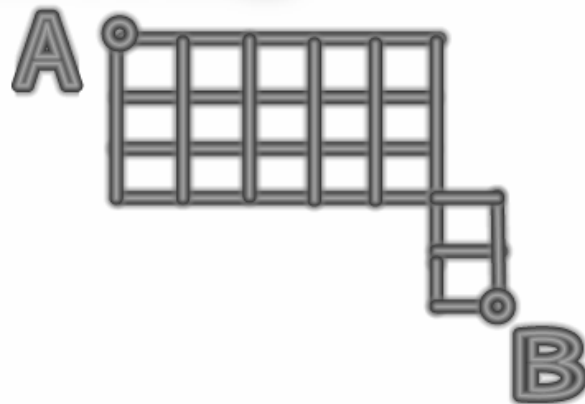


Pure Math 30:

Permutations and Combinations



Lesson 1

Permutations

Pure Math
30:

EXPLAINED!

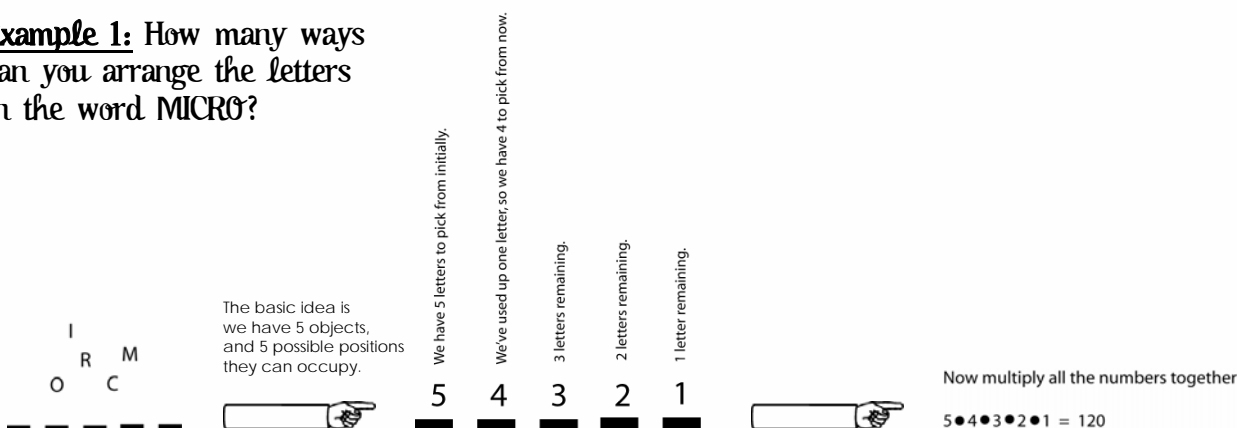
By
Barry
Mabillard

Permutations and Combinations

Lesson 1, Part One: The Fundamental Counting Principle

The Fundamental Counting Principle: This is an easy way to determine how many ways you can arrange items. The following examples illustrate how to use it:

Example 1: How many ways can you arrange the letters in the word MICRO?



Example 2: How many ways can 8 different albums be arranged?

We *could* approach this question in the same way as the last one by using the spaces and multiplying all the numbers, but there is a shorter way. The factorial function on your calculator will perform this calculation for you!

$$8! = 8 \cdot 7 \cdot 6 \cdot \dots \cdot 2 \cdot 1$$
$$8! = 40320$$

Questions:

- 1) How many ways can the letters in the word PENCIL be arranged?
- 2) If there are four different types of cookies, how many ways can you eat all of them?
- 3) If three albums are placed in a multi-disc stereo, how many ways can the albums be played?
- 4) How many ways can you arrange all the letters in the alphabet?
- 5) How many ways can you arrange the numbers 24 through 28 (inclusive)?

TI-83 Info
You can get the
factorial function
using:
Math → **Prb** → **!**

Answers:

- 1) $6! = 720$
- 2) $4! = 24$
- 3) $3! = 6$
- 4) $26! = 4.03 \cdot 10^{26}$
- 5) $5! = 120$

Permutations and Combinations

Lesson 1, Part Two: Repetitions Not Allowed

Repetitions Not Allowed: In many cases, some of the items we want to arrange are identical. For example, in the word TOOTH, if we exchange the places of the two O's, we still get TOOTH. Because of this, we have to get rid of extraneous cases by dividing out repetitions.

Example 1: How many ways can you arrange the letters in the word THESE?

Do this as a fraction. Factorial the total number of letters and put this on top. Factorial the repeated letters and put them on the bottom.

$$\frac{5!}{2!} = \frac{120}{2} = 60$$

There are 5 letters altogether
There are 2 E's

Example 2: How many ways can you arrange the letters in the word REFERENCE?

$$\frac{9!}{2! \cdot 4!} = \frac{362880}{2 \cdot 24} = 7560$$

There are 9 letters altogether
There are 2 R's and 4 E's

TI-83 Info

Make sure you put the denominator in brackets or you'll get the wrong answer!

Questions:

- 1) How many ways can the letters in the word SASKATOON be arranged?
- 2) How many ways can the letters in the word MISSISSIPPI be arranged?
- 3) How many ways can the letters in the word MATHEMATICS be arranged?
- 4) If there are eight cookies (4 chocolate chip, 2 oatmeal, and 2 chocolate) in how many different orders can you eat all of them?
- 5) If a multiple choice test has 10 questions, of which one is answered A, 4 are answered B, 3 are answered C, and 2 are answered D, how many answer sheets are possible?

Answers:

- 1) $\frac{9!}{2! \cdot 2! \cdot 2!} = 45360$
- 2) $\frac{11!}{4! \cdot 4! \cdot 2!} = 34650$
- 3) $\frac{11!}{2! \cdot 2! \cdot 2!} = 4989600$
- 4) $\frac{8!}{4! \cdot 2! \cdot 2!} = 420$
- 5) $\frac{10!}{4! \cdot 3! \cdot 2!} = 12600$

Permutations and Combinations

Lesson 1, Part Three: Repetitions Are Allowed

Repetitions Are Allowed: Sometimes we are interested in arrangements allowing the use of items more than once.

Example 1: There are 9 switches on a fuse box. How many different arrangements are there?

2 2 2 2 2 2 2 2 2

Each switch has two possible positions, on or off.

Placing a 2 in each of the 9 positions, we have $2^9 = 512$.

Example 2: How many 3 letter words can be created, if repetitions are allowed?

26 26 26

There are 26 letters to choose from, and we are allowed to have repetitions.

There are $26^3 = 17576$ possible three letter words.

Questions:

1) If there are 4 light switches on an electrical panel, how many different orders of on/off are there?

2) How many 5 letter words can be formed, if repetitions are allowed?

3) How many three digit numbers can be formed? (Zero can't be the first digit)

4) A coat hanger has four knobs. If you have 6 different colors of paint available, how many different ways can you paint the knobs?

There
are ~~ten~~ digits in
total, from zero
to nine.

Answers:

- 1) $2^4 = 16$
- 2) $26^5 = 11881376$
- 3) $9 \cdot 10 \cdot 10 = 900$
- 4) $6^4 = 1296$

Permutations and Combinations

Lesson 1, Part Four: Arranging A Subset

Arranging a subset of items: Sometimes you will be given a bunch of objects, and you want to arrange only a few of them:

Example 1: There are 10 people in a competition. How many ways can the top three be ordered?

There are 10 people, and any of them could win first place.	Nine remain who could win 2nd	Eight remain who could win 3rd
<u>10</u>	<u>9</u>	<u>8</u>

Since only 3 positions can be filled, we have 3 spaces. Multiplying, we get 720.

A fast way to do questions like this is to use the ${}_nP_r$ feature on your calculator.

n is the total number of items.
 r is the number of items you want to order.

For Example 1, you would type:

$10 \rightarrow \text{Math} \rightarrow \text{PRB} \rightarrow {}_nP_r \rightarrow \text{enter} \rightarrow 3$

Example 2: There are 12 movies playing at a theater, in how many ways can you see two of them consecutively?

12 11 **OR** ${}_{12}P_2$

You could use the spaces, but let's try this question with the permutation feature. There are 12 movies, and you want to see 2, so type ${}_{12}P_2$ into your calculator, and you'll get 132.

Example 3: How many 4 letter words can be created if repetitions are not allowed?

26 25 24 23 **OR** ${}_{26}P_4$

The answer is: ${}_{26}P_4 = 358800$

Questions:

- 1) How many three letter words can be made from the letters of the word KEYBOARD?
- 2) If there are 35 songs and you want to make a mix CD with 17 songs, how many different ways could you arrange them?
- 3) There are six different colored balls in a box, and you pull them out one at a time. How many different ways can you pull out four balls?
- 4) A committee is to be formed with a president, a vice-president, and a treasurer. There are 10 people to be selected from. How many different committees are possible?
- 5) A baseball league has 13 teams, and each team plays each other twice; once at home, and once away. How many games are scheduled?

Answers:

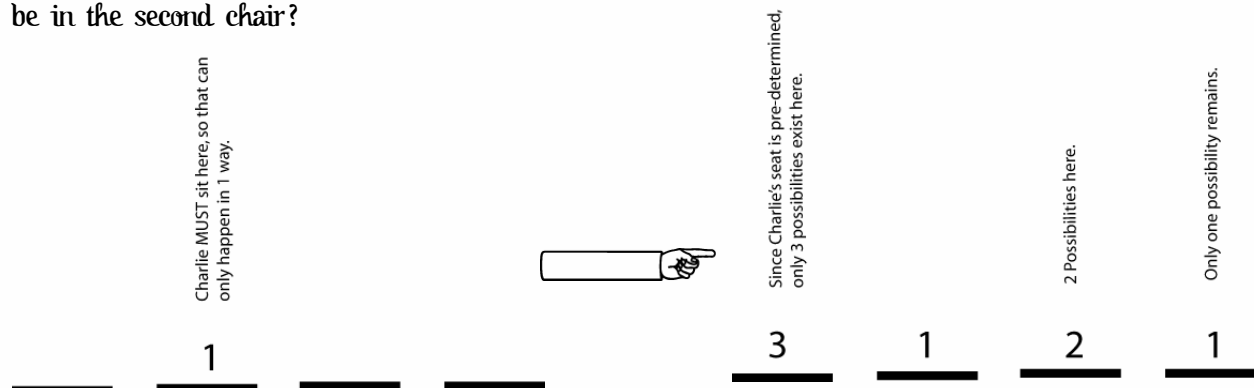
- 1) ${}_8P_3 = 336$
- 2) ${}_{35}P_{17} = 1.6 \cdot 10^{24}$
- 3) ${}_6P_4 = 360$
- 4) ${}_{10}P_3 = 720$
- 5) ${}_{13}P_2 = 156$

Permutations and Combinations

Lesson 1, Part Five: Specific Positions

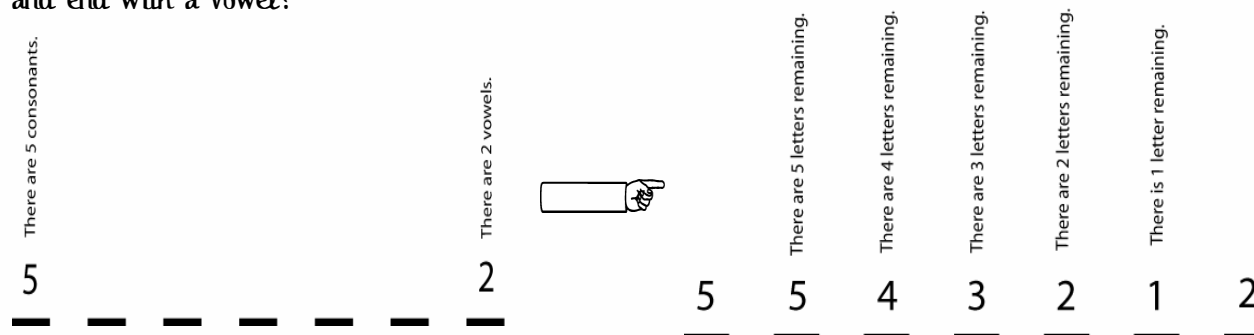
Specific Positions: Frequently when arranging items, a particular position must be occupied by a particular item. The easiest way to approach these questions is by analyzing how many possible ways each space can be filled.

Example 1: How many ways can Adam, Beth, Charlie, and Doug be seated in a row if Charlie must be in the second chair?



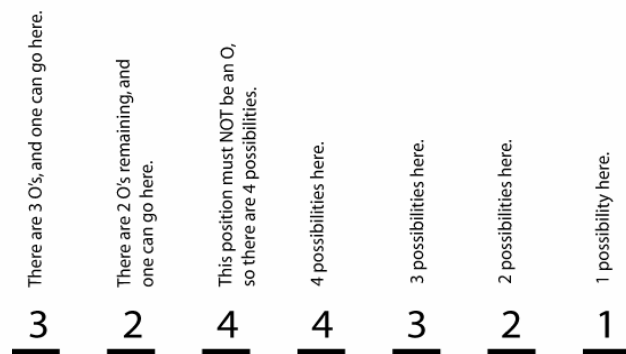
The answer is 6.

Example 2: How many ways can you order the letters of KITCHEN if it must start with a consonant and end with a vowel?



The answer is 1200.

Example 3: How many ways can you order the letters of TORONTO if it begins with exactly two O's?



Exactly Two O's means the first 2 letters must be O, and the third must NOT be an O.

If the question simply stated **two O's**, then the third letter could also be an O, since that case wasn't excluded.

Don't forget repetitions! The answer from the left will be the numerator with repetitions divided out.

$$\frac{576}{3! \cdot 2!} = 48$$

Permutations *and* Combinations

Lesson 1, Part Five: Specific Positions

Questions:

1) Six Pure Math 30 students (Brittany, Geoffrey, Jonathan, Kyle, Laura, and Stephanie) are going to stand in a line:
How many ways can they stand if:

- a) Stephanie must be in the third position?
- b) Geoffrey must be second and Laura third?
- c) Kyle can't be on either end of the line?
- d) Boys and girls alternate, with a boy starting the line?
- e) The first three positions are boys, the last three are girls?
- f) A girl must be on both ends?
- g) The row starts with two boys?
- h) The row starts with exactly two boys?
- i) Brittany must be in the second position, and a boy must be in the third?



2) How many ways can you order the letters from the word TREES if:

- a) A vowel must be at the beginning?
- b) It must start with a consonant and end with a vowel?
- c) The R must be in the middle?
- d) It begins with an E?
- e) It begins with exactly one E?
- f) Consonants & vowels alternate?

Permutations and Combinations

Lesson 1, Part Five: Specific Positions

Answers:

1) a) If Stephanie must be in the third position, place a one there to reserve her spot. You can then place the remaining 5 students in any position. $\underline{5} \underline{4} \underline{1} \underline{3} \underline{2} \underline{1} = 120$

b) Place a 1 in the second position to reserve Geoffrey's spot, and place a 1 in the third position to reserve Laura's spot. Place the remaining students in the other positions. $\underline{4} \underline{1} \underline{1} \underline{3} \underline{2} \underline{1} = 24$

c) Since Kyle can't be on either end, 5 students could be placed on one end, then 4 at the other end. Now that 2 students are used up, there are 4 that can fill out the middle. $\underline{5} \underline{4} \underline{3} \underline{2} \underline{1} \underline{4} = 480$

d) Three boys can go first, then three girls second. Two boys remain, then two girls. Then one boy and one girl remain. $\underline{3} \underline{3} \underline{2} \underline{2} \underline{1} \underline{1} = 36$

e) Three boys can go first, then place the girls in the next three spots. $\underline{3} \underline{2} \underline{1} \underline{3} \underline{2} \underline{1} = 36$

f) Three girls could be placed on one end, then 2 girls at the other end. There are four students left to fill out the middle. $\underline{3} \underline{4} \underline{3} \underline{2} \underline{1} \underline{2} = 144$

g) Three boys could go first, then 2 boys second. Once those positions are filled, four people remain for the rest of the line. $\underline{3} \underline{2} \underline{4} \underline{3} \underline{2} \underline{1} = 144$

h) Three boys could go first, then two boys second. The third position can't be a boy, so there are three girls that could go here. Then, three people remain to fill out the line. $\underline{3} \underline{2} \underline{3} \underline{3} \underline{2} \underline{1} = 108$

i) Place a 1 in the second position to reserve Brittany's spot, then 3 boys could go in the third position. Now fill out the rest of the line with the four remaining people. $\underline{4} \underline{1} \underline{3} \underline{3} \underline{2} \underline{1} = 72$

2) Note that since there are 2 E's, all answers MUST be divided by 2! to eliminate repetitions.

a) There are two vowels that can go first, then four letters remain to fill out the other positions. $\underline{2} \underline{4} \underline{3} \underline{2} \underline{1} = 48$ (Answer = $48 / 2! = 24$)

b) Three consonants could go first, and two vowels could go last. There are three letters to fill out the remaining positions. $\underline{3} \underline{3} \underline{2} \underline{1} \underline{2} = 36$ (Answer = $36 / 2! = 18$)

c) Place a 1 in the middle spot to reserve the R's spot. Then fill out the rest of the spaces with the remaining 4 letters. $\underline{4} \underline{3} \underline{1} \underline{2} \underline{1} = 24$ (Answer = $24 / 2! = 12$)

d) Two E's could go in the first spot, then fill the remaining spaces with the 4 remaining letters. $\underline{2} \underline{4} \underline{3} \underline{2} \underline{1} = 48$ (Answer = $48 / 2! = 24$)

e) Two E's could go in the first spot, but the next letter must NOT be an E, so there are 3 letters that can go here. Fill out the three last spaces with the 3 remaining letters. $\underline{2} \underline{3} \underline{3} \underline{2} \underline{1} = 36$ (Answer = $36 / 2! = 18$)

f) Three consonants could go first, then 2 vowels, and so on. $\underline{3} \underline{2} \underline{2} \underline{1} \underline{1} = 12$ (Answer = $12 / 2! = 6$)

Permutations and Combinations

Lesson 1, Part Six: Adding Permutations

More than one case (Adding): Given a set of items, it is possible to form multiple groups by ordering any 1 item from the set, any 2 items from the set, and so on. If you want the total arrangements from multiple groups, you have to *ADD* the results of each case.

Example 1: How many words (of any number of letters) can be formed from CANS.

$$\begin{array}{cccccccccccc}
 \text{If you are ordering only one letter,} & & \text{If ordering two letters, any 4 can go here.} & & & & \text{If ordering three letters, any 4 can go here.} & & & & \text{If ordering four letters, any 4 can go here.} & & & & \\
 \text{any of the 4 can go here.} & & & & \text{Any 3 can go here.} & & \text{Any 3 can go here.} & & \text{Any 2 can go here.} & & & & \text{Any 3 can go here.} & & \text{Any 2 can go here.} & & \text{Any 1 can go here.} \\
 4 & + & 4 & 3 & + & 4 & 3 & 2 & + & 4 & 3 & 2 & 1
 \end{array}$$

Since we are allowed to have any number of letters in a word, we can have a 1 letter word, a 2 letter word, a 3 letter word, and a 4 letter word.

We can't have more than 4 letters in a word, since there aren't enough letters for that!

We could also write this using permutations: ${}_4P_1 + {}_4P_2 + {}_4P_3 + {}_4P_4 = 64$

The answer is **64**

Example 2: How many four digit positive numbers less than 4670 can be formed using the digits 1, 3, 4, 5, 8, 9 if repetitions are not allowed?

We must separate this question into different cases. Numbers in the 4000's have extra restrictions.

Case 1 - Numbers in the 4000's: There is only one possibility for the first digit {4}. The next digit has three possibilities. {1, 3, 5}. There are 4 possibilities for the next digit since any remaining number can be used, and 3 possibilities for the last digit.

$$\begin{array}{cccc}
 1 & 3 & 4 & 3 \\
 \hline
 & & & = 36
 \end{array}$$

Case 2 - Numbers in the 1000's and 3000's: There are two possibilities for the first digit {1, 3}. Anything goes for the remaining digits, so there are 5, then 4, then 3 possibilities.

$$\begin{array}{cccc}
 2 & 5 & 4 & 3 \\
 \hline
 & & & = 120
 \end{array}$$

Add the results together: $36 + 120 = 156$

Questions:

1) How many one-letter, two-letter, or three-letter words can be formed from the word PENCIL?

2) How many 3-digit, 4-digit, or 5-digit numbers can be made using the digits of 46723819?

3) How many numbers between 999 and 9999 are divisible by 5 and have no repeated digits?

Answers:

1) ${}_6P_1 + {}_6P_2 + {}_6P_3 = 156$

2) ${}_8P_3 + {}_8P_4 + {}_8P_5 = 8736$

3) There are two cases: The first case has five as the last digit, the second case has zero as the last digit. Remember the first digit can't be zero!

$$\begin{array}{cccc}
 8 & 8 & 7 & 1 \\
 \hline
 & & & (5) \\
 & & & = 448
 \end{array}$$

$$\begin{array}{cccc}
 9 & 8 & 7 & 1 \\
 \hline
 & & & (0) \\
 & & & = 504
 \end{array}$$

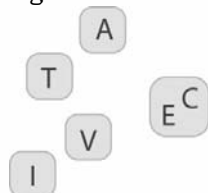
Add the results to get the total: **952**

Permutations and Combinations

Lesson 1, Part Seven: Items Always Together

Always Together: Frequently, certain items must always be kept together. To do these questions, you must treat the joined items as if they were only one object.

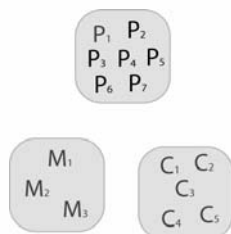
Example 1: How many arrangements of the word ACTIVE are there if C & E must always be together?



There are 5 groups in total, and they can be arranged in $5!$ ways.
The letters EC can be arranged in $2!$ ways.

The total arrangements are $5! \times 2! = 240$

Example 2: How many ways can 3 math books, 5 chemistry books, and 7 physics books be arranged on a shelf if the books of each subject must be kept together?



There are three groups, which can be arranged in $3!$ ways.
The physics books can be arranged in $7!$ ways.
The math books can be arranged in $3!$ ways.
The chemistry books can be arranged in $5!$ ways.

The total arrangements are $3! \times 7! \times 3! \times 5! = 21772800$

Questions:

- 1) How many ways can you order the letters in KEYBOARD if K and Y must always be kept together?
- 2) How many ways can the letters in OBTUSE be ordered if all the vowels must be kept together?
- 3) How many ways can 4 rock, 5 pop, & 6 classical albums be ordered if all albums of the same genre must be kept together?

Answers:

- 1) $7! \cdot 2! = 10080$
- 2) $4! \cdot 3! = 144$
- 3) $3! \cdot 4! \cdot 5! \cdot 6! = 12441600$

Permutations and Combinations

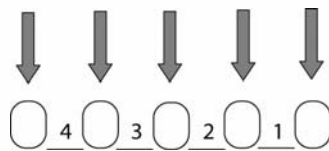
Lesson 1, Part Eight: Items Never Together

Never Together: If certain items must be kept apart, you will need to figure out how many possible positions the separate items can occupy.

Example 1: How many arrangements of the word ACTIVE are there if C & E must never be together?

Method 1:

4 3 2 1
First fill in the possible positions for the letters ATIV



Next draw empty circles representing the positions C & E can go.

You can place C & E in these spaces ${}_5P_2$ ways.
Get the answer by multiplying:

$$4! \times {}_5P_2 = 480$$

Method 2: First determine the number of ways ACTIVE can be arranged if C & E are ALWAYS together. $(5! \cdot 2!)$
Then subtract that from the total number of possible arrangements without restrictions $(6!)$
The answer is $6! - (5! \cdot 2!) = 480$

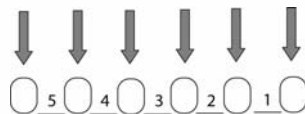
***This method does not work if there are more than two items you want to keep separate.

Example 2: How many arrangements of the word DAUGHTER are there if none the vowels can ever be together?

First fill in the possible positions for the consonants

Next draw empty circles representing the positions the vowels can go.

5 4 3 2 1



You can place the 3 vowels in the 6 spaces in ${}_6P_3$ ways.

$$5! \times {}_6P_3 = 14400$$

Example 3: In how many ways can the letters from the word EDITOR be arranged if vowels and consonants alternate positions?

First determine the number of arrangements with consonants first in the arrangement:

$$\frac{3}{C} \frac{3}{V} \frac{2}{C} \frac{2}{V} \frac{1}{C} \frac{1}{V} = 36$$

Then determine the number of arrangements with vowels coming first:

$$\frac{3}{V} \frac{3}{C} \frac{2}{V} \frac{2}{C} \frac{1}{V} \frac{1}{C} = 36$$

Add the results to get **72** possible arrangements.

Questions:

- How many ways can you order the letters in QUEST if the vowels must never be together?
- If 8 boys and 2 girls must stand in line for a picture, how many line-up's will have the girls separated from each other?
- How many ways can you order the letters in FORTUNES if the vowels must never be together?
- In how many ways can the letters AEFGQSU be arranged if vowels and consonants alternate positions?

Answers:

- $3! \cdot {}_4P_2 = 72$ or $5! - (4! \cdot 2!) = 72$
- $8! \cdot {}_6P_2 = 2903040$ or $10! - (9! \cdot 2!) = 2903040$
- $5! \cdot {}_6P_3 = 14400$
- Consonants first: $4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 576$
Vowels first: $4 \times 4 \times 3 \times 3 \times 2 \times 2 \times 1 \times 1 = 576$
Add results: $576 + 576 = 1152$