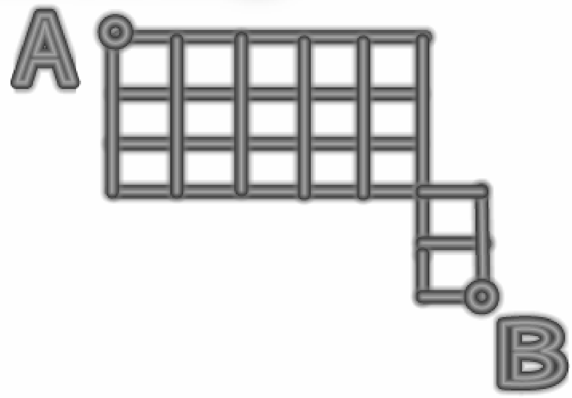
Pure Math 30:

Permutations and Combinations



LESSON 2

Combinations

Pure Math 30: EXPLAINED!

By Barry Mabillard

Basic combinations: In the previous lesson, when using the fundamental counting principal or permutations, the order of items to be arranged mattered. If all you want to do is select items, and don't care what order they're in, you can use *combinations*.

Example 1: A committee of 4 people is to be formed from a group of 9 people. How many possible committees can be formed?

This question is a *combination* since order is not important.

The answer is: ${}_9C_4 = 126$.

The way to do questions like this is to use the ${}_nC_r$ feature on your calculator.

n is the total number of items.
r is the number of items you want to choose.

For Example 1, you would type into your TI-83:
9 → Math → PRB → ${}_nC_r$ → enter → 4

Example 2: A pizza can have 3 toppings out of a possible 7 toppings. How many different pizza's can be made?

There are 7 toppings in total, and by selecting 3, we will make different types of pizza.

This question is a *combination* since having a different order of toppings will not make a different pizza.

The answer is: ${}_7C_3 = 35$.

Questions:

1) How many ways can you select 17 songs for mix CD out of a possible 38 songs?

2) If an ice cream dessert can have 2 toppings, and 9 are available, how many different topping selections can you make?

3) If there are 17 randomly placed dots on a circle, how many lines can you form using any 2 dots?

4) A committee of 4 people is to be formed from a pool of 13 people. How many different committees can be formed?

5) If there are 15 dots on a circle, how many triangles can be formed?

Answers:

1) ${}_{38}C_{17} = 2.878 \cdot 10^{10}$
2) ${}_9C_2 = 36$
3) ${}_{17}C_2 = 136$
4) ${}_{13}C_4 = 715$
5) ${}_{15}C_3 = 455$

Permutations and Combinations
Lesson 2, Part two: specific items

Combinations including specific items: Sometimes you will be forced to include or exclude particular items when making a combination. This will reduce the number of items in your selection pool, and also the number of items you can select.

Example 1: A school committee of 5 is to be formed from 12 students. How many committees can be formed if John must be on the committee?

If John must be on the committee, you'll have 11 students remaining, out of which you can choose 4.

$$_{11}C_4 = 330$$

Example 2: From a deck of 52 cards, a 5 card hand is dealt. How many distinct five card hands are there if the queen of spades & the four of diamonds must be in the hand?

If the queen of spades and the four of diamonds must be in the hand, you'll have 50 cards remaining out of which you are choosing 3.

$$_{50}C_3 = 19600$$

Questions:

1) There are 45 songs, and you want to make a mix CD of 18 songs that must include 3 particular songs. How many different selections could you make?

2) If a committee of 7 people is to be formed from a pool of 12 people, but Rachel & Megan must be on the committee, how many selections can be made?

3) There are 9 possible toppings for a sandwich, but you only want 4 toppings, one of which must be pickles. How many different sandwiches can be made?

4) A lottery has 47 numbers, and you must pick 7. How many different combinations are possible if your lucky number 8 must be on each ticket?

5) There are 8 parents and 43 students going on a school trip. Two groups are made, a large one with 30 students and 5 parents, and a small group with 13 students and 3 parents.

a) How many different ways can the parents be chosen for the small group?

b) How many ways can the students be chosen for the large group if Stefan and Dylan must be in the small group?

c) How many ways can students be chosen for the small group if Wade & both his parents must be in the small group?

Answers:

1) $_{42}C_{15} = 9.867 \cdot 10^{10}$
2) $_{10}C_5 = 252$
3) $_8C_3 = 56$
4) $_{46}C_6 = 9366819$
5) a) $_8C_3 = 56$
b) $_{41}C_{30} = 3.16 \cdot 10^9$
c) $_{42}C_{12} = 1.106 \cdot 10^{10}$

Combinations From Multiple Selection Pools: When presented with multiple groups of items from which you are required to make a selection, you will *MULTIPLY* the separate cases together.

Example 1: A committee of 3 boys and 5 girls is to be formed from a group of 10 boys and 11 girls. How many committees are possible?

Out of the 10 boys, we must choose 3: $_{10}C_3$
Out of the 11 girls, we must choose 5: $_{11}C_5$

$$_{10}C_3 \times {_{11}C_5} = 55440$$

Example 2: From a deck of 52 cards, a 7 card hand is dealt. How many distinct hands are there if the hand must contain 2 spades and 3 diamonds?

There are 13 spades, we must include 2: $_{13}C_2$
There are 13 diamonds, we must include 3: $_{13}C_3$
Since we can't have more than 2 spades and 3 diamonds, the remaining two cards must be pulled out from the 26 remaining clubs & hearts. $_{26}C_2$

$$_{13}C_2 \times {_{13}C_3} \times {_{26}C_2} = 7250100$$

Questions:

1) There are 5 meats and 9 veggies available to make a sandwich. How many sandwiches have 2 meats and 6 veggies?

2) How many committees can be formed from 11 men & 9 women if 3 men and 3 women must be on the committee?

3) How many 7 card hands are possible if all the kings must be in the hand?

4) How many 13 card Bridge hands are possible if there are 3 queens, 2 tens, and 2 aces?

5) If there are 19 rock songs and 20 pop songs, how many different ways can you select 12 rock and 8 pop songs for a mix cd?

6) If a crate of radio controlled cars contains 10 working cars and 4 defective cars, how many ways can you take out 5 cars and have only three work?

7) If a student must select two courses from Group A, two courses from Group B, and one course from group C, how many combinations are there?

Group A	Group B	Group C
Math 30 Chemistry 30 Physics 30 Biology 30	English 30 Social 30	Math 31 Science 30 French 30

Answers:

1) $_5C_2 \cdot {_9C_6} = 840$
2) $_{11}C_3 \cdot {_9C_3} = 13860$
3) $_4C_4 \cdot {_{48}C_3} = 17296$
4) $_4C_3 \cdot {_4C_2} \cdot {_4C_2} \cdot {_{40}C_6} = 552726720$
Use $_{40}C_6$ since we can't have any queens, tens, or aces in the remaining six cards.
5) $_{19}C_{12} \cdot {_{20}C_8} = 6347376360$
6) $_{10}C_3 \cdot {_4C_2} = 720$
7) $_4C_2 \cdot {_2C_2} \cdot {_3C_1} = 18$

At Least / At Most: These questions will require you to ADD all the possible cases together. Know the shortcuts too!

Example 1: A committee of 5 people is to be formed from a group of 4 men & 7 women. How many possible committees can be formed if at least 3 women are on the committee?

If at least three women are on the committee, that means we can have a committee with 3 women, 4 women, or 5 women. Find the combinations for each case separately, then add them all together.

3 Women:	4 Women:	5 Women:
Seven women to choose from, and we require 3: ${}_7C_3$	Seven women to choose from and we require 4 ${}_7C_4$	Seven women to choose from and we require 5 ${}_7C_5$
Four men to choose from and we require 2: ${}_4C_2$	Four men to choose from and we require 1 ${}_4C_1$	Four men to choose from and we require 0 ${}_4C_0 = 1$
Total Combinations: ${}_7C_3 \cdot {}_4C_2$	Total Combinations: ${}_7C_4 \cdot {}_4C_1$	Total Combinations: ${}_7C_5$

$${}_7C_3 \cdot {}_4C_2 + {}_7C_4 \cdot {}_4C_1 + {}_7C_5 = 371$$

Example 2: From a deck of 52 cards, a 5 card hand is dealt. How many distinct hands can be formed if there are at least 2 queens?

We could approach this question the same way as the last one, but let's use a shortcut instead.

At Least/At Most = Total Cases − Unwanted Cases

The shortcut works since we have the same number of cards in each hand, and the unrestricted combinations for a 5 card hand must include every possible combination you can get! Simply subtract those you don't want, and you'll be left with the ones you do want.

The total possible cases would be a 5 card hand with no restrictions: ${}_{52}C_5$

The unwanted cases are:
no queens (Out of 48 non-queen cards, we get 5) ${}_{48}C_5$
only 1 queen (Out of 4 queens we get 1, and out of 48 non-queens we get 4) ${}_4C_1 \cdot {}_{48}C_4$

$${}_{52}C_5 - \left({}_{48}C_5 + {}_4C_1 \cdot {}_{48}C_4\right) = 108336$$

Example 3: From a deck of 52 cards, a 7 card hand is dealt. How many distinct hands can be formed if there are at most 6 black cards?

The total possible cases would be a 7 card hand with no restrictions: ${}_{52}C_7$

The unwanted case is:
7 black cards (There are 26 black cards, and we get 7) ${}_{26}C_7$

$${}_{52}C_7 - {}_{26}C_7 = 133126760$$

Notice how the shortcut takes way less time than adding up all the cases you do want.

Questions:

There are
52 cards in a standard deck
with jokers removed.

There are 4 suits:
Spades, Clubs, Hearts, Diamonds

There are 26 black cards, and 26 red cards.

Each suit has 13 cards,
each of a different rank.

Face cards are Jacks, Queens, and Kings.
There are 12 face cards
in a deck.

1) A student council of 5 members is to be formed from a selection pool of 8 boys and 9 girls. How many committees will have:

a) All girls?

b) At least 4 girls?

c) At least 3 girls?

d) At most 2 boys?

e) At least 1 boy?

2) A student council of 5 members is to be formed from a selection pool of 6 boys and 8 girls. How many councils can have:

a) Jason on the council?

b) Katie, but not Alex?

c) Zach, but not Caroline, Allison, or James?

d) At least 3 boys, but one of those boys can't be Brian?

3) From a deck of 52 cards, how many 5 card hands have:

a) Exactly 2 red cards?

b) At least one red card?

c) At least two black cards?

d) At most three 9's?

e) No 5's

f) At least 1 king?

g) At most 2 face cards

4) A research team of 6 people is to be formed from 10 chemists, 5 politicians, 8 economists, and 15 biologists. How many teams have:

a) At least 5 chemists?

b) Exactly three economists?

c) Four chemists, but no economists?

d) At least 2 biologists?

e) 4 economists and 2 biologists?

Answers:

1) a) To select all girls for the committee, you can totally ignore the set of boys, leaving you with a combination of $_9C_5$ **= 126**

b) At least four girls means we can have 4 girls and 1 boy, ***or***, 5 girls. $_9C_4 \cdot {}_8C_1 + {}_9C_5 = 1134$

c) At least three girls means we can have 3 girls and 2 boys, ***or*** 4 girls and 1 boy, ***or***, 5 girls.
This gives a combination of $_9C_3 \cdot {}_8C_2 + {}_9C_4 \cdot {}_8C_1 + {}_9C_5 = 3486$

d) At most two boys will lead to exactly the same cases as above, so we get the same answer of $_9C_3 \cdot {}_8C_2 + {}_9C_4 \cdot {}_8C_1 + {}_9C_5 = 3486$

e) Since it would be time consuming to add up all the separate cases, use the formula
Total unrestricted cases – Cases with all girls: $_{17}C_5 - {}_9C_5 = 6062$

2) a) If Jason is on the council, this reduces the selection pool to only 13 people, out of which we still need to select 4. $_{13}C_4 = 715$

b) If Katie is on the council, this reduces the selection pool to 13, and we still need to select 4. If Alex must not be on the council, this further reduces the selection pool to 12. $_{12}C_4 = 495$

c) If Zach is on the council, this reduces the selection pool to 13, and we still need to select 4. If Caroline, Allison, and James must not be on the council, this further reduces the selection pool to only 10 people. $_{10}C_4 = 210$

d) Since Brian can't be on the council, this reduces the pool of boys to 5 instead of 6. Once we do this, we can have 3 boys/2 girls, ***or*** 4 boys/1 girl, ***or*** 5 boys. $_5C_3 \cdot {}_8C_2 + {}_5C_4 \cdot {}_8C_1 + {}_5C_5 = 321$

3) a) 2 red cards must be drawn from the 26 red cards, leaving 3 black cards to be drawn from the 26 black cards.
$_{26}C_2 \cdot {}_{26}C_3 = 845000$

b) Since it would be time consuming to add up all the separate cases we want, use the shortcut:
Total unrestricted cases – Cases with only black cards:
$= {}_{52}C_5 - {}_{26}C_5 = 2533180$

c) Once again, use the formula:
Total unrestricted cases – (Cases with no black cards (all red) + Cases with one back card)
$= {}_{52}C_5 - ({}_{26}C_5 + {}_{26}C_1 \cdot {}_{26}C_4) = 2144480$

d) Use the formula *Total unrestricted cases – Cases with all four 9's:*
$= {}_{52}C_5 - {}_4C_4 \cdot {}_{48}C_1 = 2598912$

e) If there are no 5's, simply remove the four 5's from the selection pool, leaving 48 cards to select from. $_{48}C_5 = 1712304$

f) To get at least one king, use the formula:
Total unrestricted cases – Cases with no kings:
$= {}_{52}C_5 - {}_{48}C_5 = 886656$

g) At most 2 face cards means we can have no face card ***or*** 1 face card/4 other cards, ***or*** 2 face cards/3 other cards. Add these cases together. (Note that in this case the shortcut would require just as many calculations for the unwanted cases, so you might as well just add everything up!)
$_{40}C_5 + {}_{12}C_1 \cdot {}_{40}C_4 + {}_{12}C_2 \cdot {}_{40}C_3 = 2406768$

4) a) $_{10}C_5 \times {}_{28}C_1 + {}_{10}C_6 = 7266$

b) $_8C_3 \times {}_{30}C_3 = 227360$

c) $_{10}C_4 \times {}_{20}C_2 = 39900$

d) $_{38}C_6 - ({}_{23}C_6 + {}_{23}C_5 \times {}_{15}C_1) = 2154999$ *Unrestricted cases – (No Biologists* **+** *Case with 1 Biologist)*

e) $_8C_4 \times {}_{15}C_2 = 7350$

Permutations & Combinations Together: In the following questions, you will see how both arranging & selecting can be used together in solving problems.

Example 1: How many arrangements of the word TRIGONAL can be made if only two vowels and three consonants are used?

First we need to choose two vowels $_3C_2$ and then three consonants $_5C_3$. Now that we have the five letters required to make the word, arrange them in 5! ways.

Answer: $_3C_2 \cdot {_5C_3} \cdot 5! = 3600$

Example 2: There are 7 men and 10 women on a committee selection pool. A committee consisting of President, Vice-President, and Treasurer is to be formed. How many ways can exactly two men be on the committee?

There are $_7C_2$ ways of selecting two men, and $_{10}C_1$ ways of selecting a woman. Since each position in the committee is different, arrange the three people in 3! ways.

Answer: $_7C_2 \cdot {_{10}C_1} \cdot 3! = 1260$

Questions:

1) How many ways can you arrange the letters of the word ORANGES if:
a) two vowels and two consonants are used to make a four letter word?
b) two vowels and three consonants are used to make a five letter word?
c) three letters from ORANG and one letter from ES are used to make a four letter word?
d) Given the sentence KITY LUVZ ORANGES, how many arrangements can you make if a seven letter word is formed using two letters from KITY, two letters from LUVZ, and three letters from ORANGES?

2) If there are 14 boys & 12 girls in a selection pool, and a school council of President, VP, Treasurer, and Secretary is to be formed, how many ways can:
a) exactly one boy be on the council?
b) exactly two girls be on the council?
c) no boys be on the council?

3) There are 5 pop CD's and 10 rock CD's that can be placed in a multi-disc changer. How many ways can three pop & two rock albums be ordered, provided all three pop albums must play first?

4) If a sports team with six unique positions is to be formed from 5 men and 7 women, in how many ways can two positions be filled by men and four positions by women?

Answers:

1) a) $_3C_2 \cdot {_4C_2} \cdot 4! = 432$
b) $_3C_2 \cdot {_4C_3} \cdot 5! = 1440$
c) $_5C_3 \cdot {_2C_1} \cdot 4! = 480$
d) $_4C_2 \cdot {_4C_2} \cdot {_7C_3} \cdot 7! = 6350400$

2) a) $_{14}C_1 \cdot {_{12}C_3} \cdot 4! = 73920$
b) $_{12}C_2 \cdot {_{14}C_2} \cdot 4! = 144144$
c) $_{12}C_4 \cdot {_{14}C_0} \cdot 4! = 11800$
(Or simply use $_{12}P_4$)

3) $_5P_3 \cdot {_{10}P_2} = 5400$

4) $_5C_2 \cdot {_7C_2} \cdot 6! = 252000$

<u>Other Types:</u> In these unique question types, memorize the following formulas and use them to get the answer.

Handshakes / Teams: ${}_nC_2$ *n = number of people or teams*

Diagonals: ${}_nC_2 - n$ *n = number of sides*

Multiple Combinations: ${}_nC_r \times k$ *k = number of times all the possible combinations must happen*

Making Shapes: ${}_nC_k$ *n = points in a circle, k = number of vertices*

Choosing One Or More: $2^n - 1$ *n = number of items in total*

<u>Example 1:</u> 12 people at a party shake hands once with everyone else in the room. How many handshakes took place?

There are 12 people, so this is our *n* value.
Using the formula for handshakes, we get $\mathbf{{}_{12}C_2 = 66}$

<u>Example 2:</u> A polygon has 7 sides. How many diagonals can be formed?

There are 7 sides, so this is our *n* value.
Using the diagonal formula, we get $\mathbf{{}_7C_2 - 7 = 14}$

<u>Example 3:</u> If each of the 8 teams in a league must play each other three times, how many games will be played?

This is similar to a handshake question, since each member of the league must meet every other member of the league. If they only played each other once, there would be ${}_8C_2$ games. Since each pairing of teams will occur three times, the answer will be triple.
Answer: $\mathbf{{}_8C_2 \times 3 = 84}$

<u>Example 4:</u> If there are 15 dots on a circle, how many triangles can be formed?

There are 15 dots in total, and to make a triangle we need to select any three of those dots.
Answer: $\mathbf{{}_{15}C_3 = 455}$

<u>Example 5:</u> In how many ways can you choose <u>one or more</u> of 12 different candies?

The way this question is worded, you could take one candy, two candies, three candies, and so on.
The solution is found by adding all possible selections using ${}_{12}C_1 + {}_{12}C_2 + {}_{12}C_3 + \dots {}_{12}C_{12}$. However, this would take a long time. Fortunately there is a shortcut for ***<u>one or more</u>*** questions!
Each candy can be dealt with in two ways. It can be chosen or not chosen. This will give 2 possibilities for the first candy, 2 for the second, and so on. By multiplying the cases together we'll get 2^{12}. Since the case of no candy being selected is not an option, we'll have to subtract 1 from our answer.
There are $\mathbf{2^{12} - 1 = 4095}$ ways of selecting one or more candies.

Questions:

1) If there are 10 people at a party and each person shakes hands once with every other person, how many handshakes took place?

2) If there are 9 sports teams on a league, and each team plays each other once, how many games were played?

3) A polygon has 8 sides, how many diagonals can be formed?

4) A polygon has 11 sides. How many diagonals can be formed?

5) If each of the 13 teams in a league plays each other twice, how many games are to be played?

6) If there are 17 randomly placed dots on a circle, how many lines can you form using any 2 dots?

7) If there are 9 toys in a toybox and a child wants to take out one or more toys, how many ways can this be done?

Answers:

1) ${}_{10}C_2 = 45$
2) ${}_{9}C_2 = 36$
3) ${}_{8}C_2 - 8 = 20$
4) ${}_{11}C_2 - 11 = 44$
5) ${}_{13}C_2 \times 2 = 156$
6) ${}_{17}C_2 = 136$
7) $2^9 - 1 = 511$