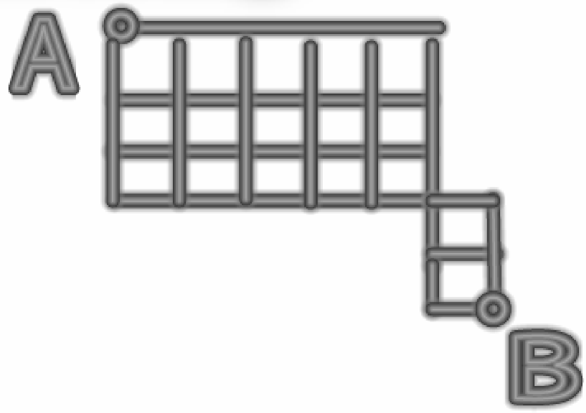
Pure Math 30:

Permutations and Combinations



Lesson 5

Expanding Binomials

Pure Math 30: EXPLAINED!

By Barry Mabillard

Expanding Binomials: The following examples will illustrate how to expand a binomial to the n^{th} power.

Example 1: Expand $(2a-3b)^5$

One way to expand this would be to write it as (2a-3b)(2a-3b)(2a-3b)(2a-3b)(2a-3b), then do all the algebra in multiplying everything together! This would obviously take a very long time. Fortunately there is a formula we can use to find each term in the expansion separately.

$$t_{k+1} = {}_nC_k x^{n-k} y^k$$

With this ***term formula***, it is possible to find any term in the expansion of a binomial. Using the formula isn't hard, all that is involved is substituting your numbers and simplifying.

There are a few simple rules to follow:

n is the power of the binomial. In this case it's 5.
x represents the first term of the binomial (*including sign*) → 2a
y represents the second term of the binomial (*including sign*) → -3b
k represents ***one less*** than the term you want. If you want the first term, *k* = 0. For the second term, *k* = 1. For the third term, *k* = 2.

*The total number of terms in the expansion is **n + 1**

First term: ${}_5C_0(2a)^{5-0}(-3b)^0 = (2a)^5 = 32a^5$

Second Term: ${}_5C_1(2a)^{5-1}(-3b)^1 = 5(2a)^4(-3b) = 5(16a^4)(-3b) = -240a^4b$

Third Term: ${}_5C_2(2a)^{5-2}(-3b)^2 = 10(2a)^3(-3b)^2 = 10(8a^3)(9b^2) = 720a^3b^2$

Fourth Term: ${}_5C_3(2a)^{5-3}(-3b)^3 = 10(2a)^2(-3b)^3 = 10(4a^2)(-27b^3) = -1080a^2b^3$

Fifth Term: ${}_5C_4(2a)^{5-4}(-3b)^4 = 5(2a)^1(-3b)^4 = 5(2a)(81b^4) = 810ab^4$

Sixth Term: ${}_5C_5(2a)^{5-5}(-3b)^5 = (2a)^0(-3b)^5 = -243b^5$

Full Expansion = $32a^5 - 240a^4b + 720a^3b^2 - 1080a^2b^3 + 810ab^4 - 243b^5$

Part One: Expanding Binomials

Example 2: Expand: $(3x + \frac{1}{4})^3$

First term: ${}_3C_0(3x)^{3-0}\left(\frac{1}{4}\right)^0 = (3x)^3 = 27x^3$

Second Term: ${}_3C_1(3x)^{3-1}\left(\frac{1}{4}\right)^1 = 3(3x)^2\left(\frac{1}{4}\right) = \frac{3(9x^2)}{4} = \frac{27}{4}x^2$

Third Term: ${}_3C_2(3x)^{3-2}\left(\frac{1}{4}\right)^2 = 3(3x)^1\left(\frac{1}{4}\right)^2 = \frac{3(3x)}{16} = \frac{9}{16}x$

Fourth Term: ${}_3C_3(3x)^{3-3}\left(\frac{1}{4}\right)^3 = (3x)^0\left(\frac{1}{4}\right)^3 = \frac{1}{64}$

Full Expansion: $27x^3 + \frac{27}{4}x^2 + \frac{9}{16}x + \frac{1}{64}$

Example 3: Expand: $(2x^3 - 3y^2)^4$

First term: ${}_4C_0(2x^3)^{4-0}(-3y^2)^0 = (2x^3)^4 = 16x^{12}$

Second Term: ${}_4C_1(2x^3)^{4-1}(-3y^2)^1 = 4(2x^3)^3(-3y^2) = 4(8x^9)(-3y^2) = -96x^9y^2$

Third Term: ${}_4C_2(2x^3)^{4-2}(-3y^2)^2 = 6(2x^3)^2(-3y^2)^2 = 6(4x^6)(9y^4) = 216x^6y^4$

Fourth Term: ${}_4C_3(2x^3)^{4-3}(-3y^2)^3 = 4(2x^3)^1(-3y^2)^3 = 4(2x^3)(-27y^6) = -216x^3y^6$

Fifth Term: ${}_4C_4(2x^3)^{4-4}(-3y^2)^4 = (2x^3)^0(-3y^2)^4 = 81y^8$

Full Expansion: $16x^{12} - 96x^9y^2 + 216x^6y^4 - 216x^3y^6 + 81y^8$

Example 4: Expand: $(2x^2 - 3/x)^4$

First term: ${}_4C_0(2x^2)^{4-0}\left(\frac{-3}{x}\right)^0 = (2x^2)^4 = 16x^8$

Second Term: ${}_4C_1(2x^2)^{4-1}\left(\frac{-3}{x}\right)^1 = 4(2x^2)^3\left(\frac{-3}{x}\right) = 4(8x^6)\left(\frac{-3}{x}\right) = \frac{-96x^6}{x} = -96x^5$

Third Term: ${}_4C_2(2x^2)^{4-2}\left(\frac{-3}{x}\right)^2 = 6(2x^2)^2\left(\frac{-3}{x}\right)^2 = 6(4x^4)\left(\frac{9}{x^2}\right) = \frac{216x^4}{x^2} = 216x^2$

Fourth Term: ${}_4C_3(2x^2)^{4-3}\left(\frac{-3}{x}\right)^3 = 4(2x^2)^1\left(\frac{-3}{x}\right)^3 = 4(2x^2)\left(\frac{-27}{x^3}\right) = \frac{-216x^2}{x^3} = \frac{-216}{x}$

Fifth Term: ${}_4C_4(2x^2)^{4-4}\left(\frac{-3}{x}\right)^4 = (2x^2)^0\left(\frac{-3}{x}\right)^4 = \frac{81}{x^4}$

Full Expansion: $16x^8 - 96x^5 + 216x^2 - \frac{216}{x} + \frac{81}{x^4}$

Part One: Expanding Binomials

Questions:

1) Expand $(2a-3b)^4$

2) Expand $(3a - \frac{1}{4})^3$

3) Expand $(2x^3 - 3y^2)^3$

4) Expand $\left(2x^2 + \frac{3}{y}\right)^3$

1) *First term:* $_4C_0(2a)^{4-0}(-3b)^0 = (2a)^4 = 16a^4$

Second Term: $_4C_1(2a)^{4-1}(-3b)^1 = 4(2a)^3(-3b) = 4(8a^3)(-3b) = -96a^3b$

Third Term: $_4C_2(2a)^{4-2}(-3b)^2 = 6(2a)^2(-3b)^2 = 6(4a^2)(9b^2) = 216a^2b^2$

Fourth Term: $_4C_3(2a)^{4-3}(-3b)^3 = 4(2a)(-3b)^3 = 4(2a)(-27b^3) = -216ab^3$

Fifth Term: $_4C_4(2a)^{4-4}(-3b)^4 = (-3b)^4 = 81b^4$

Full Expansion: $16a^4 - 96a^3b + 216a^2b^2 - 216ab^3 + 81b^4$

2) *First term:* $_3C_0(3a)^{3-0}\left(-\frac{1}{4}\right)^0 = (3a)^3 = 27a^3$

Second Term: $_3C_1(3a)^{3-1}\left(-\frac{1}{4}\right)^1 = 3(3a)^2\left(-\frac{1}{4}\right) = 3(9a^2)\left(-\frac{1}{4}\right) = -\frac{27a^2}{4}$

Third Term: $_3C_2(3a)^{3-2}\left(-\frac{1}{4}\right)^2 = 3(3a)^1\left(-\frac{1}{4}\right)^2 = 3(3a)\left(\frac{1}{16}\right) = \frac{9a}{16}$

Fourth Term: $_3C_3(3a)^{3-3}\left(-\frac{1}{4}\right)^3 = (3a)^0\left(-\frac{1}{4}\right)^3 = -\frac{1}{64}$

Full Expansion: $27a^3 - \frac{27a^2}{4} + \frac{9a}{16} - \frac{1}{64}$

3) *First term:* $_3C_0(2x^3)^{3-0}(-3y^2)^0 = (2x^3)^3 = 8x^9$

Second Term: $_3C_1(2x^3)^{3-1}(-3y^2)^1 = 3(2x^3)^2(-3y^2)^1 = 3(4x^6)(-3y^2)^1 = -36x^6y^2$

Third Term: $_3C_2(2x^3)^{3-2}(-3y^2)^2 = 3(2x^3)^1(-3y^2)^2 = 3(2x^3)(9y^4) = 54x^3y^4$

Fourth Term: $_3C_3(2x^3)^{3-3}(-3y^2)^3 = (-3y^2)^3 = -27y^6$

Full Expansion: $8x^9 - 36x^6y^2 + 54x^3y^4 - 27y^6$

4) *First term:* $_3C_0(2x^2)^{3-0}\left(\frac{3}{y}\right)^0 = (2x^2)^3 = 8x^6$

Second Term: $_3C_1(2x^2)^{3-1}\left(\frac{3}{y}\right)^1 = 3(2x^2)^2\left(\frac{3}{y}\right) = 3(4x^4)\left(\frac{3}{y}\right) = \frac{36x^4}{y}$

Third Term: $_3C_2(2x^2)^{3-2}\left(\frac{3}{y}\right)^2 = 3(2x^2)^1\left(\frac{3}{y}\right)^2 = 3(2x^2)\left(\frac{9}{y^2}\right) = \frac{54x^2}{y^2}$

Fourth Term: $_3C_3(2x^2)^{3-3}\left(\frac{3}{y}\right)^3 = \left(\frac{3}{y}\right)^3 = \frac{27}{y^3}$

Full Expansion: $8x^6 + \frac{36x^4}{y} + \frac{54x^2}{y^2} + \frac{27}{y^3}$

Part Two: Particular Terms

Finding a Particular Term: Given some information about a term, you will be expected to solve various question types.

Example 1: Given $(3x - 4)^8$, determine the middle term of the expansion.

The *k*-value of the middle term is found by dividing *n* by 2. → 8 ÷ 2 = 4

$${}_8C_4(3x)^{8-4}(-4)^4 = 70(81x^4)(256) = \mathbf{1451520x^4}$$

Example 2: Given $(5x - 2y)^9$ Find the coefficient of the term containing x^5

In order to find the *k*-value, first plug everything you know into the formula. (Use an empty box for *k*, since that's what we're trying to figure out.

$${}_9C_{\square}(5x)^{9-\square}(-2y)^{\square}$$

By inspection, we can see that to get a term with x^5, we need to put a 4 in the space.

$${}_9C_4(5x)^{9-4}(-2y)^4 = 126(5x)^5(-2y)^4 = 126(3125x^5)(16y^4) = \mathbf{6300000x^5y^4}$$

Example 3: Given $(2x^4 - 2y^2)^5$ find the coefficient of the term containing x^{12}

In order to find the *k*-value, first plug everything you know into the formula:

$${}_5C_{\square}(2x^4)^{5-\square}(-2y^2)^{\square}$$

By inspection, we can see that to get a term with x^{12}, *k* must equal 2.

$${}_5C_2(2x^4)^{5-2}(-2y^2)^2 = 10(2x^4)^3(-2y^2)^2 = 10(8x^{12})(4y^4) = \mathbf{320x^{12}y^4}$$

Example 4: Given the binomial $\left(3x^3 + \frac{1}{x^3}\right)^6$, find the constant term

In order to find the *k*-value, first plug everything you know into the formula:

$${}_6C_k(3x^3)^{6-k}\left(\frac{1}{x^3}\right)^k$$

The constant term occurs when the *x*'s completely cancel out.
Do a quick table to see what value of *k* is needed.

$k = 0$	$(x^3)^6\left(\frac{1}{x^3}\right)^0 = x^{18}$
$k = 1$	$(x^3)^5\left(\frac{1}{x^3}\right)^1 = \frac{x^{15}}{x^3} = x^{12}$
$k = 2$	$(x^3)^4\left(\frac{1}{x^3}\right)^2 = \frac{x^{12}}{x^6} = x^6$
$k = 3$	$(x^3)^3\left(\frac{1}{x^3}\right)^3 = \frac{x^9}{x^9} = 1$

Now fill in the term formula and solve with $k = 3$.

$${}_6C_3(3x^3)^{6-3}\left(\frac{1}{x^3}\right)^3$$
$$= 20(3x^3)^3\left(\frac{1}{x^3}\right)^3$$
$$= 20(27x^9)\left(\frac{1}{x^9}\right)$$
$$= \mathbf{540}$$

Example 5: A term in the expansion of $(x + a)^7$ is $\frac{21504x^5}{y^4}$. Find the value of a.

To find the *k*-value, first set up the term formula as follows:

$_7C_k(x)^{7-k}(a)^k$

By inspection, we can see that to get a term with x^5, k must equal 2.

Now that we know $k = 2$, set the given term equal to the term formula. This will set things up so you can solve for a.

$$\frac{21504x^5}{y^4} = {}_7C_k(x)^{7-k}(a)^k$$

$$\frac{21504x^5}{y^4} = {}_7C_2(x)^{7-2}(a)^2$$

$$\frac{21504x^5}{y^4} = 21x^5a^2$$

$$21504x^5 = 21x^5a^2y^4$$

$$\frac{21504}{21y^4} = a^2$$

$$\sqrt{\frac{21504}{21y^4}} = \sqrt{a^2}$$

$$\mathbf{\frac{32}{y^2} = a}$$

Questions:

1) Given $(2x - 6)^{10}$, determine the middle term of the expansion.

2) Given $(3x + 2y)^8$, determine the middle term of the expansion.

3) Given $(5z + 9y)^6$ Find the coefficient of the term containing z^2

4) Given $(8x^6 - 7y^3)^9$ Find the coefficient of the term containing x^{36}

5) Given $(4z^5 + 3y^3)^5$ Find the coefficient of the term containing y^{12}

6) If a term in the expansion of $\left(2x^2 + \frac{m}{y}\right)^3$ is $\frac{54x^2}{y^2}$, the value of m is

7) Given $\left(x^3 + \frac{1}{x^3}\right)^8$, find the constant term

8) A term in the expansion of $(mx - 4)^8$ is $1451520x^4$. The value of m is

Permutations and Combinations: Lesson 5

Part Two: Particular Terms

9) Determine the sixth term in the expansion of $\left(x^2 - \frac{3}{x}\right)^9$

10) In the expansion of $\left(x+y\right)^{14}$, what is the numerical coefficient of the term containing x^3y^{11}

11) Given $(8x^6 - 7y^3)^9$, determine the position of the term containing x^{36}

12) The fifth term in the expansion of $\left(a^4 - \frac{3}{a}\right)^n$ contains a^4. Determine the value of n.

13) The term $-1080a^2b^3$ occurs in the expansion of $(2a - 3b)^n$. Determine the value of n.

14) Find and simplify the fourth term in the expansion of $(2a - 3b)^6$

15) How many terms are in the expansion of $\left(x^3 + \frac{1}{x}\right)^7$

Part Two: Particular Terms

1) The k for the middle term is $10 \div 2 = 5$

$$\begin{aligned}&{}_{10}C_5(2x)^{10-5}(-6)^5\\&=252(2x)^5(-6)^5\\&=252(32x^5)(-7776)\\&=-62705664x^5\end{aligned}$$

2) The k for the middle term is $8 \div 2 = 4$

$$\begin{aligned}&{}_8C_4(3x)^{8-4}(2y)^4\\&=70(3x)^4(2y)^4\\&=70(81x^4)(16y^4)\\&=90720x^4y^4\end{aligned}$$

3) First find the k-value of the term containing z^2

$$\boxed{{}_6C_{\square}(5z)^{6-\square}(9y)^{\square}}$$

By inspection, we'll get z^2 when k = 4

$$\begin{aligned}&{}_6C_4(5z)^{6-4}(9y)^4\\&=15(5z)^2(9y)^4\\&=15(25z^2)(6561y^4)\\&=2460375z^2y^4\end{aligned}$$

4) First find the k-value of the term containing x^{36}

$$\boxed{{}_9C_{\square}(8x^6)^{9-\square}(-7y^3)^{\square}}$$

By inspection, we'll get x^{36} when k = 3

$$\begin{aligned}&{}_9C_3(8x^6)^{9-3}(-7y^3)^3\\&=84(8x^6)^6(-7y^3)^3\\&=84(262144x^{36})(-343y^9)\\&=-7552892928x^{36}y^9\end{aligned}$$

5) First find the k-value of the term containing y^{12}

$$\boxed{{}_5C_{\square}(4z^5)^{5-\square}(3y^3)^{\square}}$$

By inspection, we'll get y^{12} when k = 4

$$\begin{aligned}&{}_5C_4(4z^5)^{5-4}(3y^3)^4\\&=5(4z^5)(3y^3)^4\\&=5(4z^5)(81y^{12})\\&=1620z^5y^{12}\end{aligned}$$

6) Use the formula $t_{k+1} = {}_nC_k x^{n-k}y^k$ to solve this question. First place everything you know into the equation, leaving k blank for now.

$$t_{\square+1} = {}_3C_{\square}\left(2x^2\right)^{3-\square}\left(\frac{m}{y}\right)^{\square}$$

By inspection, we get a term containing $\frac{x^2}{y^2}$ when k = 2.

$$t_{2+1} = {}_3C_2\left(2x^2\right)^{3-2}\left(\frac{m}{y}\right)^2$$

$$t_{2+1} = {}_3C_2\left(2x^2\right)\left(\frac{m}{y}\right)^2$$

$$t_3 = \frac{6x^2m^2}{y^2}$$

Now plug in the known term in the left side

$$\frac{54\cancel{x^2}}{\cancel{y^2}} = \frac{6\cancel{x^2}m^2}{\cancel{y^2}}$$

$$54 = 6m^2$$

$$9 = m^2$$

$$m = 3$$

7)

$$\boxed{(x^3)^{8-k}\left(\frac{1}{x^3}\right)^k}$$

$$k=0 \quad (x^3)^8\left(\frac{1}{x^3}\right)^0 = x^{24}$$

$$k=1 \quad (x^3)^7\left(\frac{1}{x^3}\right)^1 = \frac{x^{21}}{x^3} = x^{18}$$

$$k=2 \quad (x^3)^6\left(\frac{1}{x^3}\right)^2 = \frac{x^{18}}{x^6} = x^{12}$$

$$k=3 \quad (x^3)^5\left(\frac{1}{x^3}\right)^3 = \frac{x^{15}}{x^9} = x^6$$

$$k=4 \quad (x^3)^4\left(\frac{1}{x^3}\right)^4 = \frac{x^{12}}{x^{12}} = 1$$

$$\begin{aligned}&{}_8C_4(x^3)^{8-4}\left(\frac{1}{x^3}\right)^4\\&=70(x^3)^4\left(\frac{1}{x^3}\right)^4\\&=\frac{70x^{12}}{x^{12}}\\&=70\end{aligned}$$

8) Use the formula $t_{k+1} = {}_nC_k x^{n-k}y^k$ to solve this question.
First place everything you know into the equation, leaving k blank for now.

$$t_{\square+1} = {}_8C_{\square}(mx)^{8-\square}(-4)^{\square}.$$

By inspection, we get a term containing x^4 when k = 4.

$$t_{4+1} = {}_8C_4(mx)^{8-4}(-4)^4$$

$$t_{4+1} = {}_8C_4(mx)^4(-4)^4$$

$$t_5 = 17920m^4x^4$$

Now plug in the known term in the left side

$$1451520x^4 = 17920m^4x^4$$

$$1451520 = 17920m^4$$

$$81 = m^4$$

$$m = 3$$

Permutations and Combinations: Lesson 5

Part Two: Particular Terms

9) Use the formula $t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$.

For the sixth term, $k = 5$

$$t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$$

$$t_{5+1} = {}_9C_5(x^2)^{9-5}\left(-\frac{3}{x}\right)^5$$

$$t_6 = 126(x^2)^4\left(-\frac{3}{x}\right)^5$$

$$t_6 = 126x^8\left(-\frac{243}{x^5}\right)$$

$$t_6 = -30618x^3$$

11)

First find the k value of the term containing x^{36}

$$\boxed{{}_9C_{\square}(8x^6)^{9-\square}(-7y^3)^{\square}}$$

By inspection, we'll get x^{36} when $k = 3$

This corresponds to the **fourth term** of the expansion

13)

Use the formula

$$t_{k+1} = {}_nC_k(x)^{n-k}(y)^k.$$

First predict what k has to be in order to get the required term:
To get back b^3, there is only one option for k. It has to be 3.

$$t_{3+1} = {}_nC_3(x)^{n-3}(y)^3$$

To get back a^2, **the value of n must be 5**

$$t_{3+1} = {}_5C_3(2a)^{5-3}(-3b)^3$$

$$t_4 = {}_5C_3(2a)^{5-3}(-3b)^3$$

$$t_4 = 10(4a^2)(-27b^3)$$

$$t_4 = -1080a^2b^3$$

10) Use the formula $t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$.

First predict what k has to be in order to get the required term: $t_{\square+1} = {}_{14}C_{\square}(x)^{14-\square}(y)^{\square}$

If the empty box is filled with the number 11, you will get the required term.

$$t_{11+1} = {}_{14}C_{11}(x)^{14-11}(y)^{11}$$

$$t_{12} = 364x^3y^{11}$$

12)

Use the formula $t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$.

To get the fifth term, $k = 4$.

$$t_{4+1} = {}_nC_4(a^4)^{n-4}\left(-\frac{3}{a}\right)^4$$

By inspection, we can see that ***n* = 6** will give the correct exponent for a.

$$t_5 = {}_6C_4(a^4)^{6-4}\left(-\frac{3}{a}\right)^4$$

$$t_5 = 15(a^4)^2\left(-\frac{3}{a}\right)^4$$

$$t_5 = 15a^8\left(-\frac{81}{a^4}\right)$$

$$t_5 = -1215a^4$$

14)

Use the formula $t_{k+1} = {}_nC_k(x)^{n-k}(y)^k$.

To get the fourth term, $k = 3$.

$$t_{3+1} = {}_6C_3(2a)^{6-3}(-3b)^3$$

$$t_4 = 20(2a)^3(-3b)^3$$

$$t_4 = 20(8a^3)(-27b^3)$$

$$t_4 = -4320a^3b^3$$

15) The number of terms in an expansion is one more than the value of n.
Therefore, there are 8 terms.