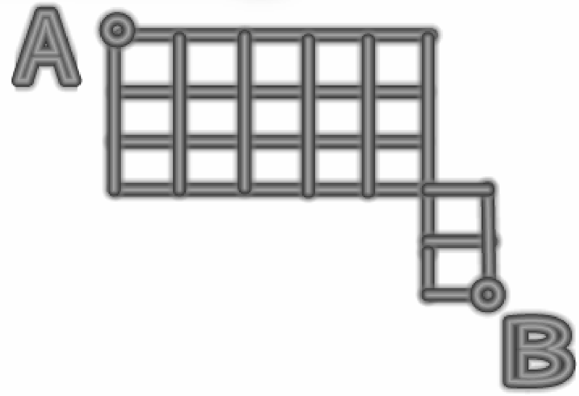
Pure Math 30:

Permutations and Combinations



Lesson 3

Pathways

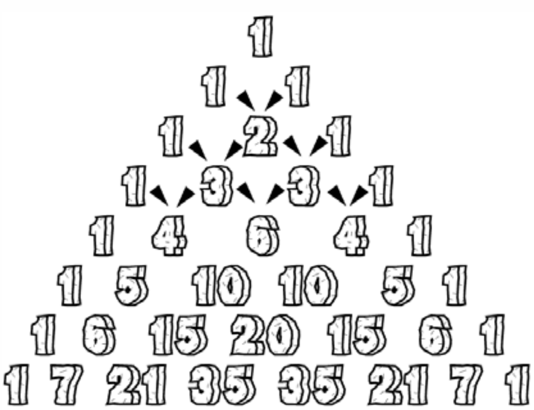
Pure Math 30: EXPLAINED!

By Barry Mabillard

Pascal's Triangle: The following triangle is used in solving pathway problems, so you will need to learn the pattern.

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1
1 6 15 20 15 6 1
1 7 21 35 35 21 7 1

- Start the triangle at the top with the number 1.
- Slide the 1 down diagonally to form the beginning & ends of each row.
- To fill in the rows, add together the numbers immediately above. (Diagram)



- Notice the symmetry in the triangle. Positions equidistant from either end will have the same value!
- The sum of each row is equal to 2^n, where n is the row number. *The top is considered to be row zero.* Recall from Lesson 1 that 2^n is the number of ways to select any number of objects from a set containing n items.

Note that Pascal's Triangle is made up entirely of combinations!

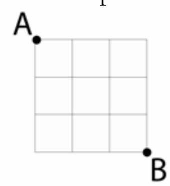
The first row is: ${}_0C_0$
The second row is ${}_1C_0$ ${}_1C_1$
The third row is: ${}_2C_0$ ${}_2C_1$ ${}_2C_2$

This pattern continues forever.

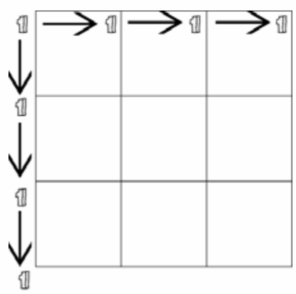
Lesson 3: Pathways

Simple Pathways: You can use the principles of Pascal's Triangle to find out how many possible pathways exist from one point in a grid to another point in the grid.

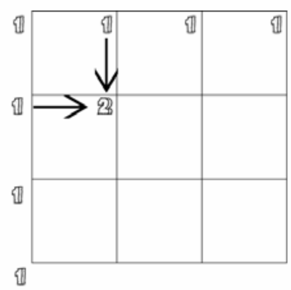
Example 1: How many paths exist from point A to B?



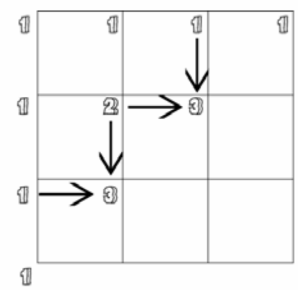
Step 1: First slide 1's across the outer edge to form the edge of Pascal's Triangle.



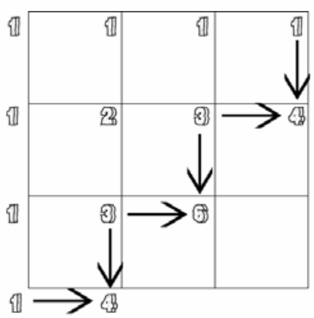
Steps 2 through 6: Fill in the Pascal Triangle Pattern.



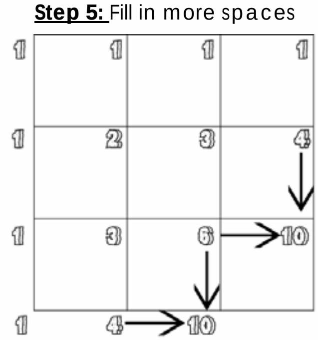
Step 3: Fill in more spaces



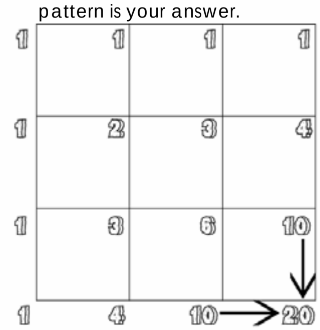
Step 4: Fill in more spaces



Step 5: Fill in more spaces



Step 6: The final number in the pattern is your answer.



There are 20 possible paths.

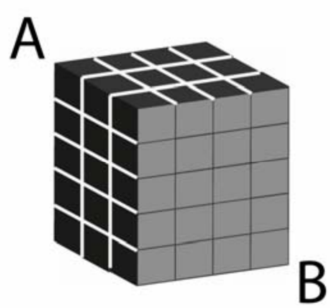
Example 1 Shortcut:

Note that you can go East 3 times, and Down 3 times. Writing out the letters gives you EEEDDD. How many ways can you arrange these letters? You can do it in:

$$\frac{6!}{3! \bullet 3!} = 20 \text{ ways.}$$

This shortcut will only work when there are no gaps or extra spaces in the pathway.

Example 2: In the following cube, how many paths exist from point A to B?



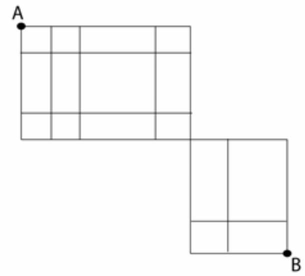
In the 3-D cube, we can write out all the possible directions we can go. We can move East 4 times, Down 5 times, and Forward 3 times. Writing out the letters gives you EEEEDDDDDFFF.

You can do it in:

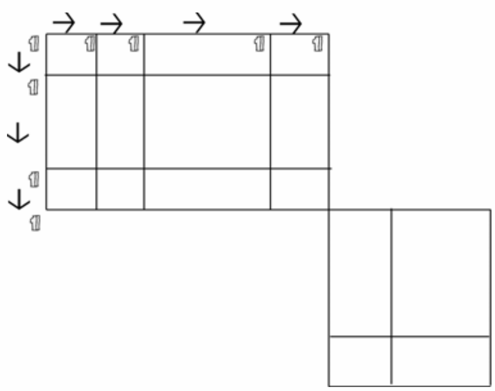
$$\frac{12!}{4! \bullet 5! \bullet 3!} = 27720 \text{ ways.}$$

Complex Pathways: More difficult pathway problems involve gaps & spaces in the grid, so you have to be careful when applying Pascal's triangle.

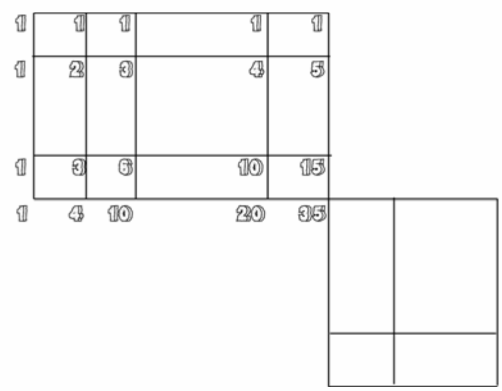
Example 3: Find the number of paths from point A to B:



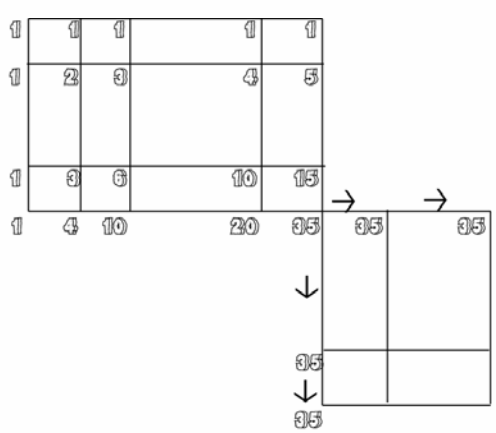
Step 1: First slide 1's across the outer edge to form the edge of Pascal's Triangle.



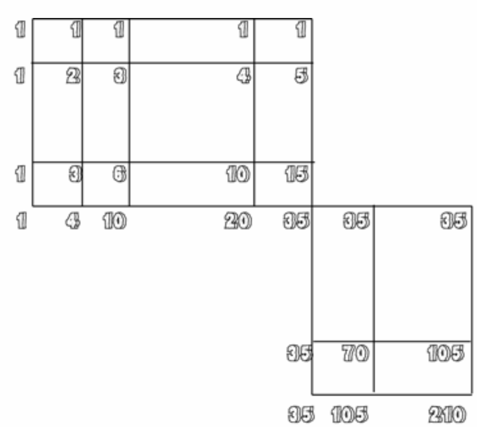
Step 2: Now fill in as many points as you can.



Step 3: Slide 35's across the second rectangle.



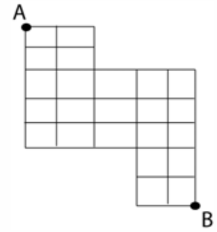
Step 4: Fill in the second rectangle.

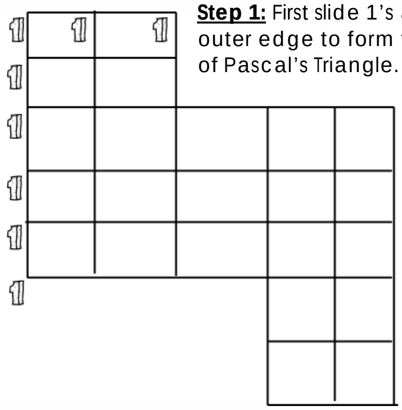


You could also do this question by considering these to be two separate rectangles and multiplying the answers together. From the first one, we have EEEESSS = $\frac{7!}{4!\bullet 3!} = 35$.

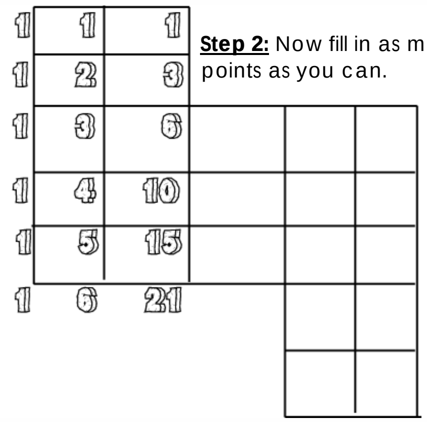
From the second, we have EESS = $\frac{4!}{2!\bullet 2!} = 6$. Multiplying these results, we get 35 • 6 = 210

Example 4: Find the number of paths from point A to B:

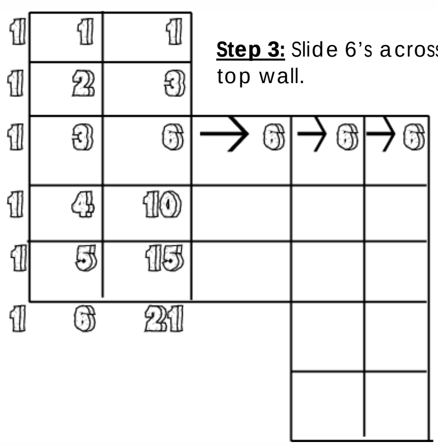




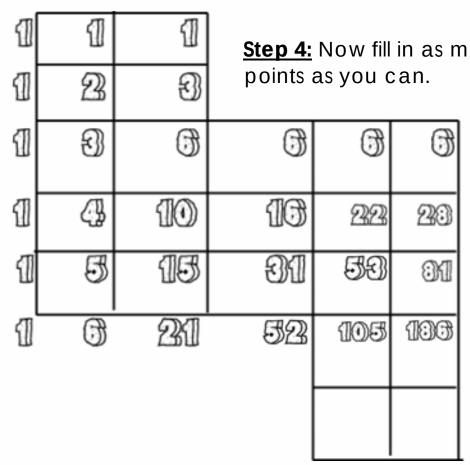
Step 1: First slide 1's across the outer edge to form the edge of Pascal's Triangle.



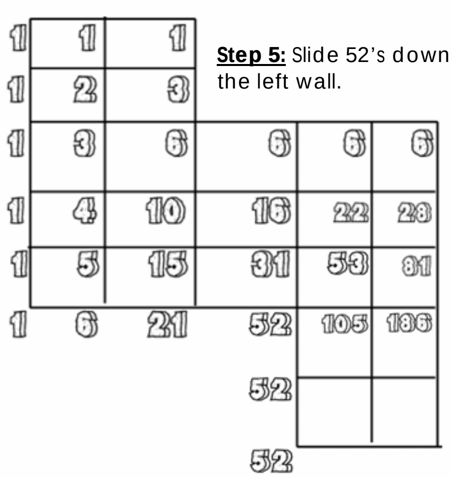
Step 2: Now fill in as many points as you can.



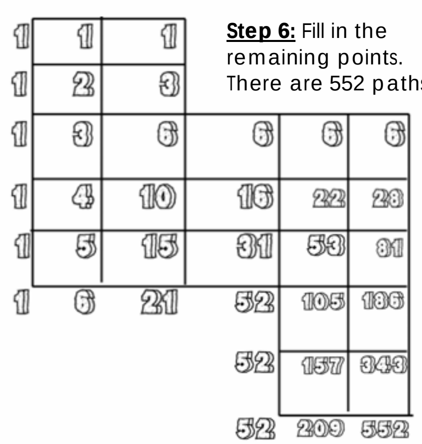
Step 3: Slide 6's across the top wall.



Step 4: Now fill in as many points as you can.

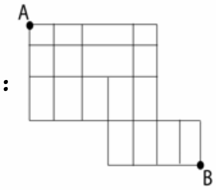


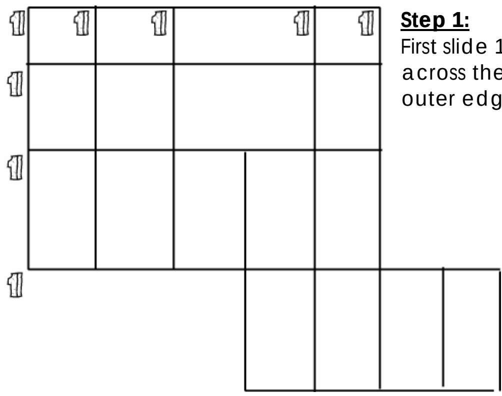
Step 5: Slide 52's down the left wall.



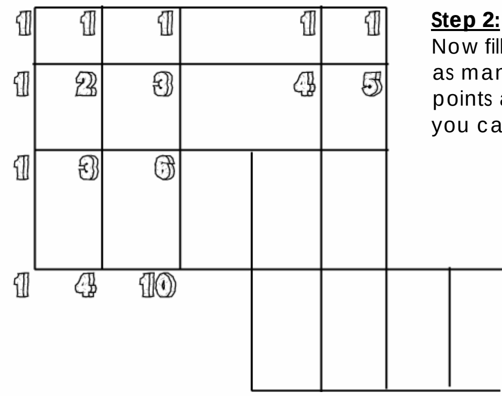
Step 6: Fill in the remaining points. There are 552 paths.

Example 5: Find the number of paths from point A to B:

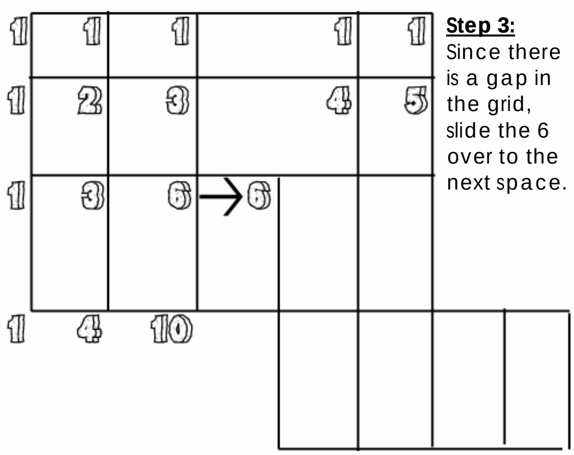




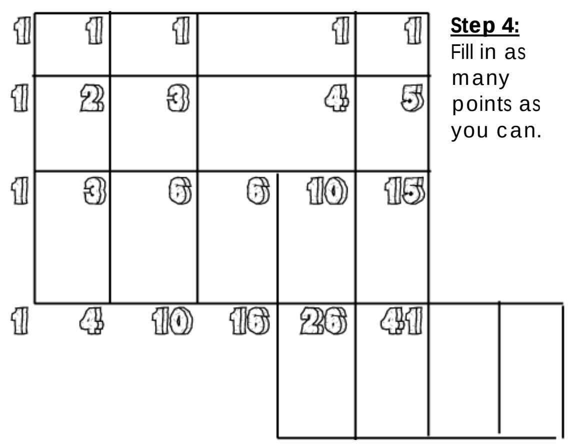
Step 1:
First slide 1's across the outer edge.



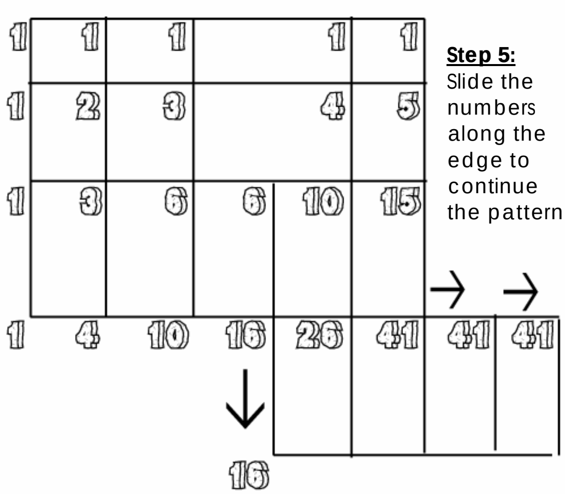
Step 2:
Now fill in as many points as you can.



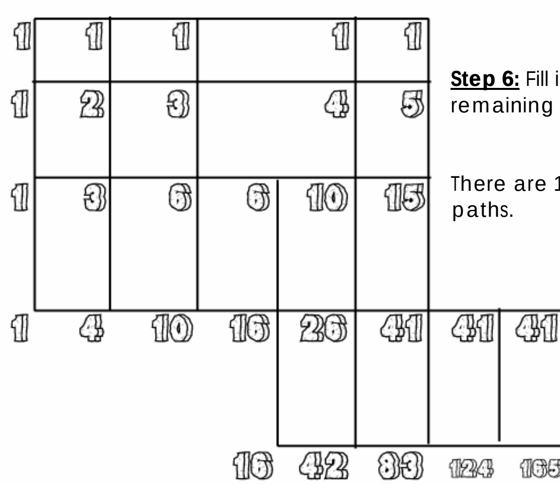
Step 3:
Since there is a gap in the grid, slide the 6 over to the next space.



Step 4:
Fill in as many points as you can.



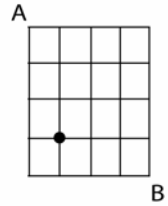
Step 5:
Slide the numbers along the edge to continue the pattern.



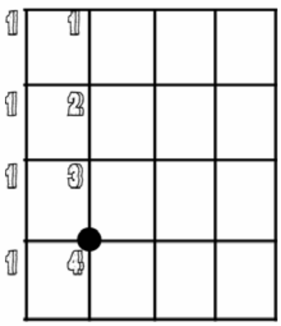
Step 6: Fill in the remaining points.

There are 165 paths.

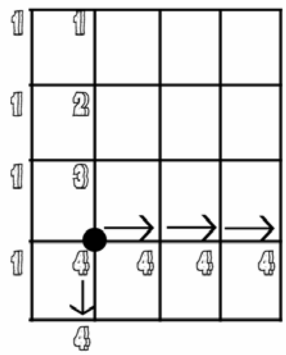
Example 6: Find the number of paths from point A to B:



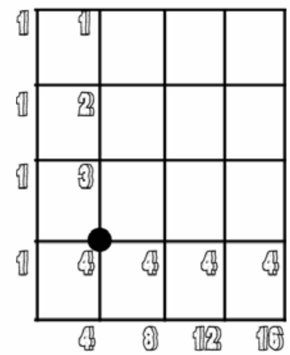
Step 1:
First complete the points required to get to the dot.



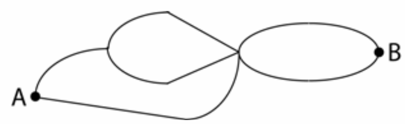
Step 2:
Slide 4's right & down.

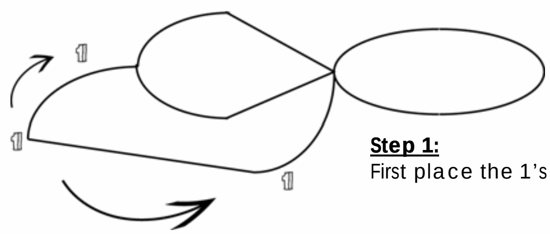


Step 3:
Fill in the remaining points. There are 16 paths.

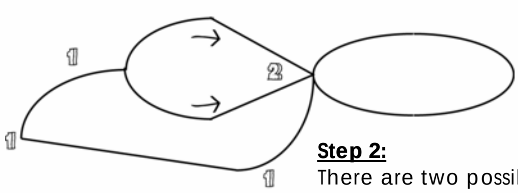


Example 7: Find the number of paths from point A to B:

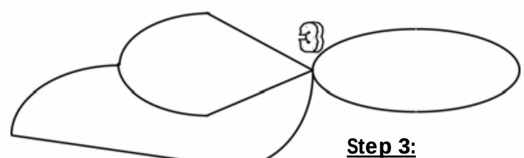




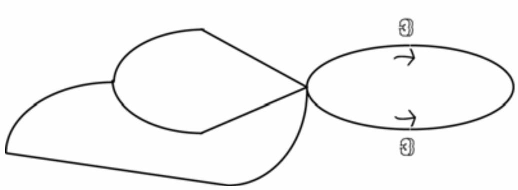
Step 1:
First place the 1's to start off Pascal's Triangle.



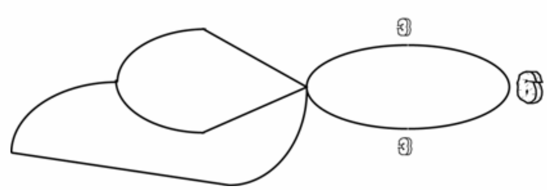
Step 2:
There are two possible paths, as indicated by the arrows.



Step 3:
There are three paths in total to get to the middle point.

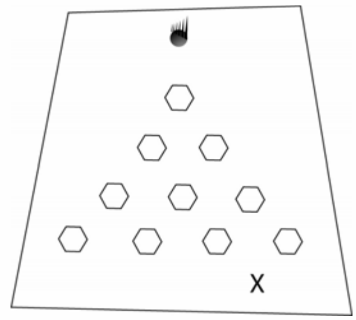


Step 5:
Slide the 3's to find the remaining paths.

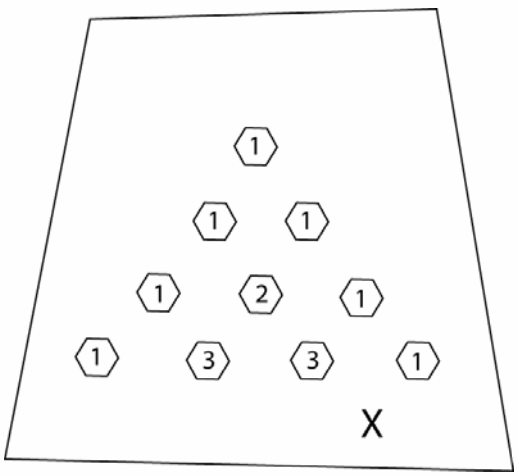


Step 6:
There are a total of six paths to B.

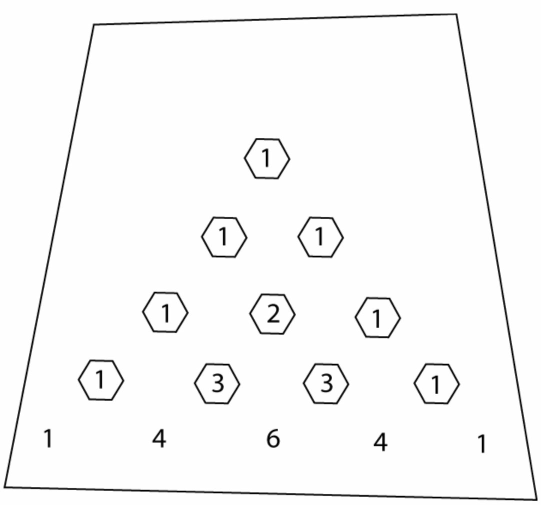
Example 8: A pinball game has a series of pegs which create multiple paths for a ball to reach the bottom. How many paths lead to X?



Step 1:
Use the pegs to create Pascal's triangle.



Step 2:
Fill in the bottom row as if there were pegs there too.



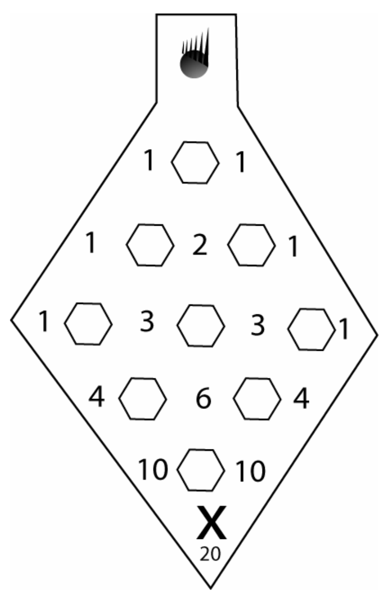
There are 4 ways the ball can reach X.

Interesting Variation:

In Example 8, the pegs are used to form Pascal's triangle since the X can be reached diagonally from the two pegs immediately above it.

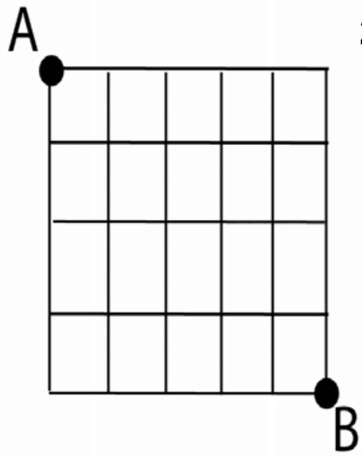
If the X can't be reached diagonally from two pegs immediately above it, the spaces between the pegs must be used to find the number of pathways.

Example:



<u>Questions:</u> Find the number of paths from A to B in each of the following:

1)



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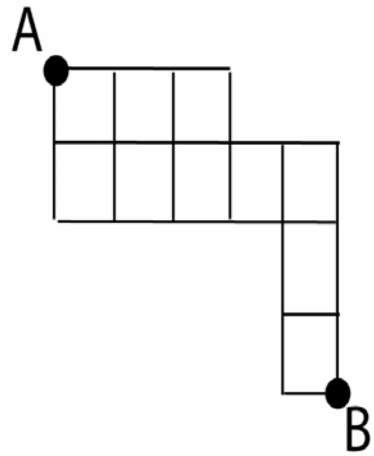
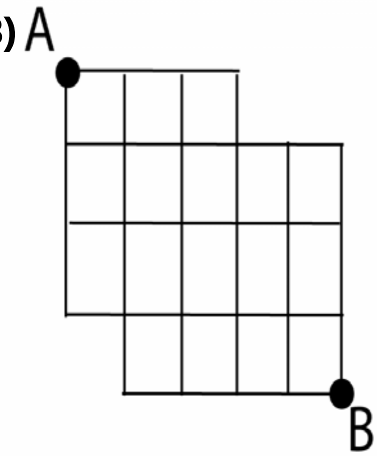


3)



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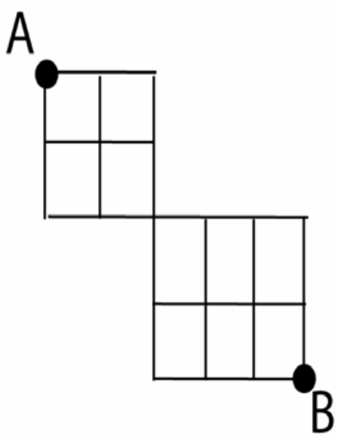
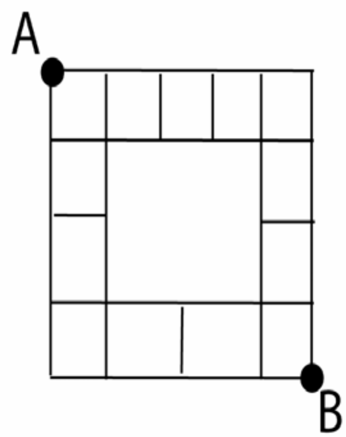


5)



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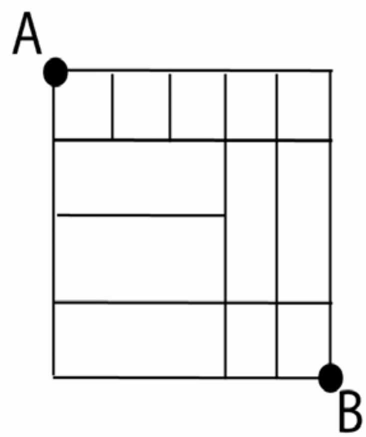
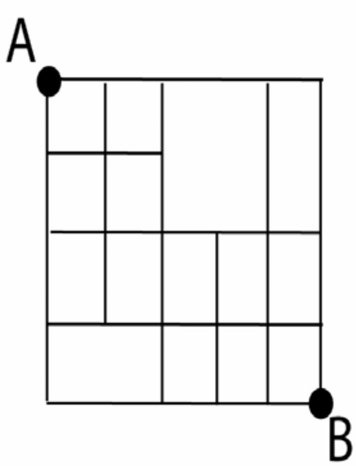


7)



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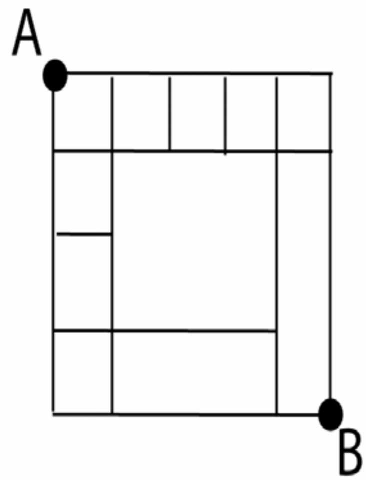

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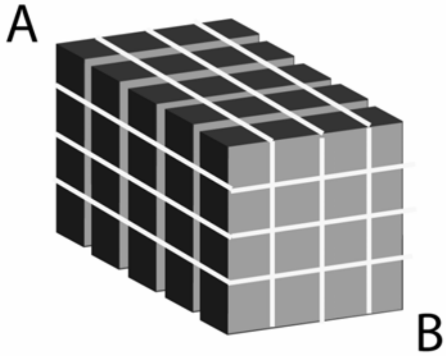
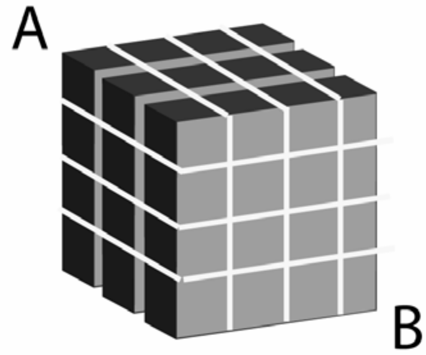
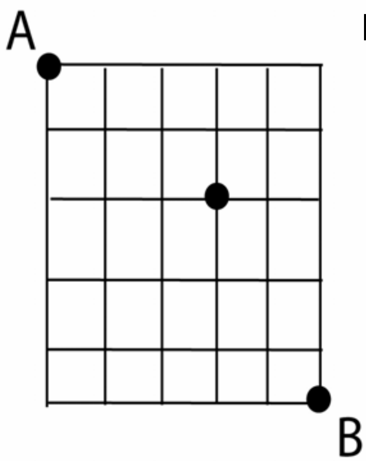


10)

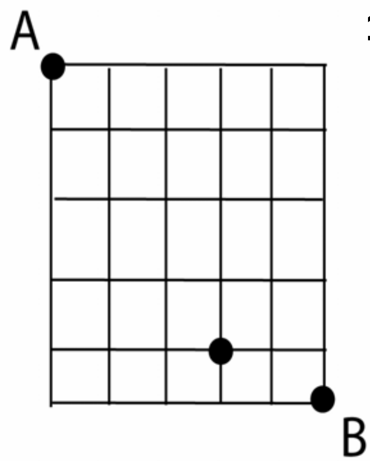


For each of the following, find the number of paths through the dot.

11)



l2)



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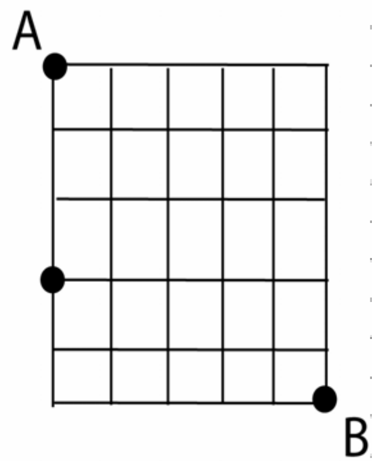
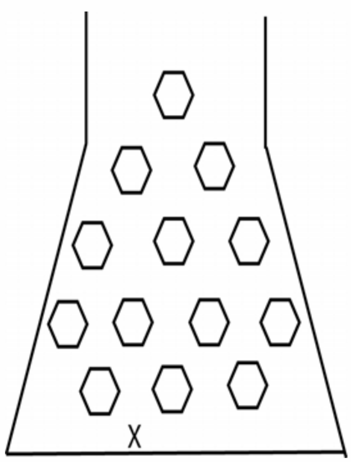


For each of the following, find the number of paths leading to X.

14)



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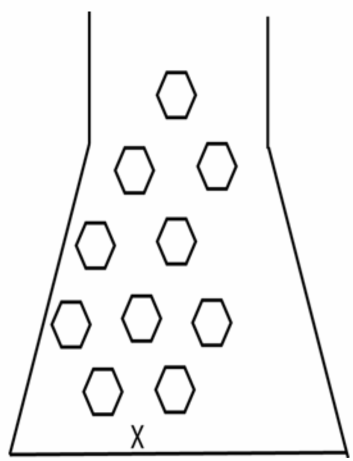

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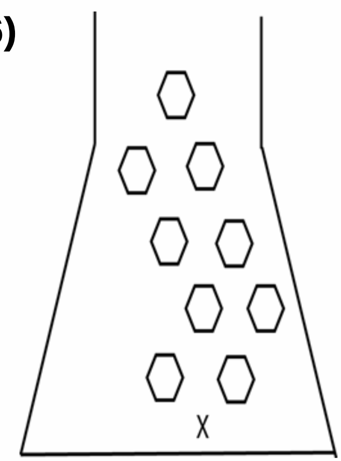
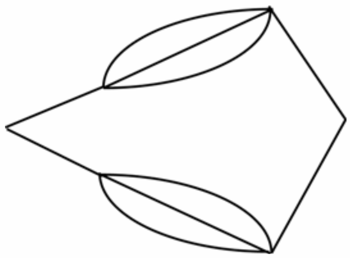


For each of the following, find the number of paths from left to right.

17)



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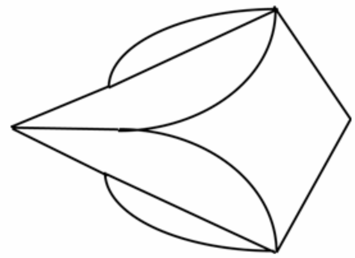

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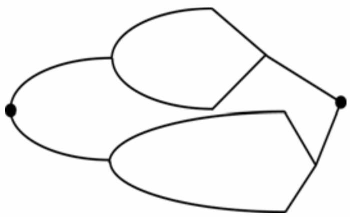

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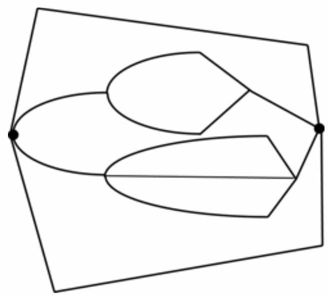
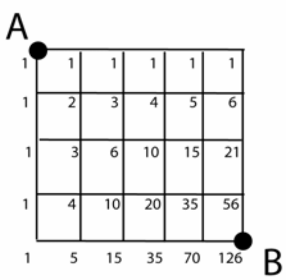


Answers:

1.



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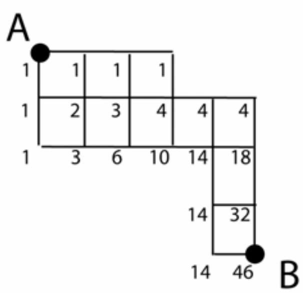

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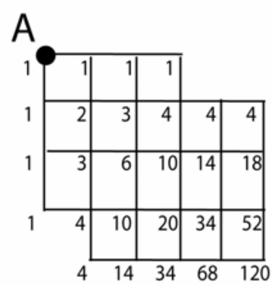

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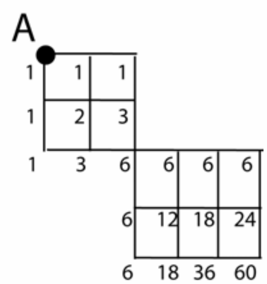

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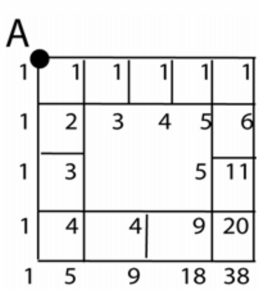

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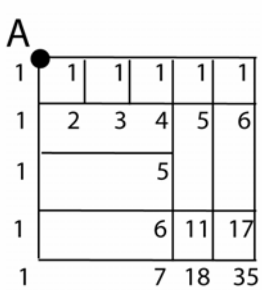

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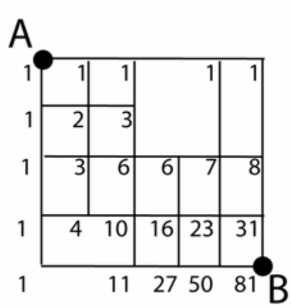

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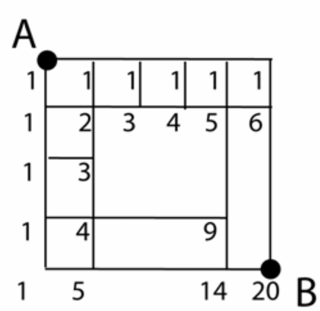
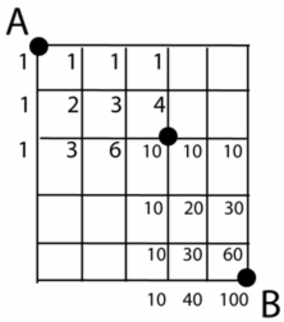


9. FFFFFEEEEDDDD = $\dfrac{13!}{5!\bullet 4!\bullet 4!} = 90090$

10. FFFEEEEDDDD = $\dfrac{11!}{3!\bullet 4!\bullet 4!} = 11550$

11.



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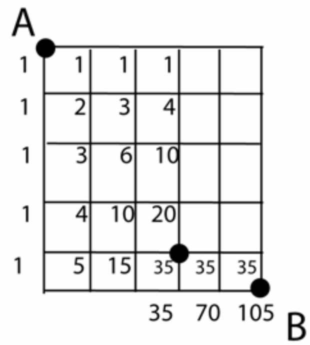

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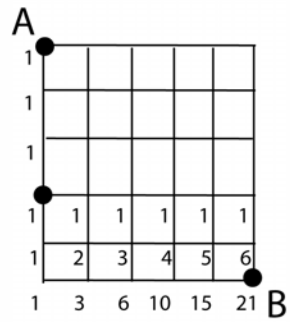

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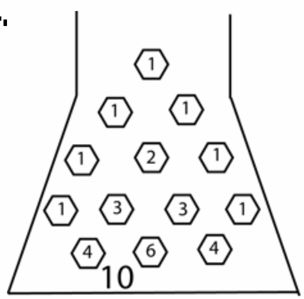

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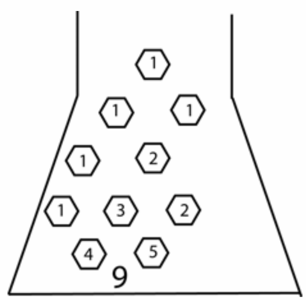

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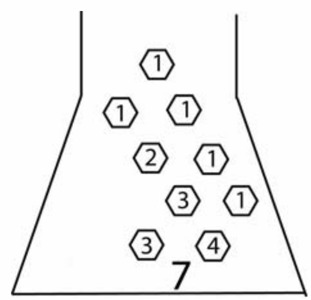

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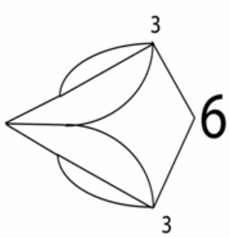

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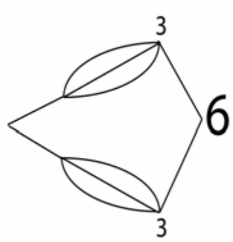

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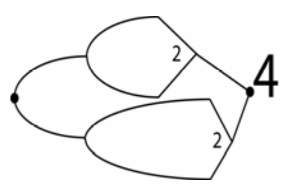

20.