqrt{2}}{2}\right)$

$=-\frac{\sqrt{2}}{2}$

8. $\cos(\frac{\pi}{2}+\frac{\pi}{3})$

$=\cos\frac{\pi}{2}\cos\frac{\pi}{3}-\sin\frac{\pi}{2}\sin\frac{\pi}{3}$

$=(0)\left(\frac{1}{2}\right)-(1)\left(\frac{\sqrt{3}}{2}\right)$

$=-\frac{\sqrt{3}}{2}$

Given an expanded form, we can work backwards to find a single trigonometric expression that can be easily solved using the unit circle.

Example 1: Express $\sin 85^\circ \cos 5^\circ + \cos 85^\circ \sin 5^\circ$ as a single trigonometric expression and solve.

We know $\sin(A+B) = \sin A \cos B + \cos A \sin B$

Looking at the question and matching to the sum formula, we can see that:
$A = 85^\circ$ and $B = 5^\circ$
Now plug A & B into the *left* side of the formula:

$\sin(A + B)$
$= \sin(85^\circ + 5^\circ)$
$= \sin(90^\circ)$
$= 1$

Example 2: Express $\dfrac{1}{\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)}$ as a single trigonometric expression and solve.

We know: $\sin(A - B) = \sin A \cos B - \cos A \sin B$

We have from the denominator of the question: $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$

Comparing the two, $A = \frac{\pi}{3}$ & $B = \frac{\pi}{6}$

Plug A & B into $\sin(A - B)$

$= \sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$

$= \sin\left(\frac{2\pi}{6} - \frac{\pi}{6}\right)$

$= \sin\left(\frac{\pi}{6}\right)$

Subtract radians by finding a common denominator. (Or think in terms of degrees.)

We now know the denominator is $\sin\left(\frac{\pi}{6}\right)$.

Thus, we have $\dfrac{1}{\sin\left(\frac{\pi}{6}\right)} = \csc\left(\frac{\pi}{6}\right)$. Solving, $\dfrac{1}{\sin\left(\frac{\pi}{6}\right)} = \dfrac{1}{\frac{1}{2}} = 2$

For each of the following, express as a single trigonometric expression and solve using the unit circle.

1) $\cos 60^\circ \cos 15^\circ + \sin 60^\circ \sin 15^\circ$

2) $\cos^2 \dfrac{\pi}{6} - \sin^2 \dfrac{\pi}{6}$

3) $\dfrac{1}{\sin\left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{\pi}{3}\right) - \cos\left(\dfrac{\pi}{2}\right)\sin\left(\dfrac{\pi}{3}\right)}$

4) $\sin\left(\dfrac{5\pi}{12}\right)\cos\left(\dfrac{\pi}{3}\right) + \cos\left(\dfrac{5\pi}{12}\right)\sin\left(\dfrac{\pi}{3}\right)$

5) $\dfrac{1}{\cos(-15^\circ)\cos(30^\circ)+\sin(-15^\circ)\sin(30^\circ)}$

6) $\sin\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{6}\right)-\sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{3}\right)$

7) $\dfrac{1}{\cos\left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{\pi}{4}\right)+\sin\left(\dfrac{\pi}{2}\right)\sin\left(\dfrac{\pi}{4}\right)}$

8) $\cos\left(\dfrac{2\pi}{3}\right)\cos\left(\dfrac{\pi}{2}\right)+\sin\left(\dfrac{2\pi}{3}\right)\sin\left(\dfrac{\pi}{2}\right)$

1. $\cos 60^\circ \cos 15^\circ + \sin 60^\circ \sin 15^\circ$

$A = 60^\circ \quad \& \quad B = 15^\circ$

$\cos(60^\circ - 15^\circ)$

$\cos(45^\circ)$

$= \dfrac{\sqrt{2}}{2}$

2. $\cos\dfrac{\pi}{6}\cos\dfrac{\pi}{6} - \sin\dfrac{\pi}{6}\sin\dfrac{\pi}{6}$

$A = \dfrac{\pi}{6} \quad \& \quad B = \dfrac{\pi}{6}$

$\cos(\dfrac{\pi}{6} + \dfrac{\pi}{6})$

$= \cos(\dfrac{2\pi}{6})$

$= \cos(\dfrac{\pi}{3})$

$= \dfrac{1}{2}$

3. $\dfrac{1}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{3}\right)}$

$A = \dfrac{\pi}{2} \quad \& \quad B = \dfrac{\pi}{3}$

$\dfrac{1}{\sin(\frac{\pi}{2} - \frac{\pi}{3})}$

$= \csc(\dfrac{\pi}{2} - \dfrac{\pi}{3})$

$= \csc(\dfrac{3\pi}{6} - \dfrac{2\pi}{6})$ Getting A Common Denominator

$= \csc(\dfrac{\pi}{6})$

$= 2$

4. $\sin\left(\dfrac{5\pi}{12}\right)\cos\left(\dfrac{\pi}{3}\right) + \cos\left(\dfrac{5\pi}{12}\right)\sin\left(\dfrac{\pi}{3}\right)$

$A = \dfrac{5\pi}{12} \quad \& \quad B = \dfrac{\pi}{3}$

$\sin(\dfrac{5\pi}{12} + \dfrac{\pi}{3})$

$= \sin(\dfrac{5\pi}{12} + \dfrac{4\pi}{12})$

$= \sin(\dfrac{9\pi}{12})$

$= \sin(\dfrac{3\pi}{4})$

$= \dfrac{\sqrt{2}}{2}$

5. $\dfrac{1}{\cos(-15^\circ)\cos(30^\circ) + \sin(-15^\circ)\sin(30^\circ)}$

$A = -15^\circ \quad \& \quad B = 30^\circ$

$\dfrac{1}{\cos(-15^\circ - 30^\circ)}$

$= \dfrac{1}{\cos\left(-45^\circ\right)}$

$= \dfrac{1}{\frac{\sqrt{2}}{2}}$

$= \dfrac{2}{\sqrt{2}}$

$= \dfrac{2}{\sqrt{2}} \times \dfrac{\sqrt{2}}{\sqrt{2}}$

$= \sqrt{2}$

6. $\sin\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{6}\right) - \sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{3}\right)$

$A = \dfrac{\pi}{3} \quad \& \quad B = \dfrac{\pi}{6}$

$\sin(\dfrac{\pi}{3} - \dfrac{\pi}{6})$

$= \sin(\dfrac{2\pi}{6} - \dfrac{\pi}{6})$

$= \sin\dfrac{\pi}{6}$

$= \dfrac{1}{2}$

7. $\dfrac{1}{\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)}$

$A = \dfrac{\pi}{2} \quad \& \quad B = \dfrac{\pi}{4}$

$\dfrac{1}{\cos(\frac{\pi}{2} - \frac{\pi}{4})}$

$= \sec(\dfrac{\pi}{2} - \dfrac{\pi}{4})$

$= \sec(\dfrac{2\pi}{4} - \dfrac{\pi}{4})$

$= \sec(\dfrac{\pi}{4})$

$= \dfrac{2}{\sqrt{2}}$

$= \sqrt{2}$

8. $\cos\left(\dfrac{2\pi}{3}\right)\cos\left(\dfrac{\pi}{2}\right) + \sin\left(\dfrac{2\pi}{3}\right)\sin\left(\dfrac{\pi}{2}\right)$

$A = \dfrac{2\pi}{3} \quad \& \quad B = \dfrac{\pi}{2}$

$\cos(\dfrac{2\pi}{3} - \dfrac{\pi}{2})$

$= \cos(\dfrac{4\pi}{6} - \dfrac{3\pi}{6})$

$= \cos\dfrac{\pi}{6}$

$= \dfrac{\sqrt{3}}{2}$

The sum & difference formulas are useful in determining the exact values of sine & cosine for angles not on the unit circle.

Example 1: Find the exact value of sin15°

First, think of how you can get 15° by using angles on the unit circle:
15° = 60° - 45°
15° = 45° - 30°
15° = 135° - 120°
15° = -30° + 45°

As you can see, there are many possibilities, you can choose any of them and still get the right answer.

We'll use the top one, 15° = 60° - 45°

$$\begin{aligned}&\sin 15^\circ\\&=\sin(60^\circ-45^\circ)\\&=\sin 60^\circ\cos 45^\circ-\cos 60^\circ\sin 45^\circ\\&=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\&=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

Example 2: Find the exact value of $\sec\left(-\frac{5\pi}{12}\right)$

It's easiest to think in terms of degrees, so convert: $-\frac{5\pi}{12}\times\frac{180^\circ}{\pi}=-75^\circ$ (-75° = -45° - 30°)

$$\begin{aligned}&\sec(-75^\circ)\\&=\sec(-45^\circ-30^\circ)\\&=\frac{1}{\cos(-45^\circ-30^\circ)}\\&=\frac{1}{\cos(-45^\circ)\cos(30^\circ)+\sin(-45^\circ)\sin(30^\circ)}\\&=\frac{1}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)}\\&=\frac{1}{\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}}\\&=\frac{1}{\frac{\sqrt{6}-\sqrt{2}}{4}}\\&=\frac{4}{\sqrt{6}-\sqrt{2}}\end{aligned}$$

Now rationalize the denominator of your answer.

$$\begin{aligned}&\frac{4}{\sqrt{6}-\sqrt{2}}\\&=\frac{4}{\sqrt{6}-\sqrt{2}}\times\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}\\&=\frac{4(\sqrt{6}+\sqrt{2})}{6-2}\\&=\frac{4(\sqrt{6}+\sqrt{2})}{4}\\&=\sqrt{6}+\sqrt{2}\end{aligned}$$

Find the exact value of each of the following:

Note that there are multiple ways of getting to the correct answer. Rationalize the denominator when necessary.

1) $\cos(-15^\circ)$

2) $\sec(105^\circ)$

3) $\csc(-105^\circ)$

4) $\sin(-\frac{5\pi}{12})$

5) $\csc(165^\circ)$

6) $\sec(\frac{13\pi}{12})$

1. $\cos(30^\circ - 45^\circ)$

$= \cos 30^\circ \cos 45^\circ + \sin 30^\circ \sin 45^\circ$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

2. $\sec(60^\circ + 45^\circ)$

$= \frac{1}{\cos(60^\circ + 45^\circ)}$

$= \frac{1}{\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ}$

$= \frac{1}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}}$

$= \frac{1}{\frac{\sqrt{2}+\sqrt{6}}{4}}$

$= \frac{4}{\sqrt{2}+\sqrt{6}}$

$= \frac{4}{\sqrt{2}+\sqrt{6}} \times \frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$

$= \frac{4(\sqrt{2}-\sqrt{6})}{2-6}$

$= \frac{4(\sqrt{2}-\sqrt{6})}{-4}$

$= -1(\sqrt{2}-\sqrt{6})$

$= \sqrt{6}-\sqrt{2}$

3. $\csc(-60^\circ - 45^\circ)$

$= \frac{1}{\sin(-60^\circ - 45^\circ)}$

$= \frac{1}{\sin(-60^\circ)\cos 45^\circ - \cos(-60^\circ)\sin 45^\circ}$

$= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$

$= \frac{1}{\frac{-\sqrt{6}-\sqrt{2}}{4}}$

$= \frac{4}{-\sqrt{6}-\sqrt{2}}$

$= \frac{4}{-\sqrt{6}-\sqrt{2}} \times \frac{-\sqrt{6}+\sqrt{2}}{-\sqrt{6}+\sqrt{2}}$

$= \frac{4(-\sqrt{6}+\sqrt{2})}{6-2}$

$= \frac{4(-\sqrt{6}+\sqrt{2})}{4}$

$= -\sqrt{6}+\sqrt{2}$

4. $\sin(-\frac{5\pi}{12})$

$= \sin(-75^\circ)$

$= \sin(-45^\circ - 30^\circ)$

$= \sin(-45^\circ)\cos 30^\circ - \cos(-45^\circ)\sin 30^\circ$

$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$= \frac{-(\sqrt{6}+\sqrt{2})}{4}$

5. $\csc(165^\circ)$

$= \frac{1}{\sin(165^\circ)}$

$= \frac{1}{\sin(120^\circ + 45^\circ)}$

$= \frac{1}{\sin(120^\circ)\cos 45^\circ + \cos(120^\circ)\sin 45^\circ}$

$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$

$= \frac{1}{\frac{\sqrt{6}-\sqrt{2}}{4}}$

$= \frac{4}{\sqrt{6}-\sqrt{2}}$

$= \frac{4}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$

$= \frac{4(\sqrt{6}+\sqrt{2})}{4}$

$= \sqrt{6}+\sqrt{2}$

6. $\sec\left(\frac{13\pi}{12}\right)$

$= \sec(195^\circ)$

$= \sec(150^\circ + 45^\circ)$

$= \frac{1}{\cos(150^\circ + 45^\circ)}$

$= \frac{1}{\cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ}$

$= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$

$= \frac{1}{\frac{-\sqrt{6}-\sqrt{2}}{4}}$

$= -\frac{4}{(\sqrt{6}+\sqrt{2})}$

$= -\frac{4}{(\sqrt{6}+\sqrt{2})} \times \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$

$= -\frac{4(\sqrt{6}-\sqrt{2})}{4}$

$= -(\sqrt{6}-\sqrt{2})$

The following examples illustrate some basic proofs you can do with the sum & difference identities:

Example 1: Prove that $\cos(\frac{\pi}{2} - x) = \sin x$

We start with the formula: $\cos(A - B) = \cos A \cos B + \sin A \sin B$

Plug in $A = \frac{\pi}{2}$ & $B = x$

$$\cos(\frac{\pi}{2} - x) = \cos A \cos B + \sin A \sin B$$
$$= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= (0)\cos x + (1)\sin x$$
$$= \sin x$$

Example 2: Prove that $\csc(\pi + x) = -\csc x$

$$\csc(\pi + x)$$
$$= \frac{1}{\sin(\pi + x)}$$
$$= \frac{1}{\sin\pi\cos x + \cos\pi\sin x}$$
$$= \frac{1}{(0)\cos x + (-1)\sin x}$$
$$= \frac{1}{-\sin x}$$
$$= -\csc x$$

Prove each of the following:

1) $\cos(\frac{3\pi}{2} - x) = -\sin x$

2) $\sin(270^\circ - x) = -\cos x$

3) $\cos(\frac{\pi}{2} + x) = -\sin x$

4) $\cos(\pi - x) =$

5) $\sin(\frac{\pi}{2} - x) =$

6) $\csc(\frac{\pi}{2} + x) =$

7) $\sec(\pi + x)$

8) $\csc(\pi - x)$

1. $\cos(\frac{3\pi}{2}-x)$
$= \cos\frac{3\pi}{2}\cos x + \sin\frac{3\pi}{2}\sin x$
$= (0)\cos x + (-1)\sin x$
$= -\sin x$

2. $\sin(270^\circ - x)$
$= \sin 270^\circ \cos x - \cos 270^\circ \sin x$
$= (-1)\cos x - (0)\sin x$
$= -\cos x$

3. $\cos(\frac{\pi}{2}+x)$
$= \cos\frac{\pi}{2}\cos x - \sin\frac{\pi}{2}\sin x$
$= (0)\cos x - (1)\sin x$
$= -\sin x$

4. $\cos(\pi - x)$
$= \cos\pi\cos x + \sin\pi\sin x$
$= (-1)\cos x + (0)\sin x$
$= -\cos x$

5. $\sin(\frac{\pi}{2}-x)$
$= \sin\frac{\pi}{2}\cos x - \cos\frac{\pi}{2}\sin x$
$= (1)\cos x - (0)\sin x$
$= \cos x$

6. $\csc(\frac{\pi}{2}+x)$
$= \dfrac{1}{\sin(\frac{\pi}{2}+x)}$
$= \dfrac{1}{\sin\frac{\pi}{2}\cos x + \cos\frac{\pi}{2}\sin x}$
$= \dfrac{1}{(1)\cos x + (0)\sin x}$
$= \dfrac{1}{\cos x}$
$= \sec x$

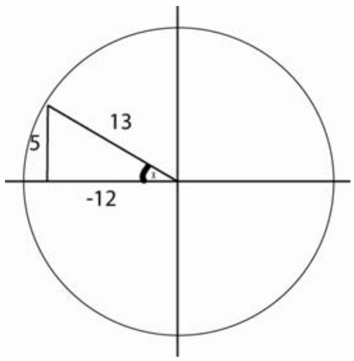
7. $\sec(\pi + x)$
$= \dfrac{1}{\cos(\pi + x)}$
$= \dfrac{1}{\cos\pi\cos x - \sin\pi\sin x}$
$= \dfrac{1}{(-1)\cos x - (0)\sin x}$
$= \dfrac{1}{-\cos x}$
$= -\sec x$

8. $\csc(\pi - x)$
$= \dfrac{1}{\sin(\pi - x)}$
$= \dfrac{1}{\sin\pi\cos x - \cos\pi\sin x}$
$= \dfrac{1}{(0)\cos x - (-1)\sin x}$
$= \dfrac{1}{\sin x}$
$= \csc x$

In some questions, you will be given incomplete information, and must use triangles to find all the trig ratios required by the sum & difference formulas.

Example 1: Given $\tan x = -\frac{5}{12}$ (In quadrant II) and $\tan y = \frac{3}{5}$ (In quadrant III), find the exact value of $\sec(x+y)$

Draw a triangle corresponding to $\tan x = -\frac{5}{12}$ and find the unknown side.

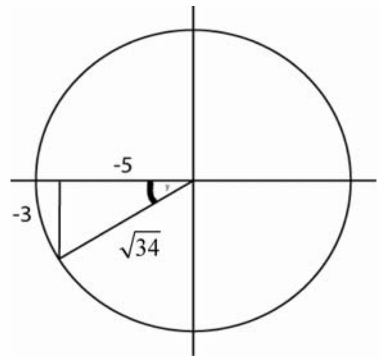


$$a^2 + b^2 = c^2$$
$$(-12)^2 + (5)^2 = c^2$$
$$169 = c^2$$
$$c = 13$$

Now find $\sin x$ & $\cos x$.

$$\sin x = \frac{5}{13}$$
$$\cos x = \frac{-12}{13}$$

Draw a triangle corresponding to $\tan y = \frac{3}{5}$ and find the unknown side.



$$a^2 + b^2 = c^2$$
$$(-5)^2 + (-3)^2 = c^2$$
$$34 = c^2$$
$$c = \sqrt{34}$$

Now find $\sin y$ & $\cos y$

$$\sin y = \frac{-3}{\sqrt{34}}$$
$$\cos y = \frac{-5}{\sqrt{34}}$$

$$\sec(x+y) = \frac{1}{\cos(x+y)}$$
$$= \frac{1}{\cos x \cos y - \sin x \sin y}$$
$$= \frac{1}{\left(\frac{-12}{13}\right)\left(\frac{-5}{\sqrt{34}}\right) - \left(\frac{5}{13}\right)\left(\frac{-3}{\sqrt{34}}\right)}$$
$$= \frac{1}{\frac{60}{13\sqrt{34}} + \frac{15}{13\sqrt{34}}}$$
$$= \frac{1}{\frac{75}{13\sqrt{34}}}$$
$$= \frac{13\sqrt{34}}{75}$$ **Final Answer**

1)

$\sin x = -\frac{1}{2}$ (Quadrant III)

$\tan y = \frac{3}{4}$ (Quadrant I)

Find $\cos(x-y)$

2)

$\cos x = \frac{12}{13}$ (Quadrant IV)

$\csc y = -\frac{5}{2}$ (Quadrant III)

Find $\csc(x+y)$

3)

$\sec x = \frac{7}{5}$ (Quadrant I)

$\cot y = -\frac{3}{4}$ (Quadrant II)

Find $\sin(x-y)$

4)
$\tan x = \frac{-6}{7}$ (Quadrant IV)

$\sin y = -\frac{2}{5}$ (Quadrant IV)

$\boxed{\text{Find } \sec(x+y)}$

5)
$\cot x = -5$ $(\cos x < 0)$

$\tan y = \frac{3}{4}$ $(\sin y > 0)$

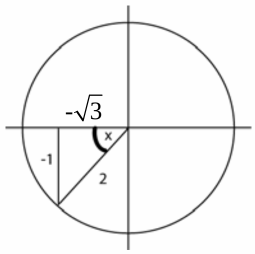
$\boxed{\text{Find } \csc(x-y)}$

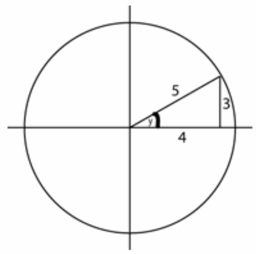
6)
$\sec x = -\frac{8}{7}$ $(\sin x > 0)$

$\tan y = -3$ $(\sin y < 0)$

$\boxed{\text{Find } \sec(x-y)}$

1) First draw out each triangle, then use Pythagoras to find the unknown side:





Now find the sine & cosine trig ratios for each triangle:

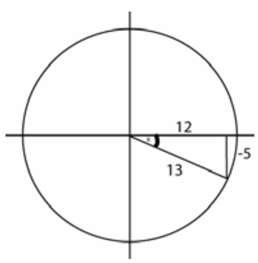
$$\sin x = \frac{-1}{2} \qquad \sin y = \frac{3}{5}$$

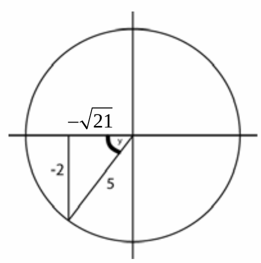
$$\cos x = \frac{-\sqrt{3}}{2} \qquad \cos y = \frac{4}{5}$$

Use these to evaluate $\cos(x-y)$

$$\cos(x-y)$$
$$= \cos x \cos y + \sin x \sin y$$
$$= \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{4}{5}\right) + \left(\frac{-1}{2}\right)\left(\frac{3}{5}\right)$$
$$= \frac{-4\sqrt{3}}{10} - \frac{3}{10}$$
$$= \frac{-4\sqrt{3}-3}{10}$$

2) First draw out each triangle, then use Pythagoras to find the unknown side:





Now state the sine cosine trig ratios for each triangle:

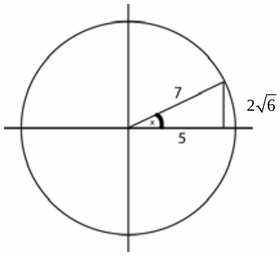
$$\sin x = \frac{-5}{13} \qquad \sin y = \frac{-2}{5}$$

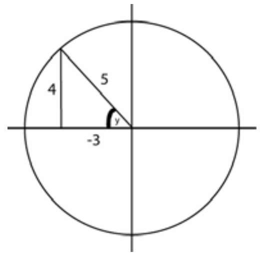
$$\cos x = \frac{12}{13} \qquad \cos y = \frac{-\sqrt{21}}{5}$$

Use these to evaluate $\csc(x+y)$

$$\csc(x+y)$$
$$= \frac{1}{\sin x \cos y + \cos x \sin y}$$
$$= \frac{1}{\left(\frac{-5}{13}\right)\left(\frac{-\sqrt{21}}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{-2}{5}\right)}$$
$$= \frac{1}{\left(\frac{-5\sqrt{21}}{65}\right) + \left(\frac{-24}{65}\right)}$$
$$= \frac{1}{\frac{-5\sqrt{21}-24}{65}}$$
$$= \frac{65}{-5\sqrt{21}-24}$$

3) First draw out each triangle, and use Pythagoras to find the unknown side:





Now find the sine & cosine trig ratios for each triangle:

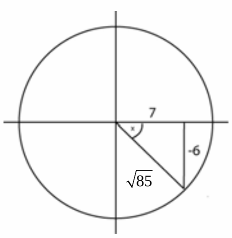
$$\sin x = \frac{2\sqrt{6}}{7} \qquad \sin y = \frac{4}{5}$$

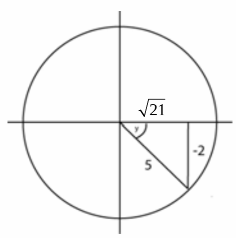
$$\cos x = \frac{5}{7} \qquad \cos y = \frac{-3}{5}$$

Use these to evaluate $\sin(x - y)$

$$\begin{aligned}
&\sin(x-y)\\
&=\sin x\cos y-\cos x\sin y\\
&=\left(\frac{2\sqrt{6}}{7}\right)\left(\frac{-3}{5}\right)-\left(\frac{5}{7}\right)\left(\frac{4}{5}\right)\\
&=\frac{-6\sqrt{6}}{35}-\frac{20}{35}\\
&=\frac{-6\sqrt{6}-20}{35}
\end{aligned}$$

4) First draw out each triangle, and use Pythagoras to find the unknown side:





Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{-6}{\sqrt{85}} \qquad \sin y = \frac{-2}{5}$$

$$\cos x = \frac{7}{\sqrt{85}} \qquad \cos y = \frac{\sqrt{21}}{5}$$

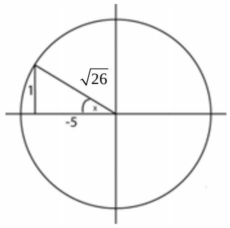
Use these to evaluate $\sec(x + y)$

$$\begin{aligned}
&\sec(x+y)\\
&=\frac{1}{\cos x\cos y-\sin x\sin y}\\
&=\frac{1}{\left(\frac{7}{\sqrt{85}}\right)\left(\frac{\sqrt{21}}{5}\right)-\left(\frac{-6}{\sqrt{85}}\right)\left(\frac{-2}{5}\right)}\\
&=\frac{1}{\left(\frac{7\sqrt{21}}{5\sqrt{85}}\right)-\left(\frac{12}{5\sqrt{85}}\right)}
\end{aligned}$$

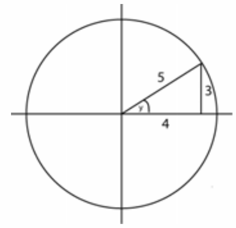
$$\begin{aligned}
&=\frac{1}{\frac{7\sqrt{21}-12}{5\sqrt{85}}}\\
&=\frac{5\sqrt{85}}{7\sqrt{21}-12}
\end{aligned}$$

5) First draw out each triangle, and use Pythagoras to find the unknown side:

$\cot x < 0$ in quadrants II & IV
$\cos x < 0$ in quadrants II & III
Overlap in II



$\tan y > 0$ in quadrants I & III
$\sin y > 0$ in quadrants I & II
Overlap in I



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{1}{\sqrt{26}} \qquad \sin y = \frac{3}{5}$$
$$\cos x = \frac{-5}{\sqrt{26}} \qquad \cos y = \frac{4}{5}$$

Use these to evaluate $\csc(x-y)$

$\csc(x-y)$

$$= \frac{1}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{26}}\right)\left(\frac{4}{5}\right) - \left(\frac{-5}{\sqrt{26}}\right)\left(\frac{3}{5}\right)}$$

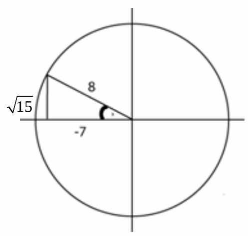
$$= \frac{1}{\left(\frac{4}{5\sqrt{26}}\right) - \left(\frac{-15}{5\sqrt{26}}\right)}$$

$$= \frac{1}{\frac{19}{5\sqrt{26}}}$$

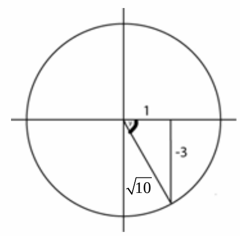
$$= \frac{5\sqrt{26}}{19}$$

6) First draw out each triangle, and use Pythagoras to find the unknown side:

$\sec x < 0$ in quadrants II & III
$\sin x > 0$ in quadrants I & II
Overlap in II



$\tan y < 0$ in quadrants II & IV
$\sin y < 0$ in quadrants III & IV
Overlap in IV



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{\sqrt{15}}{8} \qquad \sin y = \frac{-3}{\sqrt{10}}$$
$$\cos x = \frac{-7}{8} \qquad \cos y = \frac{1}{\sqrt{10}}$$

Use these to evaluate $\sec(x-y)$

$\sec(x-y)$

$$= \frac{1}{\cos x \cos y + \sin x \sin y}$$

$$= \frac{1}{\left(\frac{-7}{8}\right)\left(\frac{1}{\sqrt{10}}\right) + \left(\frac{\sqrt{15}}{8}\right)\left(\frac{-3}{\sqrt{10}}\right)}$$

$$= \frac{1}{\left(\frac{-7}{8\sqrt{10}}\right) + \left(\frac{-3\sqrt{15}}{8\sqrt{10}}\right)}$$

$$= \frac{1}{\frac{-7-3\sqrt{15}}{8\sqrt{10}}}$$

$$= \frac{8\sqrt{10}}{-7-3\sqrt{15}}$$

The following double angle identities are frequently used:

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos 2A = 1 - 2\sin^2 A$$

Example 1: Expand sin 60° using $\sin 2A = 2\sin A\cos A$:

$$\sin 2A = 2\sin A\cos A$$
$$\sin 60^\circ = 2\sin 30^\circ \cos 30^\circ$$

What goes here is always half of the angle on the left side.

Example 2: Expand cos 90° using $\cos 2A = 2\cos^2 A - 1$

$$\cos 2A = 2\cos^2 A - 1$$
$$\cos 90^\circ = 2\cos^2 45^\circ - 1$$

Example 3: Expand cos $\frac{2\pi}{3}$ using $\cos 2A = \cos^2 A - \sin^2 A$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos\frac{2\pi}{3} = \cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}$$

Example 4: Expand cos 8x using $\cos 2A = 1 - 2\sin^2 A$

$$\cos 2A = 1 - 2\sin^2 A$$
$$\cos 8x = 1 - 2\sin^2 4x$$

In the following examples, you must work backwards to condense the identity:

Example 5: Condense: $1 - 2\sin^2 150°$

$\cos 2A = 1 - 2\sin^2 A$
$\cos 300^\circ = 1 - 2\sin^2 150^\circ$

What goes on the left side is double the angle on the right.

Example 6: Condense: $2\sin 3x\cos 3x$

$\sin 2A = 2\sin A\cos A$
$\sin 6x = 2\sin 3x\cos 3x$

Example 7: Condense: $\cos^2\pi - \sin^2\pi$

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2\pi = \cos^2 \pi - \sin^2 \pi$

Questions:

1) Expand $\sin 180^\circ$ using $\sin 2A = 2\sin A\cos A$

2) Expand $\cos\frac{\pi}{3}$ using $\cos 2A = 1 - 2\sin^2 A$

3) Expand $\cos 16x$ using $\cos 2A = \cos^2 A - \sin^2 A$

4) Expand $\cos\frac{\pi}{2}$ using $\cos 2A = 2\cos^2 A - 1$

5) Condense: $2\cos^2\frac{2\pi}{3} - 1$

6) Condense: $\cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6}$

7) Condense: $\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

1) $\sin 180^\circ$

$\sin 2A = 2\sin A\cos A$

$\sin 180^\circ = 2\sin 90^\circ \cos 90^\circ$

2) $\cos\frac{\pi}{3}$

$\cos 2A = 1 - 2\sin^2 A$

$\cos\frac{\pi}{3} = 1 - 2\sin^2\frac{\pi}{6}$

3) $\cos 16x$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 16x = \cos^2 8x - \sin^2 8x$

4) $\cos\frac{\pi}{2}$

$\cos 2A = 2\cos^2 A - 1$

$\cos\frac{\pi}{2} = 2\cos^2\frac{\pi}{4} - 1$

5) $2\cos^2\frac{2\pi}{3} - 1$

$\cos 2A = 2\cos^2 A - 1$

$\cos\frac{4\pi}{3} = 2\cos^2\frac{2\pi}{3} - 1$

6) $\cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6}$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos\frac{\pi}{3} = \cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6}$

7) $\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos x = \cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

In the following examples, the double-angle identities will be used in completing proofs.

Example 1: Prove sin 2*x* + cos *x* = cos *x*(2sin *x* + 1)

$$\sin 2x + \cos x$$
$$= 2\sin x \cos x + \cos x$$
$$= \cos x(2\sin x + 1)$$

Example 2: Prove: $\cos 2x = \cos^2 x - \sin^2 x$ Hint: Write cos 2*x* as cos (*x* + *x*)

$$\cos(x + x) = \cos x \cos x - \sin x \sin x$$
$$\cos(x + x) = \cos^2 x - \sin^2 x$$

Questions:

1) $\cos 2x + \cos x = (2\cos x - 1)(\cos x + 1)$

2) $2\cos 2x - \sin x + 1 = -(4\sin x - 3)(\sin x + 1)$

3) $\cos 2x = 2\cos^2 x - 1$

4) $\cos 2x = 1 - 2\sin^2 x$

5) $\dfrac{1+\cos 2x}{\sin 2x} = \cot x$

6) $\dfrac{2}{1+\cos 2x} = \sec^2 x$

7) $\sin 2x = 2\sin x \cos x$

8) $\cos 2x - 1 + 2\sin x = 2\sin x(1-\sin x)$

9) $(\sin x + \cos x)^2 = 1 + \sin 2x$

10) $\sin(x-y)\sin(x+y) = \cos^2 y - \cos^2 x$

1) $\cos 2x + \cos x$
$= 2\cos^2 x - 1 + \cos x$
$= 2\cos^2 x + \cos x - 1$
$= (2\cos x - 1)(\cos x + 1)$

2) $2\cos 2x - \sin x + 1$
$= 2(1 - 2\sin^2 x) - \sin x + 1$
$= 2 - 4\sin^2 x - \sin x + 1$
$= -4\sin^2 x - \sin x + 3$
$= -(4\sin^2 x + \sin x - 3)$
$= -(4\sin x - 3)(\sin x + 1)$

3) $\cos 2x = \cos(x + x)$
$\cos(x + x) = \cos x\cos x - \sin x\sin x$
$\cos 2x = \cos^2 x - \sin^2 x$
$\cos 2x = \cos^2 x - (1 - \cos^2 x)$
$\cos 2x = \cos^2 x - 1 + \cos^2 x$
$\cos 2x = 2\cos^2 x - 1$

4) $\cos 2x = 1 - 2\sin^2 x$
$\cos(x + x) = \cos x\cos x - \sin x\sin x$
$\cos 2x = \cos^2 x - \sin^2 x$
$\cos 2x = (1 - \sin^2 x) - \sin^2 x$
$\cos 2x = 1 - \sin^2 x - \sin^2 x$
$\cos 2x = 1 - 2\sin^2 x$

5) $\dfrac{1 + \cos 2x}{\sin 2x}$
$= \dfrac{1 + (2\cos^2 x - 1)}{2\sin x\cos x}$
$= \dfrac{2\cos^2 x}{2\sin x\cos x}$
$= \dfrac{\cos x}{\sin x}$
$= \cot x$

6) $\dfrac{2}{1 + \cos 2x} = \sec^2 x$
$= \dfrac{2}{1 + (2\cos^2 x - 1)}$
$= \dfrac{2}{2\cos^2 x}$
$= \dfrac{1}{\cos^2 x}$
$= \sec^2 x$

7) $\sin 2x = \sin(x + x)$
$\sin(x + x) = \sin x\cos x + \cos x\sin x$
$= 2\sin x\cos x$

8) $\cos 2x - 1 + 2\sin x$
$= (1 - 2\sin^2 x) - 1 + 2\sin x$
$= -2\sin^2 x + 2\sin x$
$= 2\sin x(-\sin x + 1)$
$= 2\sin x(1 - \sin x)$

9) $(\sin x + \cos x)^2$
$= \sin^2 x + 2\sin x\cos x + \cos^2 x$
$= \sin^2 x + \cos^2 x + 2\sin x\cos x$
$= 1 + 2\sin x\cos x$

10)
$\sin(x - y)\sin(x + y) = \cos^2 y - \cos^2 x$
$= [\sin x\cos y - \cos x\sin y][\sin x\cos y + \cos x\sin y]$
$= \left(\sin x\cos y\right)^2 + \sin x\cos y\cos x\sin y - \cos x\sin y\sin x\cos y - (\cos x\sin y)^2$
$= \sin^2 x\cos^2 y - \cos^2 x\sin^2 y$
$= \sin^2 x\cos^2 y - (1 - \sin^2 x)(1 - \cos^2 y)$
$= \sin^2 x\cos^2 y - [1 - \cos^2 y - \sin^2 x + \sin^2 x\cos^2 y]$
$= \sin^2 x\cos^2 y - 1 + \cos^2 y + \sin^2 x - \sin^2 x\cos^2 y$
$= \cos^2 y + \sin^2 x - 1$
$= \cos^2 y - \left(1 - \sin^2 x\right)$
$= \cos^2 y - \cos^2 x$