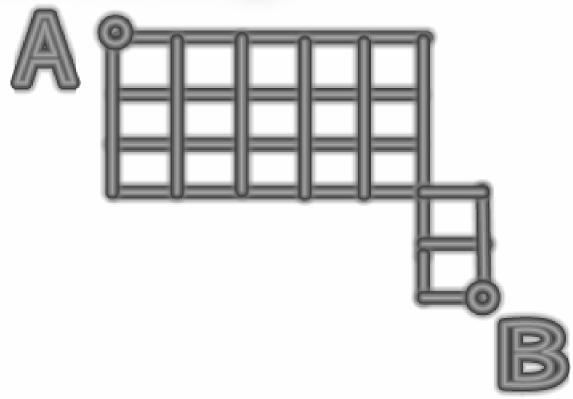
Pure Math 30:

Permutations and Combinations



Lesson 4

Formulas of Perms & Combs

Pure Math 30: EXPLAINED!

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<u>Permutations & Combinations Formulas:</u> The following examples show how to expand and simplify expressions involving permutations & combinations:

$$_nP_r = \frac{n!}{(n-r)!} \qquad \& \qquad _nC_r = \frac{n!}{r!(n-r)!}$$

Example 1: Expand & simplify $_6P_2$ without using a calculator:

$$_6P_2 = \frac{6!}{(6-2)!} = \frac{6!}{4!} = \frac{6 \bullet 5 \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{\cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}} = 6 \bullet 5 = 30$$

Example 2: Expand & simplify $_7C_5$ without using a calculator:

$$_7C_5 = \frac{7!}{5!(7-5)!} = \frac{7!}{5!2!} = \frac{7 \bullet 6 \bullet \cancel{5} \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{(\cancel{5} \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}) \times (2 \bullet 1)} = \frac{7 \bullet 6}{2} = \frac{42}{2} = 21$$

Questions:

Expand & simplify each of the following without using a calculator:

1) $_5P_3$

2) $_6P_3$

3) $_8C_6$

4) Express $\frac{7!}{3!}$ as a permutation

5) Express $\frac{8!}{2!6!}$ as a combination

<u>Answers:</u>

1) $\frac{5!}{2!} = \frac{5 \bullet 4 \bullet 3 \bullet \cancel{2} \bullet \cancel{1}}{\cancel{2} \bullet \cancel{1}} = 5 \bullet 4 \bullet 3 = 60$

2) $\frac{6!}{3!} = \frac{6 \bullet 5 \bullet 4 \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{\cancel{3} \bullet \cancel{2} \bullet \cancel{1}} = 6 \bullet 5 \bullet 4 = 120$

3) $\frac{8!}{6!2!} = \frac{8 \bullet 7 \bullet \cancel{6} \bullet \cancel{5} \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1}}{\cancel{6} \bullet \cancel{5} \bullet \cancel{4} \bullet \cancel{3} \bullet \cancel{2} \bullet \cancel{1} \bullet 2 \bullet 1} = \frac{8 \bullet 7}{2} = 28$

4) $\frac{7!}{3!} = \frac{7!}{(7-4)!} = {}_7P_4$

5) $\frac{8!}{2!6!} = \frac{8!}{2!(8-2)!} = {}_8C_2$

<u>Canceling With Factorials:</u> The following examples will introduce variables that must be manipulated to simplify expressions.

To expand factorials, you will multiply all the numbers going down to one. eg. $5! = 5 \bullet 4 \bullet 3 \bullet 2 \bullet 1$

If you ever want to stop before reaching 1, you can do so by simply placing the factorial after the number you want to stop at. eg. $5! = 5 \bullet 4 \bullet 3!$

Example 1: Expand $_7P_3$ by canceling factorials:

$$_7P_3 = \frac{7!}{(7-3)!} = \frac{7!}{4!} = \frac{7 \bullet 6 \bullet 5 \bullet \cancel{4}!}{\cancel{4}!} = 7 \bullet 6 \bullet 5 = 210$$

If you want to expand algebraic expressions, you must do so by subtracting one from each term.

n! = n(n-1)(n-2)(n-3)…1
(n-2)! = (n-2)(n-3)(n-4)…1
(n+4)! = (n+4)(n+3)(n+2)…1
(n+1)! = (n+1)n(n-1)(n-2)…1

Don't forget you can stop expanding at any term by writing your factorial symbol to indicate the end!

Example 2: Simplify the expression $\frac{(n+2)!}{(n-1)!}$

When trying to decide which factorial to expand, always choose the larger one.

n+2 would be a larger number than n-1, so expand the (n+2)!

$$\frac{(n+2)!}{(n-1)!} = \frac{(n+2)(n+1)n\cancel{(n-1)!}}{\cancel{(n-1)!}} = (n+2)(n+1)n = (n^2+3n+2)n = n^3+3n^2+2n$$

Example 3: Simplify the expression $\frac{(2n+1)!}{(2n+3)!}$

$$\frac{(2n+1)!}{(2n+3)!} = \frac{\cancel{(2n+1)}!}{(2n+3)(2n+2)\cancel{(2n+1)!}} = \frac{1}{(2n+3)(2n+2)} = \frac{1}{4n^2+10n+6}$$

Example 4: Simplify the expression $\frac{(n-3)!}{(n-4)!}$

Watch out!
n-3 is the bigger number,
so expand the numerator.

$$\frac{(n-3)!}{(n-4)!} = \frac{(n-3)\cancel{(n-4)!}}{\cancel{(n-4)!}} = n-3$$

Remember the following:

0! = 1

Example 5: Simplify the expression $\frac{(n+2)!}{n+2}$

Expand the (n+2)!,
but once you expand to the next term,
you'll be able to cancel out the (n+2)'s

Thus, stop expanding and place your
factorial, since further terms are
not needed.

$$\frac{(n+2)!}{n+2} = \frac{\cancel{(n+2)}(n+1)!}{\cancel{n+2}} = (n+1)!$$

Questions:

Expand & simplify each of the following without using a calculator:

1) $\frac{(n+2)!}{n!}$

2) $\frac{(3n+2)!}{(3n+3)!}$

3) $\frac{(n-1)!}{(n-3)!}$

4) $\frac{(n-1)!}{n-1}$

Answers:

1) $\frac{(n+2)!}{n!} = \frac{(n+2)(n+1)n!}{n!} = (n+2)(n+1)$

2) $\frac{(3n+2)!}{(3n+3)!} = \frac{(3n+2)!}{(3n+3)(3n+2)!} = \frac{1}{3n+3}$

3) $\frac{(n-1)!}{(n-3)!} = \frac{(n-1)(n-2)(n-3)!}{(n-3)!} = (n-1)(n-2)$

4) $\frac{(n-1)!}{n-1} = \frac{(n-1)(n-2)!}{n-1} = (n-2)!$

Equations with Permutations & Combinations: The following examples will require simplification in order to obtain a value for the unknown.

Example 1: Solve for n in the following equation: $\frac{(n+2)!}{(n+1)!} = 8$

$$\frac{(n+2)!}{(n+1)!} = 8$$
$$\frac{(n+2)\cancel{(n+1)!}}{\cancel{(n+1)!}} = 8$$
$$n+2 = 8$$
$$n = 6$$

Example 2: Solve for n in the following equation: $\frac{(n+1)!}{(n-1)!} = 10n$

$$\frac{(n+1)!}{(n-1)!} = 10n$$
$$\frac{(n+1)\cancel{n}\cancel{(n-1)!}}{\cancel{(n-1)!}} = 10\cancel{n}$$
$$n+1 = 10$$
$$n = 9$$

Example 3: Solve for n in the following equation: $(n+3)! = 20(n+1)!$

$$(n+3)! = 20(n+1)!$$
$$(n+3)(n+2)\cancel{(n+1)!} = 20\cancel{(n+1)!}$$
$$(n+3)(n+2) = 20$$
$$n^2 + 5n + 6 = 20$$
$$n^2 + 5n - 14 = 0$$
$$(n+7)(n-2) = 0$$
$$n = 2$$

Reject -7 since n must be a whole number.

Example 4: Solve for n in the following equation: ${}_nP_2 = 56$

$${}_nP_2 = 56$$
$$\frac{n!}{(n-2)!} = 56$$
$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 56$$
$$n(n-1) = 56$$
$$n^2 - n = 56$$
$$n^2 - n - 56 = 0$$
$$(n-8)(n+7) = 0$$
$$n = 8$$

Reject -7 since you can't have negative objects to select from.

Example 5: Simplify: $\frac{{}_{400}C_{300}}{{}_{400}C_{100}}$

$$\frac{{}_{400}C_{300}}{{}_{400}C_{100}} = \frac{\frac{400!}{(400-300)!300!}}{\frac{400!}{(400-100)!100!}}$$
$$= \frac{400!}{(400-300)!300!} \times \frac{(400-100)!100!}{400!}$$
$$= \frac{\cancel{400!}}{\cancel{100!}\,\cancel{300!}} \times \frac{\cancel{300!}\,\cancel{100!}}{\cancel{400!}}$$
$$= 1$$

Questions:

1) $\frac{(n-1)!}{(n-3)!} = 2$

2) $\frac{n!}{(n-2)!} = 5n$

3) $(n+2)! = 12n!$

4) $\frac{n!}{10} = {}_{n-1}P_{n-3}$

5) $36 \bullet {}_{(2n-1)}P_{(n-1)} = 9 \bullet {}_{2n}P_{n}$

6) ${}_{2n+2}P_{1} = \frac{1}{2} \bullet {}_{2n}P_{2}$

Answers:

1)

$$\frac{(n-1)!}{(n-3)!} = 2$$
$$\frac{(n-1)(n-2)\cancel{(n-3)!}}{\cancel{(n-3)!}} = 2$$
$$(n-1)(n-2) = 2$$
$$n^2 - 3n + 2 = 2$$
$$n^2 - 3n = 0$$
$$n(n-3) = 0$$
$$n = 3$$

2)

$$\frac{n!}{(n-2)!} = 5n$$
$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 5n$$
$$n(n-1) = 5n$$
$$n^2 - n = 5n$$
$$n^2 - 6n = 0$$
$$n(n-6) = 0$$
$$n = 6$$

3)

$$(n+2)! = 12n!$$
$$(n+2)(n+1)\cancel{n!} = 12\cancel{n!}$$
$$(n+2)(n+1) = 12$$
$$n^2 + 3n + 2 = 12$$
$$n^2 + 3n - 10 = 0$$
$$(n+5)(n-2) = 0$$
$$n = 2$$

4)

$$\frac{n!}{10} = {}_{n-1}P_{n-3}$$
$$\frac{n!}{10} = \frac{(n-1)!}{[(n-1)-(n-3)]!}$$
$$\frac{n!}{10} = \frac{(n-1)!}{2!}$$
$$\frac{n\cancel{(n-1)!}}{10} = \frac{\cancel{(n-1)!}}{2}$$
$$\frac{n}{10} = \frac{1}{2}$$
$$2n = 10$$
$$n = 5$$

5)

$$36 \bullet {}_{(2n-1)}P_{(n-1)} = 9 \bullet {}_{2n}P_n$$
$$36 \bullet \frac{(2n-1)!}{[(2n-1)-(n-1)]!} = 9 \bullet \frac{(2n)!}{(2n-n)!}$$
$$36 \bullet \frac{(2n-1)!}{n!} = 9 \bullet \frac{(2n)!}{n!}$$
$$36 \bullet \frac{\cancel{(2n-1)!}}{\cancel{n!}} = 9 \bullet \frac{(2n)\cancel{(2n-1)!}}{\cancel{n!}}$$
$$36 = 9(2n)$$
$$36 = 18n$$
$$n = 2$$

6)

$${}_{2n+2}P_1 = \frac{1}{2} \bullet {}_{2n}P_2$$
$$\frac{(2n+2)!}{[(2n+2)-1]!} = \frac{1}{2} \bullet \frac{(2n)!}{[2n-2]!}$$
$$\frac{(2n+2)!}{[2n+1]!} = \frac{1}{2} \bullet \frac{(2n)!}{[2n-2]!}$$
$$\frac{(2n+2)\cancel{(2n+1)!}}{\cancel{[2n+1]!}} = \frac{1}{\cancel{2}} \bullet \frac{\cancel{2}n(2n-1)\cancel{(2n-2)!}}{\cancel{(2n-2)!}}$$
$$2n + 2 = n(2n-1)$$
$$2n + 2 = 2n^2 - n$$
$$0 = 2n^2 - 3n - 2$$
$$0 = (2n+1)(n-2)$$
$$n = 2$$

Word Problems with Permutations & Combinations: The following examples will require the use of algebra in order to solve the question.

Example 1: If there are 78 handshakes in a room, and each person shook every other person's hand one time, how many people are in the room?

$${}_nC_2 = 78$$
$$\frac{n!}{2!(n-2)!} = 78$$
$$\frac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}} = 78$$
$$\frac{n(n-1)}{2} = 78$$
$$n^2 - n = 156$$
$$n^2 - n - 156 = 0$$
$$(n-13)(n+12) = 0$$
$$n = 13$$

Example 2: If there are 56 games in a series, and each team played every other team twice, once at home and once away, how many teams are there?

$${}_nP_2 = 56$$
$$\frac{n!}{(n-2)!} = 56$$
$$\frac{n(n-1)\cancel{(n-2)!}}{\cancel{(n-2)!}} = 56$$
$$n(n-1) = 56$$
$$n^2 - n = 56$$
$$n^2 - n - 56 = 0$$
$$(n-8)(n+7) = 0$$
$$n = 8$$

Example 3: If a polygon has 54 diagonals, how many sides are there?

$${}_nC_2 - n = 54$$
$$\frac{n!}{2!(n-2)!} - n = 54$$
$$\frac{n(n-1)\cancel{(n-2)!}}{2\cancel{(n-2)!}} - n = 54$$
$$\frac{n(n-1)}{2} - n = 54$$
$$n(n-1) - 2n = 108$$
$$n^2 - n - 2n = 108$$
$$n^2 - 3n - 108 = 0$$
$$(n-12)(n+9) = 0$$
$$n = 12$$

Lesson Four, Part Three: Word Problems

Questions:

1) If there are 190 handshakes in a room, and each person shook every other person's hand one time, how many people are in the room?

2) If there are 72 games in a series, and each team played every other team twice, once at home and once away, how many teams are there?

3) If a polygon has 119 diagonals, how many sides are there?

It is usually pretty easy to find the value of n by trial and error through your calculator.

Answers:
1) 20
2) 9(Hint: Use a permutation)
3) 17