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Absolute Convergence
Absolutely Convergent

Describes a series that converges when all terms are replaced by their absolute values. To see if a series converges absolutely, replace any subtraction in the series with addition. If the new series converges, then the original series converges absolutely.

Note: Any series that converges absolutely is itself convergent.

Definition: A series $\sum_{n=1}^{\infty} a_n$ is *absolutely convergent* if the series $\sum_{n=1}^{\infty} |a_n|$ converges.

Example: Determine if $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n} = 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \cdots$ is absolutely convergent.

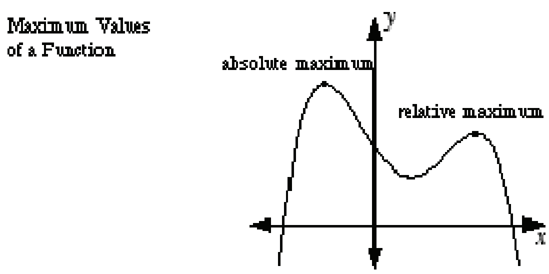
Solution: To find out, consider the series $\sum_{n=0}^{\infty} \left|\frac{(-1)^n}{2^n}\right| = 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$.

This is an infinite geometric series with ratio $r = \frac{1}{2}$, so it converges to $\frac{1}{1-\frac{1}{2}}$ or 2. As a result, we know that $\sum_{n=0}^{\infty} \frac{(-1)^n}{2^n}$ converges absolutely.

Absolute Maximum, Absolute Max
Global Maximum, Global Max

The highest point over the entire domain of a function or relation.

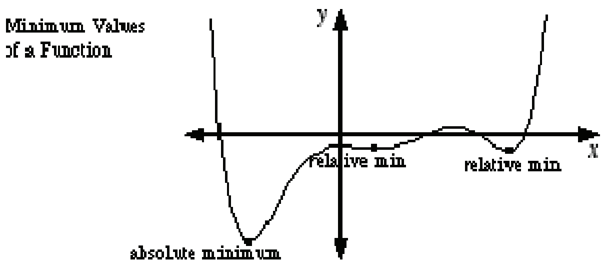
Note: The first derivative test and the second derivative test are common methods used to find maximum values of a function.



Absolute Minimum, Absolute Min
Global Minimum, Golbal Min

The lowest point over the entire domain of a function or relation.

Note: The first derivative test and the second derivative test are common methods used to find minimum values of a function.



Acceleration

The rate of change of velocity over time. For motion along the number line, acceleration is a scalar. For motion on a plane or through space, acceleration is a vector.

Absolutely Convergent
See Absolute Convergence

Alternating Series

A series which alternates between positive and negative terms. For example, the series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\cdots$ is alternating.

Alternating Series Remainder

A quantity that measures how accurately the nth partial sum of an alternating series estimates the sum of the series.

Consider the following alternating series (where an > 0 for all n) and/or its equivalents.

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

If the series converges to S by the alternating series test, then the remainder

$$R_n = S - \sum_{k=1}^{n} (-1)^{k+1} a_k$$

can be estimated as follows for all n ≥ N:

$$|R_n| \leq a_{n+1}$$

Here, N is the point at which the values of an become non-increasing:

$$a_n \geq a_{n+1} \text{ for all } n \geq N \text{ where } N \geq 1$$

Alternating Series Test

A convergence test for alternating series.

Consider the following alternating series (where $a_n > 0$ for all n) and/or its equivalents:

$$\sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

The series converges if the following conditions are met:

1. $a_n \geq a_{n+1}$ for all $n \geq N$ where $N \geq 1$, and
2. $\lim\limits_{n \to \infty} a_n = 0$

Analytic Methods

The use of algebraic and/or numeric methods as the main technique for solving a math problem. The instructions "solve using analytic methods" and "solve analytically" usually mean that no calculator is allowed.

Annulus

See Washer

Antiderivative of a Function

A function that has a given function as its derivative. For example, $F(x) = x^3 - 8$ is an antiderivative of $f(x) = 3x^2$.

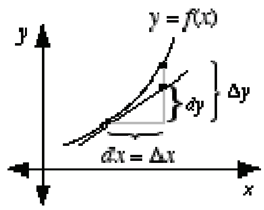
Approximation by Differentials

A method for approximating the value of a function near a known value. The method uses the tangent line at the known value of the function to approximate the function's graph. In this method Δx and Δy represent the changes in x and y for the function, and dx and dy represent the changes in x and y for the tangent line.

$$\left.\begin{aligned} dy &= f'(x)dx \\ \Delta x &= dx \\ \Delta y &\approx dy \end{aligned}\right\} \Rightarrow \Delta y \approx f'(x)\Delta x$$

This is also written

$$f(x+\Delta x) = f(x) + \Delta y \approx f(x) + f'(x)\Delta x$$



Example: Approximate $\sqrt{10}$ by differentials.

Solution: $\sqrt{10}$ is near $\sqrt{9}$, so we will use $f(x) = \sqrt{x}$ with x = 9 and Δx = 1.

Note that $f'(x) = \dfrac{1}{2\sqrt{x}}$.

$$\begin{aligned} \sqrt{10} &= f(x+\Delta x) \\ &\approx f(x) + f'(x)\Delta x \\ &= \sqrt{x} + \frac{1}{2\sqrt{x}}\Delta x \\ &= \sqrt{9} + \frac{1}{2\sqrt{9}}(1) \\ &= 3\frac{1}{6} \end{aligned}$$

Thus we see that $\sqrt{10} \approx 3\frac{1}{6} = 3.1\bar{6}.$

This is very close to the correct value of $\sqrt{10} \approx 3.1623.$

Arc Length of a Curve

The length of a curve or line.

The length of an arc can be found by one of the formulas below for any differentiable curve defined by rectangular, polar, or parametric equations.

For the length of a circular arc, see arc of a circle.

Formula:

$$\text{Arc length} = \int_a^b ds$$

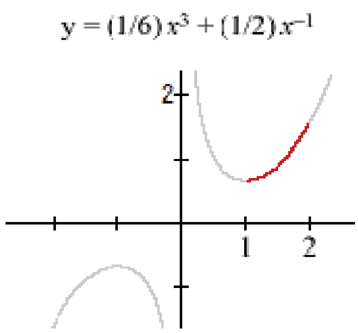
where *a* and *b* represent *x*, *y*, *t*, or θ-values as appropriate, and *ds* can be found as follows.

1. In rectangular form, use whichever of the following is easier:

$$ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \quad \text{or} \quad ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example) Find the length of an arc of the curve $y = (1/6)\, x^3 + (1/2)\, x^{-1}$ from $x = 1$ to $x = 2$.

$$\begin{aligned}
\text{Arc length} &= \int_a^b \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \\
&= \int_1^2 \sqrt{1 + \left(\frac{1}{2}x^2 - \frac{1}{2}x^{-2}\right)^2}\, dx \\
&= \int_1^2 \sqrt{1 + \frac{1}{4}x^4 - \frac{1}{2} + \frac{1}{4}x^{-4}}\, dx \\
&= \int_1^2 \sqrt{\frac{1}{4}x^4 + \frac{1}{2} + \frac{1}{4}x^{-4}}\, dx \\
&= \int_1^2 \sqrt{\left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right)^2}\, dx \\
&= \int_1^2 \left(\frac{1}{2}x^2 + \frac{1}{2}x^{-2}\right) dx \\
&= \left(\frac{1}{6}x^3 - \frac{1}{2}x^{-1}\right)\Big|_1^2 \\
&= \left(\frac{8}{6} - \frac{1}{4}\right) - \left(\frac{1}{6} - \frac{1}{2}\right) \\
&= \frac{17}{12}
\end{aligned}$$

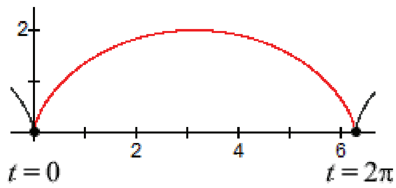


2. In parametric form, use

$$ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Example) Find the length of the arc in one period of the cycloid x = t − sin t, y = 1 − cos t. The values of t run from 0 to 2π.

$$\begin{aligned}
\text{Arc length} &= \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{(1-\cos t)^2 + (\sin t)^2}\, dt \\
&= \int_0^{2\pi} \sqrt{1 - 2\cos t + \cos^2 t + \sin^2 t}\, dt \\
&= \int_0^{2\pi} \sqrt{2 - 2\cos t}\, dt \\
&= \int_0^{2\pi} 2\,|\sin 2t|\, dt \\
&= 8\int_0^{\pi/4} 2\sin 2t\, dt \\
&= 8\,(-\cos 2t)\big|_0^{\pi/4} \\
&= 8
\end{aligned}$$

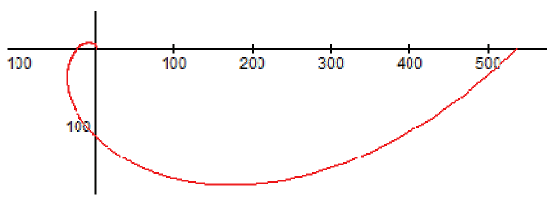


3. In polar form, use

$$ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta$$

Example) Find the length of the first rotation of the logarithmic spiral $r = e^{\theta}$. The values of θ run from 0 to 2π.

$$\begin{aligned}
\text{Arc length} &= \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2}\, d\theta \\
&= \int_0^{2\pi} \sqrt{(e^{\theta})^2 + (e^{\theta})^2}\, d\theta \\
&= \int_0^{2\pi} e^{\theta}\sqrt{2}\, d\theta \\
&= \left(e^{\theta}\sqrt{2}\right)\Big|_0^{2\pi} \\
&= \sqrt{2}\,(e^{2\pi} - 1)
\end{aligned}$$



Area between Curves

The area between curves is given by the formulas below.

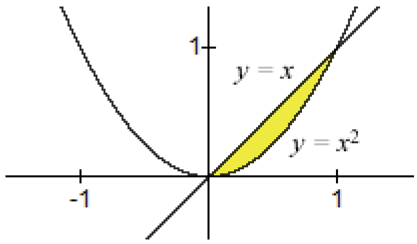
Formula 1: $$\text{Area} = \int_a^b |f(x) - g(x)|\,dx$$

for a region bounded above and below by y = f(x) and y = g(x), and on the left and right by x = a and x = b.

Formula 2: $$\text{Area} = \int_c^d |f(y) - g(y)|\,dy$$

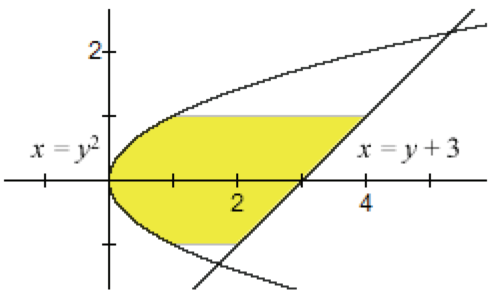
for a region bounded left and right by x = f(y) and x = g(y), and above and below by y = c and y = d.

Example 1: Find the area between $y = x$ and $y = x^2$ from $x = 1$ to $x = 2$.



$$\begin{aligned}\text{Area} &= \int_0^1 \left|x - x^2\right| dx \\ &= \int_0^1 \left(x - x^2\right) dx \\ &= \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 \\ &= \frac{1}{6}\end{aligned}$$

Example 2: Find the area between $x = y + 3$ and $x = y^2$ from $y = -1$ to $y = 1$.



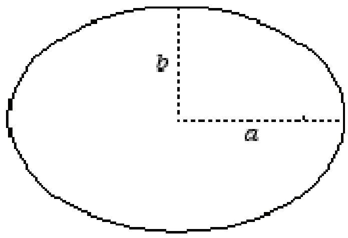
$$\begin{aligned}\text{Area} &= \int_{-1}^1 \left|y + 3 - y^2\right| dy \\ &= \int_{-1}^1 \left(y + 3 - y^2\right) dy \\ &= \left(\frac{1}{2}y^2 + 3y - \frac{1}{3}y^3\right)\Big|_{-1}^1 \\ &= \left(\frac{1}{2} + 3 - \frac{1}{3}\right) - \left(\frac{1}{2} - 3 + \frac{1}{3}\right) \\ &= \frac{16}{3}\end{aligned}$$

Area of an Ellipse

The formula is given below.

$$\text{Area} = \pi ab$$

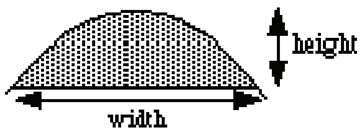
$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$



Area of a Parabolic Segment

The formula is given below.

$$\text{Area} = \frac{2}{3}(\text{width})(\text{height})$$

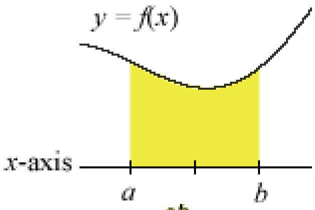


Area under a Curve

The area between the graph of y = f(x) and the x-axis is given by the definite integral below. This formula gives a positive result for a graph above the x-axis, and a negative result for a graph below the x-axis.

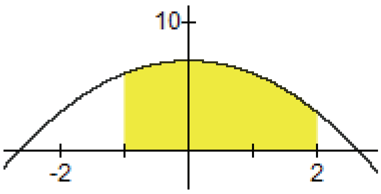
Note: If the graph of y = f(x) is partly above and partly below the x-axis, the formula given below generates the net area. That is, the area above the axis minus the area below the axis.

Formula:



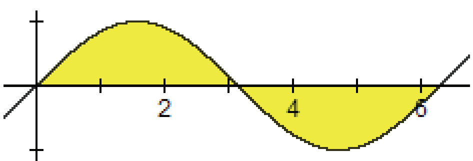
$$\text{Area} = \int_a^b f(x)\,dx$$

Example 1: Find the area between $y = 7 - x^2$ and the x-axis between the values $x = -1$ and $x = 2$.



$$\begin{aligned}\text{Area} &= \int_{-1}^{2} \left(7 - x^2\right) dx \\ &= \left(7x - \frac{1}{3}x^3\right)\Big|_{-1}^{2} \\ &= \left[7\cdot 2 - \frac{1}{3}(8)\right] - \left[7(-1) - \frac{1}{3}(-1)\right] \\ &= 18\end{aligned}$$

Example 2: Find the net area between y = sin x and the x-axis between the values x = 0 and x = 2π.



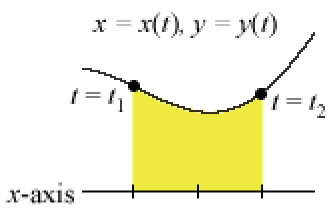
$$\begin{aligned}\text{Net area} &= \int_0^{2\pi} \sin x\,dx \\ &= (-\cos x)\big|_0^{2\pi} \\ &= (-1) - (-1) \\ &= 0\end{aligned}$$

Area Using Parametric Equations
Parametric Integral Formula

The area between the x–axis and the graph of x = x(t), y = y(t) and the x–axis is given by the definite integral below. This formula gives a positive result for a graph above the x–axis, and a negative result for a graph below the x–axis.

Note: If the graph of x = x(t), y = y(t) is partly above and partly below the x–axis, the formula given below generates the net area. That is, the area above the axis minus the area below the axis.

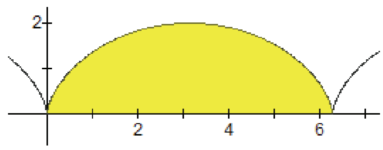
Formula:



$$\text{Area} = \int_{t_1}^{t_2} y\,dx$$

Example: Find the area of the between the x–axis and the first period of the cycloid x = t – sin t, y = 1 – cos t. The values of t run from 0 to 2π.

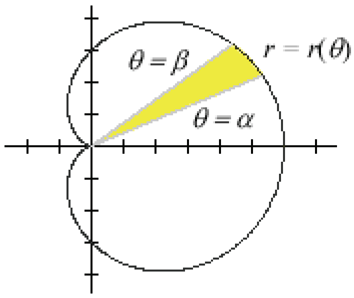
$$\begin{aligned}\text{Area} &= \int_{t_1}^{t_2} y\,dx \\ &= \int_0^{2\pi} (1-\cos t)\,d\,(t-\sin t) \\ &= \int_0^{2\pi} (1-\cos t)^2\,dt \\ &= \int_0^{2\pi} \left(1-2\cos t+\cos^2 t\right)dt \\ &= \int_0^{2\pi} \left(1-2\cos t+\frac{1+\cos 2t}{2}\right)dt \\ &= \int_0^{2\pi} \left(\frac{3}{2}-2\cos t+\frac{1}{2}\cos 2t\right)dt \\ &= \left(\frac{3}{2}t-2\sin t+\frac{1}{4}\sin 2t\right)\Big|_0^{2\pi} \\ &= 3\pi\end{aligned}$$



Area Using Polar Coordinates
Polar Integral Formula

The area between the graph of r = r(Θ) and the origin and also between the rays Θ = α and Θ = β is given by the formula below (assuming α ≤ β).

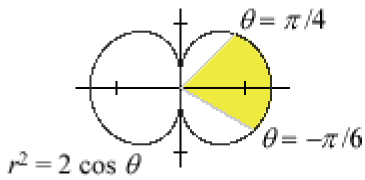
Formula:



$$\text{Area} = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

Example: Find the area of the region bounded by the graph of the lemniscate $r^2 = 2\cos\Theta$, the origin, and between the rays $\Theta = -\pi/6$ and $\Theta = \pi/4$.

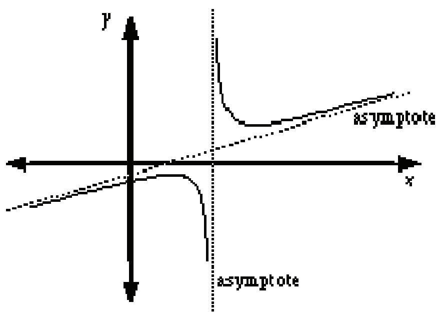
$$\begin{aligned}
\text{Area} &= \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta \\
&= \int_{-\pi/6}^{\pi/4} \frac{1}{2} (2\cos\theta)\, d\theta \\
&= (\sin\theta)\,\Big|_{-\pi/6}^{\pi/4} \\
&= \left(\frac{\sqrt{2}}{2}\right) - \left(-\frac{1}{2}\right) \\
&= \frac{1+\sqrt{2}}{2}
\end{aligned}$$



Asymptote

A line or curve that the graph of a relation approaches more and more closely the further the graph is followed.

Note: Sometimes a graph will cross a horizontal asymptote or an oblique asymptote. The graph of a function, however, will never cross a vertical asymptote.



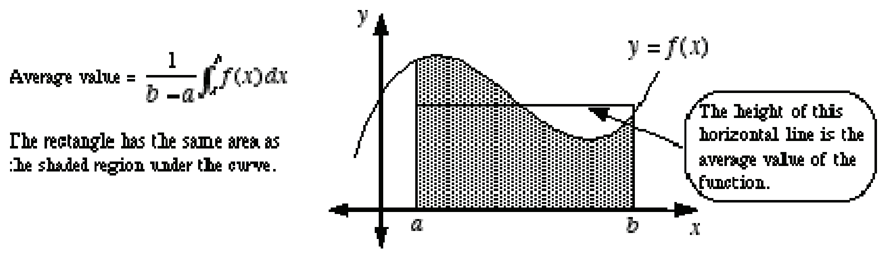
Average Rate of Change

The change in the value of a quantity divided by the elapsed time. For a function, this is the change in the y-value divided by the change in the x-value for two distinct points on the graph.

Note: This is the same thing as the slope of the secant line that passes through the two points.

Average Value of a Function

The average height of the graph of a function. For $y = f(x)$ over the domain $[a, b]$, the formula for average value is given below.



Axis of Rotation

A line about which a plane figure is rotated in three dimensional space to create a solid or surface.

Boundary Value Problem
BVP

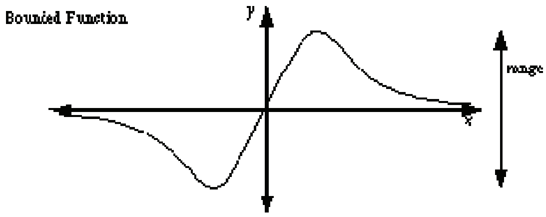
A differential equation or partial differential equation accompanied by conditions for the value of the function but with no conditions for the value of any derivatives.

Note: Boundary value problem is often abbreviated BVP.

Differential Equation	$y'' + y = \sin x$
Initial Value Problem (IVP)	$y'' + y = \sin x$, $y(0) = 1$, $y'(0) = -2$
Boundary Value Problem (BVP)	$y'' + y = \sin x$, $y(0) = 1$, $y(1) = -2$

Bounded Function

A function with a range that is a bounded set. The range must have both an upper bound and a lower bound.



Bounded Sequence

A sequence with terms that have an upper bound and a lower bound. For example, the harmonic sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$ is bounded since no term is greater than 1 or less than 0.

Bounds of Integration
Limits of Integration

For the definite integral $\int_a^b f(x)dx$, the bounds (or limits) of integration are a and b.

Calculus

The branch of mathematics dealing with limits, derivatives, definite integrals, indefinite integrals, and power series.

Common problems from calculus include finding the slope of a curve, finding extrema, finding the instantaneous rate of change of a function, finding the area under a curve, and finding volumes by parallel cross-sections.

Cartesian Form
Rectangular Form

A function (or relation) written using (x, y) or (x, y, z) coordinates.

Cavalieri's Principle

A method, with formula given below, of finding the volume of any solid for which cross-sections by parallel planes have equal areas. This includes, but is not limited to, cylinders and prisms.

Formula: Volume = Bh, where B is the area of a cross-section and h is the height of the solid.

Center of Mass Formula

The coordinates $(\bar{x}, \bar{y})$ of the center of mass of a plane figure are given by the formulas below. The formulas only apply for figures of uniform (constant) density.

1. Plane region bounded above by $y = f(x)$, below by the x-axis, on the left by $x = a$, and on the right by $x = b$.

$$\bar{x} = \frac{M_y}{m} = \frac{1}{A}\int_a^b x f(x)\,dx \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{A}\int_a^b \frac{1}{2}[f(x)]^2\,dx$$

Note: M_y is the moment about the y-axis, M_x is the moment about the x-axis, m is the mass, and A is the area.

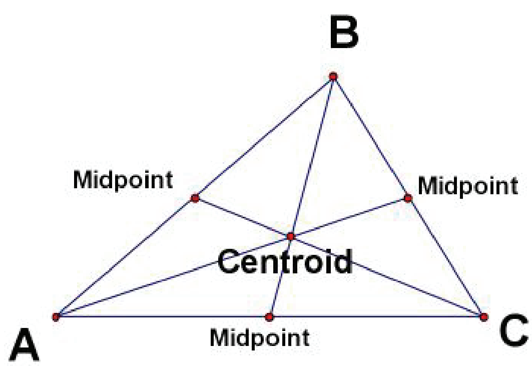
2. Plane region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by $x = a$, and on the right by $x = b$.

$$\bar{x} = \frac{M_y}{m} = \frac{1}{A}\int_a^b x[f(x) - g(x)]\,dx \qquad \bar{y} = \frac{M_x}{m} = \frac{1}{A}\int_a^b \frac{1}{2}\left([f(x)]^2 - [g(x)]^2\right)dx$$

Note: M_y is the moment about the y-axis, M_x is the moment about the x-axis, m is the mass, and A is the area.

Centroid

For a triangle, this is the point at which the three medians intersect. In general, the centroid is the center of mass of a figure of uniform (constant) density.



Centroid of a Triangle

Chain Rule

A method for finding the derivative of a composition of functions. The formula is $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$.

Another form of the chain rule is $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$.

Chain Rule: $\frac{d}{dx} f(g(x)) = f'(g(x))g'(x)$ or $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

Examples:

1. $\frac{d}{dx}(x^2+5)^8 = 8(x^2+5)^7 \cdot 2x = 16x(x^2+5)^7$
2. For $y = u^8$ and $u = x^2+5$,
$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (8u^7)(2x) = 8(x^2+5)^7 \cdot 2x = 16x(x^2+5)^7$

Comparison Test

A convergence test which compares the series under consideration to a known series. Essentially, the test determines whether a series is "better" than a "good" series or "worse" than a "bad" series. The "good" or "bad" series is often a p-series.

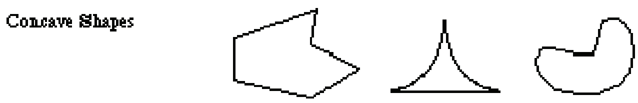
If $\sum a_n$, $\sum c_n$, and $\sum d_n$ are all positive series, where $\sum c_n$ converges and $\sum d_n$ diverges, then:

1. If $a_n \le c_n$ for all $n \ge N$ for some fixed N, then $\sum a_n$ converges.

2. If $a_n \ge d_n$ for all $n \ge N$ for some fixed N, then $\sum a_n$ diverges.

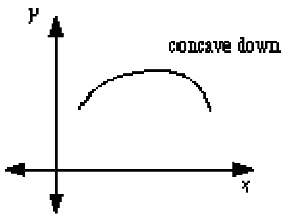
Concave
Non-Convex

A shape or solid which has an indentation or "cave". Formally, a geometric figure is concave if there is at least one line segment connecting interior points which passes outside of the figure.



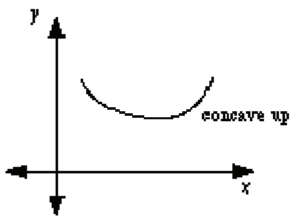
Concave Down

A graph or part of a graph which looks like an upside-down bowl or part of an upside-down bowl.



Concave Up

A graph or part of a graph which looks like a right-side up bowl or part of an right-side up bowl.



Conditional Convergence

Describes a series that converges but does not converge absolutely. That is, a convergent series that will become a divergent series if all negative terms are made positive.

Example: The series $1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots$ converges conditionally. It converges, but $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots$ diverges. (The latter is the harmonic series.)

Constant Term

The term in a simplified algebraic expression or equation which contains no variable(s). If there is no such term, the constant term is 0.

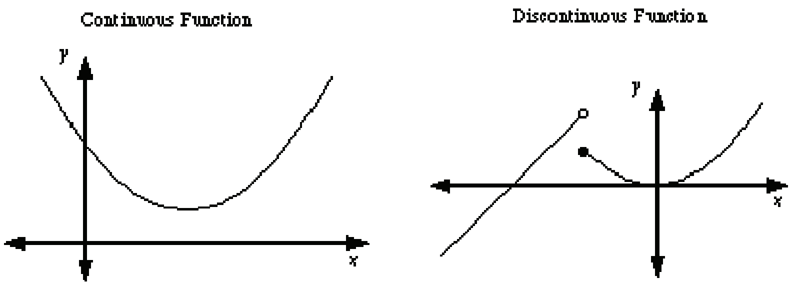
Example: -5 is the constant term in $p(x) = 2x^3 - 4x^2 + 9x - 5$

Continued Sum

See Sigma Notation

Continuous Function

A function with a connected graph.



Definition of Continuity: A function is continuous at $x = a$ if

1. $\lim_{x\to a} f(x)$ exists
2. $f(a)$ exists
3. $\lim_{x\to a} f(x) = f(a)$

Continuously Differentiable Function

A function which has a derivative that is itself a continuous function.

Converge

To approach a finite limit. There are convergent limits, convergent series, convergent sequences, and convergent improper integrals.

Converge Absolutely

See Absolute Convergence

Converge Conditionally

See Conditional Converge

Convergent Series

An infinite series for which the sequence of partial sums converges. For example, the sequence of partial sums of the series 0.9 + 0.09 + 0.009 + 0.0009 + ⋯ is 0.9, 0.99, 0.999, 0.9999, ... This sequence converges to 1, so the series 0.9 + 0.09 + 0.009 + 0.0009 + ⋯ is convergent.

Convergent Sequence

A sequence with a limit that is a real number. For example, the sequence 2.1, 2.01, 2.001, 2.0001, . . . has limit 2, so the sequence converges to 2. On the other hand, the sequence 1, 2, 3, 4, 5, 6, . . . has a limit of infinity (∞). This is not a real number, so the sequence does not converge. It is a divergent sequence.

Convergence Tests

Limit test for divergence
Integral test
Comparison test
Limit comparison test
Alternating series test
Ratio test
Root test

Critical Number
Critical Value

The x-value of a critical point.

Critical Point

A point (*x*, *y*) on the graph of a function at which the derivative is either 0 or undefined. A critical point will often be a minimum or maximum, but it may be neither.

Note: Finding critical points is an important step in the process of curve sketching.

Critical Value

See Critical Point

Curly d

The symbol ∂ used in the notation for partial derivatives.

Curve

A word used to indicate any path, whether actually curved or straight, closed or open. A curve can be on a plane or in three-dimensional space (or n-dimensional space, for that matter). Lines, circles, arcs, parabolas, polygons, and helixes are all types of curves.

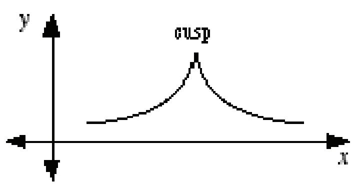
Note: Typically curves are thought of as the set of all geometric figures that can be parametrized using a single parameter. This is not in fact accurate, but it is a useful way to conceptualize curves. The exceptions to this rule require some cleverness, or at least some exposure to space-filling curves.

Curve Sketching

The process of using the first derivative and second derivative to graph a function or relation. As a result the coordinates of all discontinuities, extrema, and inflection points can be accurately plotted.

Cusp

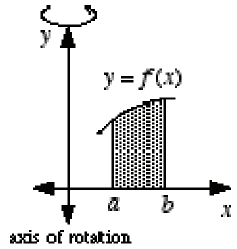
A sharp point on a curve. Note: Cusps are points at which functions and relations are not differentiable.



Cylindrical Shell Method
Shell Method

A technique for finding the volume of a solid of revolution.

$$\text{Volume} = \int_a^b 2\pi x f(x)\, dx$$



Decreasing Function

A function with a graph that moves downward as it is followed from left to right. For example, any line with a negative slope is decreasing.

Note: If a function is differentiable, then it is decreasing at all points where its derivative is negative.

Definite Integral

An integral which is evaluated over an interval. A definite integral is written $\int_a^b f(x)dx$. Definite integrals are used to find the area between the graph of a function and the x-axis. There are many other applications.

Formally, a definite integral is the limit of a Riemann sum as the norm of the partition approaches zero. That is, $\int_a^b f(x)dx = \lim_{\text{norm} \to 0} \sum_{k=1}^{n} f(c_k)(x_k - x_{k-1})$.

Example: $\int_2^5 x^2 dx = \frac{1}{3}x^3\Big|_2^5 = \frac{1}{3}(5)^3 - \frac{1}{3}(2)^3 = \frac{117}{3} = 39$

Definite Integral Rules
See Integral Rules

Degenerate

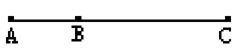
An example of a definition that stretches the definition to an absurd degree.

A degenerate triangle is the "triangle" formed by three collinear points. It doesn't look like a triangle, it looks like a line segment.

A parabola may be thought of as a degenerate ellipse with one vertex at an infinitely distant point.

Degenerate examples can be used to test the general applicability of formulas or concepts. Many of the formulas developed for triangles (such as area formulas) apply to degenerate triangles as well.

Degenerate ΔABC



Del Operator

The symbol ∇, which stands for the "vector" $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}\right)$ or $\left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$.

Deleted Neighborhood

The proper name for a set such as $\{x: 0 < |x - a| < \delta\}$. Deleted neighborhoods are encountered in the study of limits. It is the set of all numbers less than δ units away from a, omitting the number a itself.

Using interval notation the set $\{x: 0 < |x - a| < \delta\}$ would be $(a - \delta, a) \cup (a, a + \delta)$. In general, a deleted neighborhood of a is any set $(c, a) \cup (a, d)$ where $c < a < d$.

For example, one deleted neighborhood of 2 is the set $\{x: 0 < |x - 2| < 0.1\}$, which is the same as $(1.9, 2) \cup (2, 2.1)$.

The deleted neighborhood $\{x : 0 < |x - 2| < 0.1\}$

Derivative

A function which gives the slope of a curve; that is, the slope of the line tangent to a function. The derivative of a function f at a point x is commonly written $f'(x)$. For example, if $f(x) = x^3$ then $f'(x) = 3x^2$. The slope of the tangent line when $x = 5$ is $f'(x) = 3 \cdot 5^2 = 75$.

Definitions of Derivatives:

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \qquad f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \qquad \frac{dy}{dx} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

Some of the notations for the derivative of $y = f(x)$:

$$\begin{array}{ccccc} f'(x) & f' & \frac{df}{dx} & \frac{d}{dx}f(x) & Df \\ y'(x) & y' & \frac{dy}{dx} & \frac{d}{dx}y & Dy \end{array}$$

Derivative of a Power Series

The derivative of a function defined by a power series can be found by differentiating the series term-by-term.

Suppose

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

converges over the interval (c, d). Then the function f defined by this series

$$f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots + c_n(x-a)^n + \cdots$$

is differentiable over the interval (c, d) with derivative

$$f'(x) = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1} = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + \cdots + nc_n(x-a)^{n-1} + \cdots.$$

Derivative Rules

A list of common derivative rules is given below.

1. Constant function: $\frac{d}{dx}\alpha = 0$, where α = any constant

2. Scalar multiple: $\frac{d}{dx}(\alpha u) = \alpha\frac{du}{dx}$, where α = any constant

3. Sum: $\frac{d}{dx}(u+v) = \frac{du}{dx} + \frac{dv}{dx}$

4. Difference: $\frac{d}{dx}(u-v) = \frac{du}{dx} - \frac{dv}{dx}$

5. Power rule: $\frac{d}{dx}u^n = nu^{n-1}\frac{du}{dx}$

6. Product rule: $\frac{d}{dx}(uv) = u'v + uv'$

7. Quotient rule: $\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{u'v - uv'}{v^2}$

8. Reciprocal rule: $\frac{d}{dx}\left(\frac{1}{v}\right) = -\frac{v'}{v^2}$

9. Chain rule: $\frac{d}{dx}f(g(x)) = f'(g(x))g'(x)$ or $\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$

10. Trig functions: $\frac{d}{dx}\sin u = \cos u\frac{du}{dx}$ $\frac{d}{dx}\cos u = -\sin u\frac{du}{dx}$

    $\frac{d}{dx}\tan u = \sec^2 u\frac{du}{dx}$ $\frac{d}{dx}\cot u = -\csc^2 u\frac{du}{dx}$

    $\frac{d}{dx}\sec u = \sec u\tan u\frac{du}{dx}$ $\frac{d}{dx}\csc u = -\csc u\cot u\frac{du}{dx}$

11. Inverse trig functions: $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ $\frac{d}{dx}\cos^{-1}u = -\frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$

    $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$ $\frac{d}{dx}\cot^{-1}u = -\frac{1}{1+u^2}\frac{du}{dx}$

    $\frac{d}{dx}\sec^{-1}u = \frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$ $\frac{d}{dx}\csc^{-1}u = -\frac{1}{|u|\sqrt{u^2-1}}\frac{du}{dx}$

12. Exponential functions: $\frac{d}{dx}(e^u) = e^u\frac{du}{dx}$ and $\frac{d}{dx}(a^u) = a^u(\ln a)\frac{du}{dx}$

13. Log functions: $\frac{d}{dx}(\ln u) = \frac{1}{u}\frac{du}{dx}$ and $\frac{d}{dx}(\log_a u) = \frac{1}{(\ln a)u}\frac{du}{dx}$

14. Inverse Functions: $\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$

Difference Quotient

For a function f, the formula $\frac{f(x+h)-f(x)}{h}$. This formula computes the slope of the secant line through two points on the graph of f. These are the points with x–coordinates x and x + h. The difference quotient is used in the definition the derivative.

Example: $f(x)=3x^2-5x+4$

$$\begin{aligned}\frac{f(x+h)-f(x)}{h}&=\frac{3(x+h)^2-5(x+h)+4-(3x^2-5x+4)}{h}\\&=\frac{3x^2+6xh+3h^2-5x-5h+4-3x^2+5x-4}{h}\\&=\frac{6xh+3h^2-5h}{h}\\&=6x+3h-5\end{aligned}$$

Differentiable

A curve that is smooth and contains no discontinuities or cusps. Formally, a curve is differentiable at all values of the domain variable(s) for which the derivative exists.

Differential

An tiny or infinitesimal change in the value of a variable. Differentials are commonly written in the form *dx* or *dy*.

Differential Equation

An equation showing a relationship between a function and its derivative(s). For example, $\frac{dy}{dx}+y=0$ is a differential equation with solutions $y = Ce^{-x}$.

Differentiation

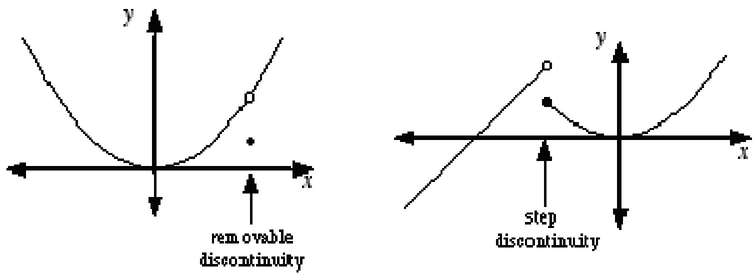
The process of finding a derivative.

Differentiation Rules

See Derivative Rules

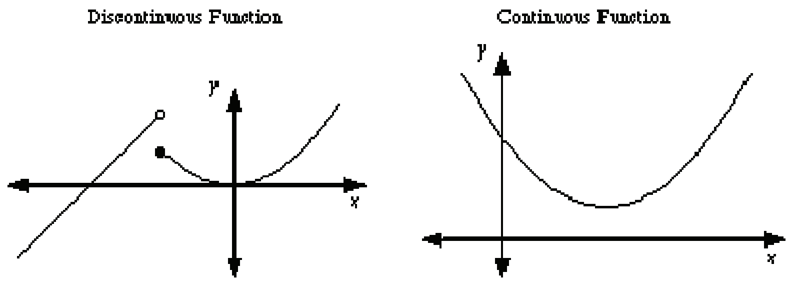
Discontinuity

A point at which the graph of a relation or function is not connected. Discontinuities can be classified as either removable or essential. There are several kinds of essential discontinuities, one of which is the step discontinuity.



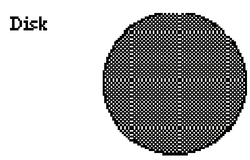
Discontinuous Function

A function with a graph that is not connected.



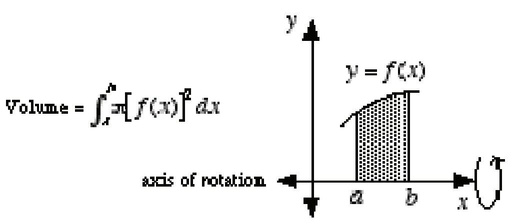
Disk

The union of a circle and its interior.



Disk Method

A technique for finding the volume of a solid of revolution. This method is a specific case of volume by parallel cross-sections.

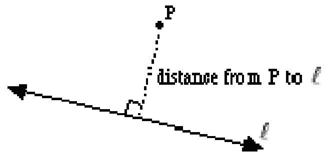


Distance from a Point to a Line

The length of the shortest segment from a given point to a given line. A formula is given below.

Distance from (x_0, y_0) to the line $Ax + By + C = 0$:

$$\text{distance} = \frac{|Ax_0 + By_0 + C|}{\sqrt{A^2 + B^2}}$$



Diverge

To fail to approach a finite limit. There are divergent limits, divergent series, divergent sequences, and divergent improper integrals.

Divergent Sequence

A sequence that does not converge. For example, the sequence 1, 2, 3, 4, 5, 6, 7, ... diverges since its limit is infinity (∞). The limit of a convergent sequence must be a real number.

Divergent Series

A series that does not converge. For example, the series 1 + 2 + 3 + 4 + 5 + ⋯ diverges. Its sequence of partial sums 1, 1 + 2, 1 + 2 + 3 , 1 + 2 + 3 + 4 , 1 + 2 + 3 + 4 + 5, ... diverges.

e

e ≈ 2.7182818284.... is a transcendental number commonly encountered when working with exponential models (growth, decay,and logistic models, and continuously compounded interest, for example) and exponential functions. e is also the base of the natural logarithm.

Definitions of $e = 2.718$:

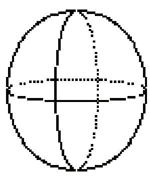
$$e = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{n} \qquad e = \lim_{x\to 0}(1+x)^{1/x}$$

$$e = \frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = \sum_{n=0}^{\infty}\frac{1}{n!}$$

Ellipsoid

A sphere-like surface for which all cross-sections are ellipses.

Ellipsoid



End Behavior

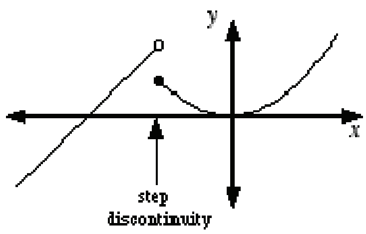
The appearance of a graph as it is followed farther and farther in either direction. For polynomials, the end behavior is indicated by drawing the positions of the arms of the graph, which may be pointed up or down. Other graphs may also have end behavior indicated in terms of the arms, or in terms of asymptotes or limits.

Polynomial End Behavior:
1. If the degree n of a polynomial is even, then the arms of the graph are either both up or both down.
2. If the degree n is odd, then one arm of the graph is up and one is down.
3. If the leading coefficient a_n is positive, the right arm of the graph is up.
4. If the leading coefficient a_n is negative, the right arm of the graph is down.

Essential Discontinuity

Any discontinuity that is not removable. That is, a place where a graph is not connected and cannot be made connected simply by filling in a single point. Step discontinuities and vertical asymptotes are two types of essential discontinuities.

Formally, an essential discontinuity is a discontinuity at which the limit of the function does not exist.



Explicit Differentiation

The process of finding the derivative of an explicit function. For example, the explicit function $y = x^2 - 7x + 1$ has derivative $y' = 2x - 7$.

Explicit Function

A function in which the dependent variable can be written explicitly in terms of the independent variable.

For example, the following are explicit functions: $y = x^2 - 3$, $f(x) = \sqrt{x+7}$, and $y = \log_2 x$.

Exponential Decay

A model for decay of a quantity for which the rate of decay is directly proportional to the amount present. The equation for the model is $A = A_0 b^t$ (where $0 < b < 1$) or $A = A_0 e^{kt}$ (where k is a negative number representing the rate of decay). In both formulas A_0 is the original amount present at time $t = 0$.

This model is used for phenomena such as radioactivity or depreciation. For example, $A = 50e^{-0.01t}$ is a model for exponential decay of 50 grams of a radioactive element that decays at a rate of 1% per year.

Exponential Growth

A model for growth of a quantity for which the rate of growth is directly proportional to the amount present. The equation for the model is $A = A_0 b^t$ (where $b > 1$) or $A = A_0 e^{kt}$ (where k is a positive number representing the rate of growth). In both formulas A_0 is the original amount present at time $t = 0$.

This model is used for such phenomena as inflation or population growth. For example, $A = 7000e^{0.05t}$ is a model for the exponential growth of $7000 invested at 5% per year compounded continuously.

Exponential Function
Exponential Model

A function of the form $y = a \cdot b^x$ where $a > 0$ and either $0 < b < 1$ or $b > 1$. The variables do not have to be x and y. For example, $A = 3.2 \cdot (1.02)^t$ is an exponential function.

Note: Exponential functions are used to model exponential growth, exponential decay, compound interest, and continuously compounded interest.

Extreme Value Theorem
Min/Max Theorem

A theorem which guarantees the existence of an absolute max and an absolute min for any continuous function over a closed interval.

Theorem: If a function f is continuous over $[a, b]$ then there are numbers c and d in $[a, b]$ such that $f(c)$ is an absolute minimum over $[a, b]$ and $f(d)$ is an absolute maximum over $[a, b]$.

Extreme Values of a Polynomial

The graph of a polynomial of degree n has at most $n - 1$ extreme values (minima and/or maxima). The total number of extreme values could be $n - 1$ or $n - 3$ or $n - 5$ etc.

For example, a degree 9 polynomial could have 8, 6, 4, 2, or 0 extreme values. A degree 2 (quadratic) polynomial must have 1 extreme value.

Extremum

An extreme value of a function. In other words, the minima and maxima of a function. Extrema may be either relative (local) or absolute (global).

Note: The first derivative test and the second derivative test are common methods used to find extrema.

Equivalent Names of Extrema

relative minimum = local minimum
relative maximum = local maximum
absolute minimum = global minimum
absolute maximum = global maximum

Factorial

The product of a given integer and all smaller positive integers. The factorial of n is written n! and is read aloud "n factorial".

Note: By definition, 0! = 1.

Formula: $n! = n\cdot(n-1)\cdot(n-2)\cdot\cdot\cdot 3\cdot2\cdot1$

Example: $6! = 6\cdot5\cdot4\cdot3\cdot2\cdot1 = 720$

Falling Bodies

See Projectile Motion

First Derivative

Same as the derivative. We say *first derivative* instead of just *derivative* whenever there may be confusion between the first derivative and the second derivative (or the nth derivative).

First Derivative Test

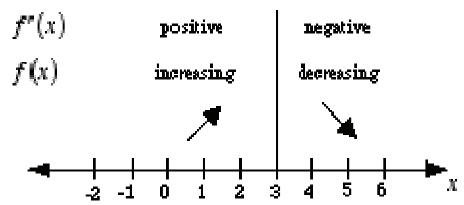
A method for determining whether an inflection point is a minimum, maximum, or neither.

First Derivative Test: For a given critical point,

1. If the derivative is negative on the left of the critical point and positive on the right, then the critical point is a minimum.
2. If the derivative is positive on the left of the critical point and negative on the right, then the critical point is a maximum.
3. In any other case, the critical point is neither a minimum nor a maximum.

Example: Consider $f(x) = 6x - x^2$, which has derivative $f'(x) = 6 - 2x$.

f has a critical number at $x = 3$ since 3 is the solution to $6 - 2x = 0$.
To determine whether this critical number is a max, min, or neither, consider where f is increasing and decreasing.



As the diagram shows, $x = 3$ is a maximum.

First Order Differential Equation

An ordinary differential equation of order 1. That is, a differential equation in which the highest derivative is a first derivative. For example, $y' + xy = 1$ is a first order differential equation.

Fixed

Constant. Not changing or moving.

Function Operations

Definitions for combining functions by adding, subtracting, multiplying, dividing, and composing them.

1. addition $(f+g)(x) = f(x) + g(x)$
2. subtraction $(f-g)(x) = f(x) - g(x)$
3. multiplication $(fg)(x) = f(x) \cdot g(x)$
4. division $\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$
5. composition $(f \circ g)(x) = f(g(x))$

Fundamental Theorem of Calculus

The theorem that establishes the connection between derivatives, antiderivatives, and definite integrals. The fundamental theorem of calculus is typically given in two parts.

Fundamental Theorem of Calculus

Part 1: If f is a function which is continuous on $[a, b]$, then the function $g(x) = \int_a^x f(t)dt$ is continuous on $[a, b]$, differentiable on (a, b), and $g'(x) = f(x)$.

Part 2: If f is a function which is continuous on $[a, b]$, then $\int_a^b f(x)dx = F(b) - F(a)$, where F is any antiderivative of f.

GLB

See Greatest Lower Bound of a Set

Global Maximum, Global Max

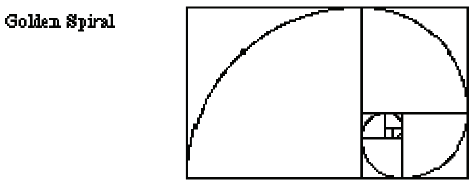
See Absolute Maximum, Absolute Max

Global Minimum, Golbal Min

See Absolute Minimum, Absolute Min

Golden Spiral

A spiral that can be drawn in a golden rectangle as shown below. The figure forming the structure for the spiral is made up entirely of squares and golden rectangles.



Graphic Methods

The use of graphs and/or pictures as the main technique for solving a math problem. When a problem is solved graphically, it is common to use a graphing calculator.

Greatest Lower Bound of a Set
GLB

The greatest of all lower bounds of a set of numbers. For example, the greatest lower bound of (5, 7) is 5. The greatest lower bound of the interval [5, 7] is also 5.

Greek Alphabet

The letters of ancient Greece, which are frequently used in math and science.

Α α	alpha	Ν ν	nu
Β β	beta	Ξ ξ	xi
Γ γ	gamma	Ο ο	omicron
Δ δ	delta	Π π	pi
Ε ε	epsilon	Ρ ρ	rho
Ζ ζ	zeta	Σ σ	sigma
Η η	eta	Τ τ	tau
Θ θ	theta	Υ υ	upsilon
Ι ι	iota	Φ φ	phi
Κ κ	kappa	Χ χ	chi
Λ λ	lambda	Ψ ψ	psi
Μ μ	mu	Ω ω	omega

Harmonic Sequence
Harmonic Progression

The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \ldots$.

Note: The harmonic mean of two terms of the harmonic sequence is the term halfway between the two original terms. For example, the harmonic mean of $\frac{1}{2}$ and $\frac{1}{6}$ is $\frac{1}{4}$.

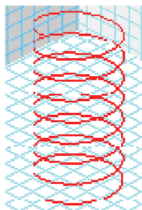
Harmonic Series

The series $1+\frac{1}{2}+\frac{1}{3}+\frac{1}{4}+\cdots+\frac{1}{n}+\cdots$. Note: The harmonic series diverges. Its sequence of partial sums is unbounded.

Helix

A curve shaped like a spring. A helix can be made by coiling a wire around the outside of a right circular cylinder.

Helix



Higher Derivative

Any derivative beyond the first derivative. That is, the second, third, fourth, fifth etc. derivatives.

Hole

See Removable Discontinuity

Homogeneous System of Equations

A system, usually a linear system, in which every constant term is zero.

Homogeneous System of Equations

$$\begin{aligned} 2x - 4y + 5z &= 0 \\ x + 7y + z &= 0 \\ -x + 2y + 3z &= 0 \end{aligned}$$

Hyperbolic Trigonometry

A variation of trigonometry. Hyperbolic trig functions are defined using e^x and e^{-x}. The six hyperbolic trig functions relate to each other in ways that are similar to conventional trig functions. Hyperbolic trig plays an important role when trig functions have imaginary or complex arguments.

Note: Hyperbolic trigonometry has no relation whatsoever to hyperbolic geometry.

Definitions:

$$\sinh x = \frac{e^x \quad e^{-x}}{2} \qquad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$\cosh x = \frac{e^x + e^{-x}}{2} \qquad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}} \qquad \coth x = \frac{\cosh x}{\sinh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

Identity Function

The function $f(x) = x$. More generally, an identity function is one which does not change the domain values at all.

Note: This is called the identity function since it is the identity for composition of functions. That is, if $f(x) = x$ and g is any function, then $(f \circ g)(x) = g(x)$ and $(g \circ f)(x) = g(x)$.

Implicit Differentiation

A method for finding the derivative of an implicitly defined function or relation.

Example: Implicit differentiation is used to find $\frac{dy}{dx}$ for $x^2 + xy - y^2 = 1$.

$$\frac{d}{dx}\left(x^2 + xy - y^2\right) = \frac{d}{dx}(1)$$

$$2x + \left(1 \cdot y + x\frac{dy}{dx}\right) - 2y\frac{dy}{dx} = 0$$

$$2x + y = 2y\frac{dy}{dx} - x\frac{dy}{dx}$$

$$2x + y = \frac{dy}{dx}(2y - x)$$

$$\frac{2x + y}{2y - x} = \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{2x + y}{-x + 2y} \text{ or } \frac{2x + y}{2y - x}$$

Implicit Function or Relation

A function or relation in which the dependent variable is not isolated on one side of the equation. For example, the equation $x^2 + xy - y^2 = 1$ represents an implicit relation

Improper Integral

A definite integral for which the integrand has a discontinuity between the bounds of integration, or which has ∞ and/or –∞ as a bound. Improper integrals are evaluated using limits as shown below. If the limit exists and is finite, we say the integral converges. If the limit does not exist or is infinite, we say the integral diverges.

Examples:

1. $$\int_1^{\infty}\frac{dx}{x^2}=\lim_{b\to\infty}\int_1^{b}\frac{dx}{x^2}=\lim_{b\to\infty}\left[-\frac{1}{x}\right]_1^{b}=\lim_{b\to\infty}\left[-\frac{1}{b}-\left(-\frac{1}{1}\right)\right]=1$$

2. $$\int_{-\infty}^{\infty}\frac{dx}{x^2+1}=\lim_{a\to-\infty}\lim_{b\to\infty}\int_a^{b}\frac{dx}{x^2+1}=\lim_{a\to-\infty}\lim_{b\to\infty}\left[\tan^{-1}x\right]_a^{b}=\lim_{a\to-\infty}\lim_{b\to\infty}\left[\tan^{-1}b-\tan^{-1}a\right]=\pi$$

3. $$\begin{aligned}\int_1^{3}\frac{dx}{(x-2)^{1/3}}&=\left(\lim_{b\to2^-}\int_1^{b}\frac{dx}{(x-2)^{1/3}}\right)+\left(\lim_{a\to2^+}\int_a^{3}\frac{dx}{(x-2)^{1/3}}\right)\\&=\left(\lim_{b\to2^-}\left[\frac{3}{2}(x-2)^{2/3}\right]_1^{b}\right)+\left(\lim_{a\to2^+}\left[\frac{3}{2}(x-2)^{2/3}\right]_a^{3}\right)\\&=\left(\lim_{b\to2^-}\left[\frac{3}{2}(b-2)^{2/3}-\frac{3}{2}(1-2)^{2/3}\right]\right)+\left(\lim_{a\to2^+}\left[\frac{3}{2}(3-2)^{2/3}-\frac{3}{2}(a-2)^{2/3}\right]\right)\\&=-\frac{3}{2}+\frac{3}{2}\\&=0\end{aligned}$$

Increasing Function

A function with a graph that goes up as it is followed from left to right. For example, any line with a positive slope is increasing.

Note: If a function is differentiable, then it is increasing at all points where its derivative is positive.

Indefinite Integral

The family of functions that have a given function as a common derivative. The indefinite integral of $f(x)$ is written
$\int f(x)\ dx$.

Example: $$\int x^2dx=\frac{1}{3}x^3+C$$

Indefinite Integral Rules

See Integral Rules

Indeterminate Expression

An undefined expression which can have a value if arrived at as a limit.

Note: Another way to think about indeterminate expressions is to see them as a disagreement between two rules for simplifying an expression. For example, one way to think about $\frac{0}{0}$ is this: The 0 in the numerator makes the fraction "equal" 0, but the 0 in the denominator makes the fraction $\frac{0}{0}$ "equal" ±∞. This conflict makes the expression indeterminate.

Common indeterminate expressions:

$\frac{0}{0}$ $\frac{\pm\infty}{\pm\infty}$ 00 1∞ ∞0 ∞ − ∞

Example:

The limit $\lim_{x\to 0} \frac{\sin x}{x}$ seems to evaluate to $\frac{0}{0}$, which is indeterminate. In fact,

$$\lim_{x\to 0} \frac{\sin x}{x} = 1$$

since sin x and x are approximately equal to each other for values of x near 0.

Note that this limit can also be computed using l'Hôpital's rule.

Infinite Geometric Series

An infinite series that is geometric. An infinite geometric series converges if its common ratio *r* satisfies –1 < *r* < 1. Otherwise it diverges.

$$\text{General Form} = a_1 + a_1 r + a_1 r^2 + a_1 r^3 + \cdots + a_1 r^{n-1} + \cdots$$

$$\text{Sum} = \frac{a_1}{1-r} \quad \text{as long as} \quad -1 < r < 1$$

Example 1: $3 + 1 + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \cdots$ has $a_1 = 3$ and $r = \frac{1}{3}$.

$$\text{Sum} = \frac{3}{1-\frac{1}{3}} = \frac{9}{2} = 4.5$$

Example 2: $1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \frac{1}{32} + \cdots$ has $a_1 = 1$ and $r = -\frac{1}{2}$.

$$\text{Sum} = \frac{1}{1-\left(-\frac{1}{2}\right)} = \frac{2}{3}$$

Infinite Limit

A limit that has an infinite result (either ∞ or –∞), or a limit taken as the variable approaches ∞ (infinity) or –∞ (minus infinity). The limit can be one-sided.

Infinite Series

A series that has no last term, such as $1+\frac{1}{4}+\frac{1}{9}+\frac{1}{16}+\cdots+\frac{1}{n^2}+\cdots$. The sum of an infinite series is defined as the limit of the sequence of partial sums.

Note: The infinite series above happens to have a sum of $\pi^2/6$.

Example: What is the sum of $.9 + .09 + .009 + .0009 + \cdots + 9\cdot 10^{-n} + \cdots$?

Sequence of partial sums:
$.9,\ .99,\ .999,\ .9999,\ \ldots$
The limit of this sequence is 1.
Thus we say $.9 + .09 + .009 + .0009 + \cdots = 1$.

Infinitesimal

A hypothetical number that is larger than zero but smaller than any positive real number. Although the existence of such numbers makes no sense in the real number system, many worthwhile results can be obtained by overlooking this obstacle.

Note: Sometimes numbers that aren't really infinitesimals are called infinitesimals anyway. The word infinitesimal is occasionally used for tiny positive real numbers that are nearly equal to zero.

Infinity

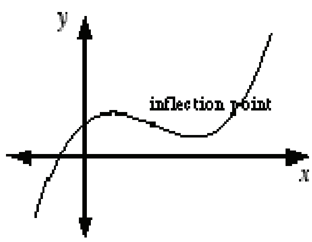
A "number" which indicates a quantity, size, or magnitude that is larger than any real number. The number infinity is written as a sideways eight: ∞. Negative infinity is written –∞.

Note: Neither ∞ nor –∞ is a real number.

Inflection Point

A point at which a curve changes from concave up to concave down, or vice-versa.

Note: If a function has a second derivative, the value of the second derivative is either 0 or undefined at each of that function's inflection points.



Initial Value Problem
IVP

A differential equation or partial differential equation accompanied by conditions for the value of the function and possibly its derivatives at one particular point in the domain.

Differential Equation	$y'' + y = \sin x$
Initial Value Problem (IVP)	$y'' + y = \sin x,\ y(0) = 1,\ y'(0) = -2$
Boundary Value Problem (BVP)	$y'' + y = \sin x,\ y(0) = 1,\ y(1) = -2$

Instantaneous Acceleration

The rate at which an object's instantaneous velocity is changing at a particular moment. This is found by taking the derivative of the velocity function.

Note: For motion on the number line, instantaneous acceleration is a scalar. For motion on a plane or in space, it is a vector.

Instantaneous Rate of Change

The rate of change at a particular moment. Same as the value of the derivative at a particular point.

For a function, the instantaneous rate of change at a point is the same as the slope of the tangent line. That is, it's the slope of a curve.

Note: Over short intervals of time, the average rate of change is approximately equal to the instantaneous rate of change

Instantaneous Velocity

The rate at which an object is moving at a particular moment. Same as the derivative of the function describing the position of the object at a particular time.

Note: For motion on the number line, instantaneous velocity is a scalar. For motion on a plane or in space, it is a vector.

Integrable Function

A function for which the definite integral exists. Piecewise continuous functions are integrable, and so are many functions that are not piecewise continuous.

Note: Non-integrable functions are seldom studied in the first two years of calculus.

Integral

As a noun, it means the integral of a function.

As an adjective, it means "in the form of an integer." For example, saying a polynomial has integral coefficients means the coefficients of the polynomial are all integers.

Integration Methods

The basic methods are listed below. Other more advanced and/or specialized methods exist as well.

u-substitution
integration by parts
partial fractions
trig substitution
rationalizing substitutions

Integral of a Function

The result of either a definite integral or an indefinite integral.

Examples: $\int x^2 dx = \frac{1}{3}x^3 + C$

$\int_2^5 x^2 dx = \frac{1}{3}x^3\Big|_2^5 = \frac{1}{3}(5)^3 - \frac{1}{3}(2)^3 = \frac{117}{3} = 39$

Integral Rules

For the following, *a*, *b*, *c*, and *C* are constants; for definite integrals, these represent real number constants. The rules only apply when the integrals exist.

Indefinite integrals (These rules all apply to definite integrals as well)

1. $\int af(u)\,du = a\int f(u)\,du$

2. $\int [f(u)+g(u)]\,du = \int f(u)\,du + \int g(u)\,du$

3. $\int [f(u)-g(u)]\,du = \int f(u)\,du - \int g(u)\,du$

4. $\int [af(u)+bg(u)]\,du = a\int f(u)\,du + b\int g(u)\,du$

5. Integration by parts: $\int u\,dv = uv - \int v\,du$

Definite integrals

1. $\int_a^b f(u)\,du = -\int_b^a f(u)\,du$

2. $\int_a^c f(u)\,du = \int_a^b f(u)\,du + \int_b^c f(u)\,du$

3. If $f(u) \le g(u)$ for all $a \le u \le b$, then $\int_a^b f(u)\,du \le \int_a^b g(u)\,du$

4. If $f(u) \le M$ for all $a \le u \le b$, then $\int_a^b f(u)\,du \le M(b-a)$

5. If $m \le f(u)$ for all $a \le u \le b$, then $m(b-a) \le \int_a^b f(u)\,du$

6. If $a \le b$, then $\left|\int_a^b f(u)\,du\right| \le \int_a^b |f(u)|\,du$

Integral Test

A convergence test used for positive series which with decreasing terms.

If $a_n = f(n)$, where f is continuous, positive, and decreasing over $[1, \infty)$, then:

1. If $\int_1^\infty f(x)dx$ converges then $\sum_{n=1}^\infty a_n$ converges.

2. If $\int_1^\infty f(x)dx$ diverges then $\sum_{n=1}^\infty a_n$ diverges.

Integral Test Remainder

For a series that converges by the integral test, this is a quantity that measures how accurately the nth partial sum estimates the overall sum.

If $\sum_{n=1}^\infty a_n$ converges to S by the integral test where $a_n = f(n)$, then the remainder $R_n = S - \sum_{k=1}^n a_k$ can be estimated as follows:

$$\int_{n+1}^\infty f(x)dx \le R_n \le \int_n^\infty f(x)dx$$

Integrand

The function being integrated in either a definite or indefinite integral.

Example: $x^2\cos 3x$ is the integrand in $\int x^2\cos 3x\, dx$.

Integration

The process of finding an integral, either a definite integral or an indefinite integral.

Integration by Parts

A formula used to integrate the product of two functions.

Formula: $$\int u\, dv = uv - \int v\, du$$

Example 1: Evaluate $\int xe^{x/2}\, dx$.

Use $u = x$ and $dv = e^{x/2}\, dx$. Then we get $du = dx$ and $v = 2e^{x/2}$. This can be summarized:

$u = x$, $dv = e^{x/2}\, dx$, $du = dx$, $v = 2e^{x/2}$

It follows that

$$\begin{aligned}\int xe^{x/2}\, dx &= 2xe^{x/2} - \int 2e^{x/2}\, dx \\ &= 2xe^{x/2} - 4e^{x/2} + C\end{aligned}$$

Example 2: Evaluate $\int \tan^{-1} x\, dx$.

Use the following: $u = \tan^{-1} x$, $dv = dx$, $du = \frac{1}{1+x^2}dx$, $v = x$

Thus

$$\begin{aligned}\int \tan^{-1} x\, dx &= x\tan^{-1} x - \int \frac{x}{1+x^2}\, dx \\ &= x\tan^{-1} x - \frac{1}{2}\int \frac{2x}{1+x^2}\, dx \\ &= x\tan^{-1} x - \frac{1}{2}\ln\left|1+x^2\right| + C \\ &= x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C\end{aligned}$$

Example 3: Evaluate $\int e^x \sin x \, dx$.

Let I = $\int e^x \sin x \, dx$. Proceed as follows: $u = \sin x$, $dv = e^x\, dx$, $du = \cos x\, dx$, $v = e^x$

Thus $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

Now use integration by parts on the remaining integral. Use the following assignments:

$u = \cos x$, $dv = e^x\, dx$, $du = -\sin x\, dx$, $v = e^x$

Thus

$$\begin{aligned} \int e^x \sin x \, dx &= e^x \sin x - \left[e^x \cos x - \int e^x (-\sin x) \, dx \right] \\ &= e^x \sin x - e^x \cos x - \int e^x \sin x \, dx \end{aligned}$$

Note that $\int e^x \sin x \, dx$ appears on both sides of this equation. Replace it with I and then solve.

$$\begin{aligned} I &= e^x \sin x - e^x \cos x - I \\ 2I &= e^x \sin x - e^x \cos x \\ I &= \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x \end{aligned}$$

We finally obtain

$$\int e^x \sin x dx = \frac{1}{2} e^x \sin x - \frac{1}{2} e^x \cos x + C$$

Integration by Substitution

An integration method that essentially involves using the chain rule in reverse.

Example: For $\int x\sqrt{5+x^2}\, dx$ let $u = 5 + x^2$.

That means $du = 2x\, dx$

$\frac{1}{2} du = x\, dx$

It follows that

$$\begin{aligned} \int x\sqrt{5+x^2}\, dx &= \int \sqrt{u} \cdot \frac{1}{2} du \\ &= \frac{1}{2} \int u^{\frac{1}{2}}\, du \\ &= \frac{1}{2} \cdot \frac{2}{3} u^{\frac{3}{2}} + C \\ &= \frac{1}{3} (5 + x^2)^{\frac{3}{2}} + C \end{aligned}$$

Integration Methods

See Integral Methods

Intermediate Value Theorem
IVT

A theorem verifying that the graph of a continuous function is connected.

Intermediate Value Theorem:

If f is a function that is continuous over the domain $[a, b]$ and if m is a number between $f(a)$ and $f(b)$, then there is some number c between a and b such that $f(c) = m$.

Interval of Convergence

For a power series in one variable, the set of values of the variable for which the series converges. The interval of convergence may be as small as a single point or as large as the set of all real numbers.

Example: The series $\sum_{n=1}^{\infty} \frac{x^n}{n} = x + \frac{x^2}{2} + \frac{x^3}{3} + \frac{x^4}{4} + \cdots$ converges only for $-1 \le x < 1$, so the interval of convergence is $[-1, 1)$.

Iterative Process

An algorithm which involves repeated use of the same formula or steps. Typically, the process begins with a starting value which is plugged into the formula. The result is then taken as the new starting point which is then plugged into the formula again. The process continues to repeat.

Examples of iterative processes are factor trees, recursive formulas, and Newton's method.

IVP

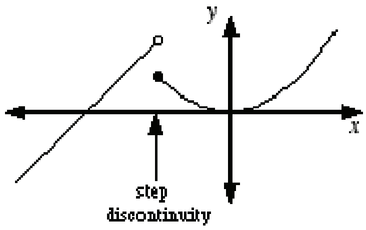
See Initial Value Problem

IVT

See Intermediate Value Theorem

Jump Discontinuity
Step Discontinuity

A discontinuity for which the graph steps or jumps from one connected piece of the graph to another. Formally, it is a discontinuity for which the limits from the left and right both exist but are not equal to each other.



L'Hôpital's Rule
L'Hospital's Rule

A technique used to evaluate limits of fractions that evaluate to the indeterminate expressions $\frac{0}{0}$ and $\frac{\pm\infty}{\pm\infty}$. This is done by finding the limit of the derivatives of the numerator and denominator.

Note: Most limits involving other indeterminate expressions can be manipulated into fraction form so that l'Hôpital's rule can be used.

L'Hôpital's Rule: If f and g are differentiable on an open interval containing a such that g(x) ≠ 0 for all x ≠ a in the interval, and if either

$$\lim_{x \to a} f(x) = 0 \text{ and } \lim_{x \to a} g(x) = 0$$

Or $$\lim_{x \to a} f(x) = \pm\infty \text{ and } \lim_{x \to a} g(x) = \pm\infty$$

Then $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$$

Example: $$\lim_{x \to 0} \frac{1 - \cos x}{x^2} = \lim_{x \to 0} \frac{\sin x}{2x} = \lim_{x \to 0} \frac{\cos x}{2} = \frac{1}{2}$$

Least Upper Bound of a Set
LUB

The smallest of all upper bounds of a set of numbers. For example, the least upper bound of the interval (5, 7) is 7. The least upper bound of [5, 7] is also 7.

Limit

The value that a function or expression approaches as the domain variable(s) approach a specific value. Limits are written in the form $\lim_{x \to a} f(x)$. For example, the limit of $f(x) = \frac{1}{x}$ as x approaches 3 is $\frac{1}{3}$. This is written $\lim_{x \to 3} \frac{1}{x} = \frac{1}{3}$.

Formal definitions:

1. $\lim_{x \to a} f(x) = L$ if and only if for each $\varepsilon > 0$ there exists a $\delta > 0$ such that
   $0 < |x - a| < \delta \Rightarrow |f(x) - L| < \varepsilon$.
2. $\lim_{x \to a} f(x) = \infty$ if and only if for each N there exists a $\delta > 0$ such that
   $0 < |x - a| < \delta \Rightarrow f(x) > N$.
3. $\lim_{x \to a} f(x) = -\infty$ if and only if for each N there exists a $\delta > 0$ such that
   $0 < |x - a| < \delta \Rightarrow f(x) < N$.
4. $\lim_{x \to \infty} f(x) = L$ if and only if for each $\varepsilon > 0$ there exists a number M such that
   $x > M \Rightarrow |f(x) - L| < \varepsilon$.
5. $\lim_{x \to -\infty} f(x) = L$ if and only if for each $\varepsilon > 0$ there exists a number M such that
   $x < M \Rightarrow |f(x) - L| < \varepsilon$.
6. $\lim_{x \to \infty} f(x) = \infty$ if and only if for each N there exists a number M such that
   $x > M \Rightarrow f(x) > N$.
7. $\lim_{x \to \infty} f(x) = -\infty$ if and only if for each N there exists a number M such that
   $x > M \Rightarrow f(x) < N$.
8. $\lim_{x \to -\infty} f(x) = \infty$ if and only if for each N there exists a number M such that
   $x < M \Rightarrow f(x) > N$.
9. $\lim_{x \to -\infty} f(x) = -\infty$ if and only if for each N there exists a number M such that
   $x < M \Rightarrow f(x) < N$.

Limit Comparison Test

A convergence test often used when the terms of a series are rational functions. Essentially, the test determines whether a series is "about as good" as a "good" series or "about as bad" as a "bad" series. The "good" or "bad" series is often a p-series.

A known positive series $\sum b_n$ can be used to test whether the positive series $\sum a_n$ converges or diverges.

1. If $\lim_{n \to \infty} \frac{a_n}{b_n} > 0$, then $\sum a_n$ and $\sum b_n$ both converge or both diverge.
2. If $\lim_{n \to \infty} \frac{a_n}{b_n} = 0$ and $\sum b_n$ converges, then $\sum a_n$ converges.
3. If $\lim_{n \to \infty} \frac{a_n}{b_n} = \infty$ and $\sum b_n$ diverges, then $\sum a_n$ diverges.
4. Otherwise, the test is inconclusive.

Limit from the Left
Limit from Below

A one-sided limit which, in the example $\lim_{x\to 0^-}\frac{1}{x}=-\infty$, restricts x such that $x<0$.

In general, a limit from the left restricts the domain variable to values less than the number the domain variable approaches. When a limit is taken from the left it is written $\lim_{x\to a^-} f(x)$ or $\lim_{x\uparrow a} f(x)$.

For example, $\lim_{x\to 0^-}\frac{1}{x}=-\infty$ since $\frac{1}{x}$ tends toward $-\infty$ as x gets closer and closer to 0 from the left.

Formal Definitions:

1. $\lim_{x\to a^-} f(x)=L$ if and only if for each $\varepsilon>0$ there exists a $\delta>0$ such that
$$a-\delta<x<a \Rightarrow |f(x)-L|<\varepsilon.$$
2. $\lim_{x\to a^-} f(x)=\infty$ if and only if for each N there exists a $\delta>0$ such that
$$a-\delta<x<a \Rightarrow f(x)>N.$$
3. $\lim_{x\to a^-} f(x)=-\infty$ if and only if for each N there exists a $\delta>0$ such that
$$a-\delta<x<a \Rightarrow f(x)<N.$$

Limit from the Right
Limit from Above

A one-sided limit which, in the example $\lim_{x\to 0^+}\frac{1}{x}=\infty$, restricts x such that $x>0$.

In general, a limit from the right restricts domain variable to values greater than the number the domain variable approaches. When a limit is taken from the right it is written $\lim_{x\to a^+} f(x)$ or $\lim_{x\downarrow a} f(x)$.

For example, $\lim_{x\to 0^+}\frac{1}{x}=\infty$ since $\frac{1}{x}$ tends toward ∞ as x gets closer and closer to 0 from the right.

Formal Definitions:

1. $\lim_{x\to a^+} f(x)=L$ if and only if for each $\varepsilon>0$ there exists a $\delta>0$ such that
$$a<x<a+\delta \Rightarrow |f(x)-L|<\varepsilon.$$
2. $\lim_{x\to a^+} f(x)=\infty$ if and only if for each N there exists a $\delta>0$ such that
$$a<x<a+\delta \Rightarrow f(x)>N.$$
3. $\lim_{x\to a^+} f(x)=-\infty$ if and only if for each N there exists a $\delta>0$ such that
$$a<x<a+\delta \Rightarrow f(x)<N.$$

Limit Test for Divergence

A convergence test that uses the fact that the terms of a convergent series must have a limit of zero.

If $\lim_{n\to\infty} a_n$ does not exist or $\lim_{n\to\infty} a_n\neq 0$, the series $\sum a_n$ diverges.
Otherwise the test is inconclusive.

Bounds of Integration
Limits of Integration

For the definite integral $\int_a^b f(x)dx$, the bounds (or limits) of integration are *a* and *b*.

Local Behavior

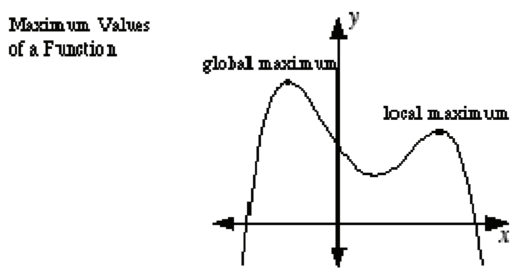
The appearance or properties of a function, graph, or geometric figure in the immediate neighborhood of a particular point. Usually this refers to any appearance or property that becomes more apparent as you zoom in on the point.

For example, as you zoom in to the graph of $y = x^2$ at any point, the graph looks more and more like a line. Thus we say that $y = x^2$ is locally linear. We say this even though the graph is not actually a straight line.

Relative Maximum, Relative Max
Local Maximum, Local Max

The highest point in a particular section of a graph.

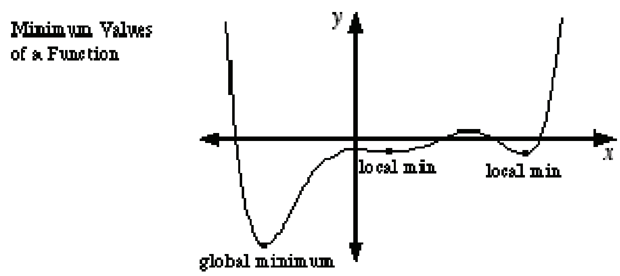
Note: The first derivative test and the second derivative test are common methods used to find maximum values of a function.



Relative Minimum, Relative Min
Local Minimum, Local Min

The lowest point in a particular section of a graph.

Note: The first derivative test and the second derivative test are common methods used to find minimum values of a function.



Logarithmic Differentiation

A method for finding the derivative of functions such as $y = x^{\sin x}$ and $y = \dfrac{(x-7)^5\sqrt{2x+3}}{\sqrt[6]{x+1}}$.

Example 1: $y = x^{\sin x}$

First take the ln of both sides and simplify.

$$\ln y = \ln x^{\sin x} = (\sin x)(\ln x)$$

Then take the derivative of both sides.

$$\frac{1}{y}y' = (\cos x)(\ln x) + (\sin x)\left(\frac{1}{x}\right)$$

Now solve for y' and simplify.

$$y' = y\left[(\cos x)(\ln x) + \frac{\sin x}{x}\right]$$

$$y' = x^{\sin x}\left[(\cos x)(\ln x) + \frac{\sin x}{x}\right]$$

Example 2: $y = \dfrac{(x-7)^5\sqrt{2x+3}}{\sqrt[6]{x+1}}$

First take the ln of both sides and simplify.

$$\ln y = \ln\frac{(x-7)^5\sqrt{2x+3}}{\sqrt[6]{x+1}}$$

$$= 5\ln(x-7) + \frac{1}{2}\ln(2x+3) - \frac{1}{6}\ln(x+1)$$

Then take the derivative of both sides.

$$\frac{1}{y}y' = \frac{5}{x-7} + \frac{1}{2}\cdot\frac{2}{2x+3} - \frac{1}{6}\cdot\frac{1}{x+1}$$

$$= \frac{5}{x-7} + \frac{1}{2x+3} - \frac{1}{6(x+1)}$$

Now solve for y' and simplify.

$$y' = y\left[\frac{5}{x-7} + \frac{1}{2x+3} - \frac{1}{6(x+1)}\right]$$

$$= \frac{(x-7)^5\sqrt{2x+3}}{\sqrt[6]{x+1}}\left[\frac{5}{x-7} + \frac{1}{2x+3} - \frac{1}{6(x+1)}\right]$$

Logistic Growth

A model for a quantity that increases quickly at first and then more slowly as the quantity approaches an upper limit. This model is used for such phenomena as the increasing use of a new technology, spread of a disease, or saturation of a market (sales).

The equation for the logistic model is $N = \dfrac{N_0K}{N_0 + (K - N_0)e^{-rt}}$. Here, t is time, N stands for the amount at time t, N_0 is the initial amount (at time 0), K is the maximum amount that can be sustained, and r is the rate of growth when N is very small compared to K.

Note: The logistic growth model can be obtained by solving the differential equation $\dfrac{dN}{dt} = rN\left(1 - \dfrac{N}{K}\right)$

LUB

See Least Upper Bound of a Set

Model
Mathematical Model

An equation or a system of equations representing real-world phenomena. Models also represent patterns found in graphs and/or data. Usually models are not exact matches the objects or behavior they represent. A good model should capture the essential character of whatever is being modeled.

Maximize

To find the largest possible value.

Example: Maximize $f(x) = 4 - x^2$.
The largest possible value of $4 - x^2$ is 4.
This occurs when $x = 0$.

Maximum of a Function:

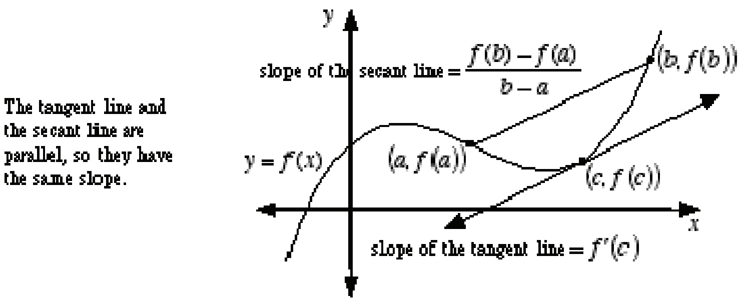
Either a relative (local) maximum or an absolute (global) maximum.

Mean Value Theorem

A major theorem of calculus that relates values of a function to a value of its derivative. Essentially the theorem states that for a "nice" function, there is a tangent line parallel to any secant line.

Mean Value Theorem: If function f is continuous on $[a, b]$ and differentiable on (a, b), then there exists a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$



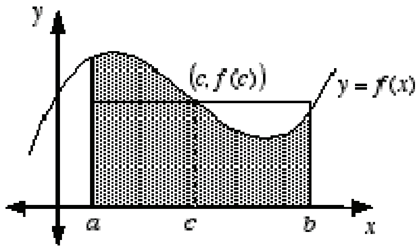
Mean Value Theorem for Integrals

A variation of the mean value theorem which guarantees that a continuous function has at least one point where the function equals the average value of the function.

Mean Value Theorem for Integrals:

If function f is continuous on $[a,b]$, then there exists a number c in $[a,b]$ such that $f(c)=\frac{1}{b-a}\int_a^b f(x)dx$.

The rectangle has the same area as the shaded region under the curve.



Mesh of a Partition
Norm of a Partition

The width of the largest sub-interval in a partition.

Example: Consider the partition {0, 0.2, 0.9, 1.1, 1.6, 2} of the interval [0,2].

The sub-intervals are [0,0.2], [0.2,0.9], [0.9,1.1], [1.1,1.6], and [1.6,2].
The widest of these is [0.2,0.9]. It has width = 0.7.
Thus the mesh of the partition is 0.7.

Min/Max Theorem
See Extreme Value Theorem

Minimize

To find the smallest possible value.

Example: Minimize $f(x)=x^2+3$.

The smallest possible value of x^2+3 is 3.
This occurs when $x=0$.

Minimum of a Function

Either a relative (local) minimum or an absolute (global) minimum.

Mode

The number that occurs the most often in a list.

Example: 5 is the mode of 2, 3, 3, 4, 5, 5, 5

Model

See Mathematical Model

Moment

A number indicating the degree to which a figure tends to balance on a given line (axis). A moment of zero indicates perfect balance, and a large moment indicates a strong tendency to tip over.

Formally, the moment of a point P about a fixed axis is the mass of P times the distance from P to the axis. For a figure, the moment is the cumulative sum of the moments of all the figure's points. This cumulative sum is the same as the mass of the figure times the distance from the figure's center of mass to the fixed axis.

Note: This is similar to, but not the same as, the physics quantity known as moment of inertia.

1. Plane region bounded above by $y = f(x)$, below by the x-axis, on the left by $x = a$, and on the right by $x = b$. The density of the region is uniform (constant) and equals ρ.

$$M_y = \rho\int_a^b x f(x)\,dx \qquad M_x = \rho\int_a^b \frac{1}{2}\left[f(x)\right]^2 dx$$

Note: M_y is the moment about the y-axis. M_x is the moment about the x-axis.

2. Plane region bounded above by $y = f(x)$, below by $y = g(x)$, on the left by $x = a$, and on the right by $x = b$. The density of the region is uniform (constant) and equals ρ.

$$M_y = \rho\int_a^b x\left[f(x) - g(x)\right]dx \qquad M_x = \rho\int_a^b \frac{1}{2}\left(\left[f(x)\right]^2 - \left[g(x)\right]^2\right)dx$$

Note: M_y is the moment about the y-axis. M_x is the moment about the x-axis.

Multivariable
Multivariate

An adjective describing any problem that uses more than one variable.

Multivariable Calculus
Multivariable Analysis
Vector Calculus

The use of calculus (limits, derivatives, and integrals) with two or more independent variables, or two or more dependent variables. This can be thought of as the calculus of three dimensional figures.

Common elements of multivariable calculus include parametric equations, vectors, partial derivatives, multiple integrals, line integrals, and surface integrals. Most of multivariable calculus is beyond the scope of this website.

MVT

See Mean Value Theorem

Neighborhood

A neighborhood of a number a is any open interval containing a. One common notation for a neighborhood of a is $\{x: |x - a| < \delta\}$. Using interval notation this would be $(a - \delta, a + \delta)$.

Newton's Method

An iterative process using derivatives that can often (but not always) be used to find zeros of a differentiable function. The basic idea is to start with an approximate guess for the zero, then use the formula below to turn that guess into a better approximation. This process is repeated until, after only a few steps, the approximation is extremely close to the actual value of the zero.

Note: In some circumstances, Newton's method backfires and gives successively worse and worse approximations.

Newton's method: To obtain successive approximations $x_1, x_2, x_3, x_4, \ldots$ of a zero, start with a guess x_1 and use

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$$

Example: Use Newton's method to approximate $\sqrt{2}$ by finding the positive zero of $f(x) = x^2 - 2$.

Starting with $x_1 = 1.5$, use $x_{n+1} = x_n - \dfrac{x_n^2 - 2}{2x_n}$.

$$x_2 = x_1 - \frac{x_1^2 - 2}{2x_1} = 1.5 - \frac{1.5^2 - 2}{2(1.5)} = 1.4167$$

$$x_3 = x_2 - \frac{x_2^2 - 2}{2x_2} = 1.4167 - \frac{1.4167^2 - 2}{2(1.4167)} = 1.4142$$

After only two iterations, $\sqrt{2} = 1.4142$ is accurate to four decimal places.

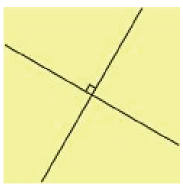
Norm of a Partition

See Mesh of a Partition

Normal
Perpendicular
Orthogonal

At a 90° angle. Note: Perpendicular lines have slopes that are negative reciprocals.

Example: Perpendicular Lines



nth Degree Taylor Polynomial

See Taylor Polynomial

nth Derivative

The result of taking the derivative of the derivative of the derivative etc. of a function a total of n times. Written $f^{(n)}(x)$ or $\frac{d^n y}{dx^n}$.

Note: $f^{(0)}(x)$ is the same thing as $f(x)$.

nth Partial Sum

The sum of the first n terms of an infinite series.

Example: For the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$ we have

1st partial sum $S_1 = 1$

2nd partial sum $S_2 = 1 + \frac{1}{4}$

3rd partial sum $S_3 = 1 + \frac{1}{4} + \frac{1}{9}$

4th partial sum $S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$

etc.

n-tuple
Coordinates / Ordered Pair / Ordered Triple

On the coordinate plane, the pair of numbers giving the location of a point (ordered pair). In three-dimensional coordinates, the triple of numbers giving the location of a point (ordered triple). In n-dimensional space, a sequence of n numbers written in parentheses.

Ordered pair: Two numbers written in the form (x, y).

Ordered triple: Three numbers written in the form (x, y, z).

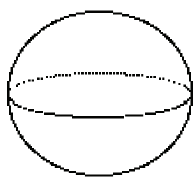
n-tuple: n numbers written in the form $(x_1, x_2, x_3, \ldots, x_n)$.

Oblate Spheroid

A flattened sphere. More formally, an oblate spheroid is a surface of revolution obtained by revolving an ellipse about its minor axis.

Note: The earth is shaped like an oblate spheroid.

Oblate Spheroid



One-Sided Limit

Either a limit from the left or a limit from the right.

Operations on Functions
See Function Operations

Order of a Differential Equation

The number of the highest derivative in a differential equation. A differential equation of order 1 is called first order, order 2 second order, etc.

Example: The differential equation $y'' + xy' - x^3y = \sin x$ is second order since the highest derivative is y'' or the second derivative.

Ordinary Differential Equation

A differential equation which does not include any partial derivatives.

Orthogonal
See Normal

p-series

A series of the form $\frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$ or $\sum_{n=1}^{\infty} \frac{1}{n^p}$, where $p > 0$. Often employed when using the comparison test and the limit comparison test.

Note: The harmonic series is a p-series with $p = 1$.

$$\text{p-series: } \sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \cdots + \frac{1}{n^p} + \cdots$$

If $0 < p \le 1$, the series diverges.
If $p > 1$, the series converges.

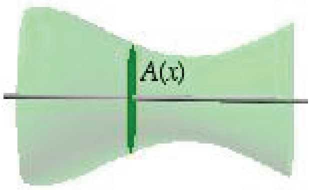
Parallel Cross Sections

The formula below gives the volume of a solid. $A(x)$ is the formula for the area of parallel cross-sections over the entire length of the solid.

Note: The disk method and the washer method are both derived from this formula.

$$\text{Volume} = \int_a^b A(x)\,dx$$

$A(x)$ = the formula for the area of parallel cross-sections over the entire length of the solid.



Parameter (algebra)

The independent variable or variables in a set of parametric equations.

System	Parameter(s)
$\begin{cases} x = t^3 \\ y = 2^t \end{cases}$	t
$\begin{cases} x = s + t + 1 \\ y = 2s - t \\ z = 4s + 7 \end{cases}$	s and t

Parametric Derivative Formulas

The formulas for the first derivative $\frac{dy}{dx}$ and second derivative $\frac{d^2y}{dx^2}$ of a parametrically defined curve are given below.

$$\text{Slope of tangent line} = \frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

$$\text{Second derivative} = \frac{d^2y}{dx^2} = \frac{\frac{d}{dt}\left(\frac{dy}{dx}\right)}{\frac{dx}{dt}} = \frac{\frac{dx}{dt}\frac{d^2y}{dt^2} - \frac{dy}{dt}\frac{d^2x}{dt^2}}{\left(\frac{dx}{dt}\right)^3}$$

Parametric Equations

A system of equations with more than one dependent variable. Often parametric equations are used to represent the position of a moving point.

Examples: $\begin{cases} x = 4t - t^3 \\ y = 2^t \end{cases}$ represents a moving point on the x-y plane.

$\begin{cases} x = s + t + 1 \\ y = 2s - t \\ z = 4s + 7 \end{cases}$ represents a plane in space.

Parametric Integral Formula
See Area Using Parametric Equations

Parametrize

To write in terms of parametric equations.

Example: The line $x + y = 2$ can be parametrized as $x = 1 + t$, $y = 1 - t$.

Partial Fractions

The process of writing any proper rational expression as a sum of proper rational expressions. This method is use in integration as shown below.

Note: Improper rational expressions can also be rewritten using partial fractions. You must, however, use polynomial long division first before finding a partial fractions representation.

Example 1: Write $\dfrac{4x+1}{x^2-x-2}$ using partial fractions.

$$\frac{4x+1}{x^2-x-2} = \frac{4x+1}{(x+1)(x-2)} = \frac{A}{x+1} + \frac{B}{x-2} = \frac{A(x-2)+B(x+1)}{(x+1)(x-2)}$$

$$4x+1 = A(x-2) + B(x+1)$$

$$x=2 \Rightarrow 4\cdot 2+1 = A(0)+B(3) \Rightarrow B=3$$

$$x=-1 \Rightarrow 4(-1)+1 = A(-3)+B(0) \Rightarrow A=1$$

Thus $\dfrac{4x+1}{x^2-x-2} = \dfrac{1}{x+1} + \dfrac{3}{x-2}$.

Example 2: Use partial fractions to find $\displaystyle\int \frac{4x+1}{x^2-x-2}\,dx$.

$$\int \frac{4x+1}{x^2-x-2}\,dx = \int\left(\frac{1}{x+1}+\frac{3}{x-2}\right)dx$$

$$= \ln(x+1) + 3\ln(x-2) + C$$

$$= \ln\left[(x+1)(x-2)^3\right] + C \text{ or } \ln(x+1)(x-2)^3 + C$$

Partial Sum of a Series

The sum of a finite number of terms of a series.

Partition of an Interval

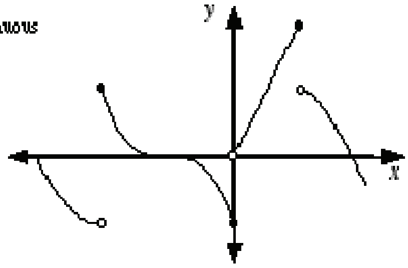
A division of an interval into a finite number of sub-intervals. Specifically, the partition itself is the set of endpoints of each of the sub-intervals.

Example: {0, 0.2, 0.9, 1.1, 1.6, 2} is a partition of the interval [0,2].

The sub-intervals are [0,0.2], [0.2,0.9], [0.9,1.1], [1.1,1.6], and [1.6,2].

Piecewise Continuous Function

A function made up of a finite number of continuous pieces. Piecewise continuous functions may not have vertical asymptotes. In fact, the only possible types of discontinuities for a piecewise continuous function are removable and step discontinuities.



Pinching Theorem

See Sandwich Theorem

Squeeze Theorem

Polar Derivative Formulas

The formula for the first derivative $\frac{dy}{dx}$ of a polar curve is given below.

$$\text{Slope of tangent line } = \frac{dy}{dx} = \frac{\frac{dr}{d\theta}\sin\theta + r\cos\theta}{\frac{dr}{d\theta}\cos\theta - r\sin\theta}$$

Polar Integral Formula

See Area Using Polar Coordinates

Positive Series

A series with terms that are all positive.

Power Rule

The formula for finding the derivative of a power of a variable.

Power Rule: $\frac{d}{dx}x^n = nx^{n-1}$

Examples: $\frac{d}{dx}x^8 = 8x^7$

$$\frac{d}{dx}x^{-5} = -5x^{-6} = -\frac{5}{x^6}$$

$$\frac{d}{dx}\sqrt{x} = \frac{d}{dx}x^{\frac{1}{2}} = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

Power Series

A series which represents a function as a polynomial that goes on forever and has no highest power of *x*.

Power series in x: $\sum_{k=0}^{\infty} c_k x^k = c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \cdots$

Power series in $x-a$: $\sum_{k=0}^{\infty} c_k (x-a)^k = c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \cdots$

Example: $e^x = \sum_{k=0}^{\infty} \frac{x^k}{k!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \cdots$

Power Series Convergence

A theorem that states the three alternatives for the way a power series may converge.

One of the following statements is true for any power series $\sum c_n (x-a)^n$:

1. $\sum c_n (x-a)^n$ converges only for $x = a$.
2. $\sum c_n (x-a)^n$ converges for all real values of x.
3. $\sum c_n (x-a)^n$ converges within a radius of convergence R and diverges outside R.
   That is, the series converges for $|x-a| < R$ and diverges for $|x-a| > R$.

Product Rule

A formula for the derivative of the product of two functions.

Product Rule: $(uv)' = u'v + uv'$

Examples:

$$\frac{d}{dx}(x\sin x) = 1\cdot\sin x + x\cos x = \sin x + x\cos x$$

$$\frac{d}{dx}\left(e^{2x}\tan x\right) = 2e^{2x}\tan x + e^{2x}\sec^2 x = e^{2x}\left(2\tan x + \sec^2 x\right)$$

$$\begin{aligned}\frac{d}{dx}\left(x^2\sqrt{1+x}\right) &= 2x\sqrt{1+x} + x^2\cdot\frac{1}{2}(1+x)^{-\frac{1}{2}}\cdot 1\\ &= 2x\sqrt{1+x} + \frac{x^2}{2\sqrt{1+x}}\\ &= \frac{4x+4x^2}{2\sqrt{1+x}} + \frac{x^2}{2\sqrt{1+x}}\\ &= \frac{4x+5x^2}{2\sqrt{1+x}}\end{aligned}$$

Projectile Motion
Falling Bodies

A formula used to model the vertical motion of an object that is dropped, thrown straight up, or thrown straight down.

Projectile Motion: $y = \frac{1}{2}at^2 + v_0 t + y_0$

$y =$ height
$t =$ time
$a =$ acceleration due to gravity
$v_0 =$ initial velocity
$y_0 =$ initial height

Example: Leslie, standing on top of a cliff, throws a rock upwards at 15 ft/sec from an initial height of 50 feet. Given $a = -32$ ft/sec^2, how high is the rock after 2 seconds?

Solution: $a = -32$, $v_0 = 15$, and $y_0 = 50$, so after t seconds the height y, in feet, is

$$y = -16t^2 + 15t + 50 .$$

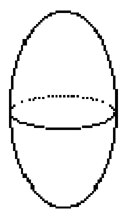
After $t = 2$ seconds, the height is

$$y = -16(2)^2 + 15(2) + 50 = -64 + 30 + 50 = 16 \text{ feet.}$$

Prolate Spheroid

A stretched sphere shaped like a watermelon. Formally, a prolate spheroid is a surface of revolution obtained by revolving an ellipse about its major axis.

Prolate Spheroid



Quotient Rule

A formula for the derivative of the quotient of two functions.

Quotient Rule: $\left(\frac{u}{v}\right)' = \frac{u'v - uv'}{v^2}$

Examples:

$$\left(\frac{1-5x}{4x+7}\right)' = \frac{-5(4x+7)-(1-5x)\cdot 4}{(4x+7)^2} = \frac{-20x-35-4+20x}{(4x+7)^2} = -\frac{39}{(4x+7)^2}$$

$$\frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x\cdot\cos x - \sin x\cdot(-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} = \sec^2 x$$

Radius of Convergence

The distance between the center of a power series' interval of convergence and its endpoints. If the series only converges at a single point, the radius of convergence is 0. If the series converges over all real numbers, the radius of convergence is ∞.

Example: The series $\sum_{n=1}^{\infty} \frac{(x-3)^n}{n} = (x-3) + \frac{(x-3)^2}{2} + \frac{(x-3)^3}{3} + \frac{(x-3)^4}{4} + \cdots$ has interval of convergence $[2,4)$. Thus the radius of convergence is 1.

Ratio Test

A convergence test used when terms of a series contain factorials and/or nth powers.

Compute $r = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

1. If $r < 1$, the series converges absolutely.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. Otherwise ($r = 1$ or the limit does not exist) the test is inconclusive.

Rationalizing Substitutions

An integration method which is often useful when the integrand is a fraction including more than one kind of root, such as $\frac{\sqrt{x}}{1+\sqrt[3]{x}}$. A different type of rationalizing substitution can be used to work with integrands such as $\frac{1}{1+e^x}$.

Note: This method transforms the integrand into a rational function, hence the name *rationalizing*.

Example 1: Find $\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx$.

Let $u = \sqrt[6]{x} = x^{\frac{1}{6}}$, so $u^6 = x$. That means $6u^5\,du = dx$.

$$\begin{aligned}
\int \frac{\sqrt{x}}{1+\sqrt[3]{x}}\,dx &= \int \frac{u^3}{1+u^2}\cdot 6u^5\,du \\
&= \int \frac{6u^8}{1+u^2}\,du \qquad \text{(use polynomial long division)} \\
&= \int \left(6u^6 - 6u^4 + 6u^2 - 6 + \frac{6}{1+u^2}\right) du \\
&= \frac{6}{7}u^7 - \frac{6}{5}u^5 + 2u^3 - 6u + 6\tan^{-1}u + C \\
&= \frac{6}{7}x^{\frac{7}{6}} - \frac{6}{5}x^{\frac{5}{6}} + 2x^{\frac{1}{2}} - 6x^{\frac{1}{6}} + 6\tan^{-1}x^{\frac{1}{6}} + C
\end{aligned}$$

Example 2: Find $\int \frac{1}{1+e^x}\,dx$.

Let $u = 1+e^x$, so $\ln(u-1) = x$. That means $\frac{1}{u-1}\,du = dx$.

$$\begin{aligned}\int \frac{1}{1+e^x}\,dx &= \int \frac{1}{u}\cdot\frac{1}{u-1}\,du \\ &= \int\left(\frac{1}{u-1}-\frac{1}{u}\right)du \qquad \text{(partial fractions)} \\ &= \ln|u-1| - \ln|u| + C \\ &= \ln\left|\frac{u-1}{u}\right| + C \\ &= \ln\left|\frac{e^x}{1+e^x}\right| + C \text{ or } \ln\left(\frac{e^x}{1+e^x}\right) + C.\end{aligned}$$

Reciprocal Rule

A formula for the derivative of the reciprocal of a function.

Reciprocal Rule: $\left(\frac{1}{v}\right)' = -\frac{v'}{v^2}$

Examples: $\left(\frac{1}{x^3-4}\right)' = -\frac{3x^2}{(x^3-4)^2}$

$$\left(\frac{1}{\sin x}\right)' = -\frac{\cos x}{\sin^2 x} = -\frac{1}{\sin x}\cdot\frac{\cos x}{\sin x} = -\csc x\cot x$$

Rectangular Form

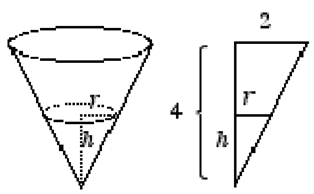
See Cartesian Form

Related Rates

A class of problems in which rates of change are related by means of differentiation. Standard examples include water dripping from a cone-shaped tank and a man's shadow lengthening as he walks away from a street lamp.

Example: A leaky water tank is shaped like an upside-down cone. The tank is 4 m high and the base has a radius of 2 m. Water leaks out at a rate of 0.15 $\mathrm{m}^3/\mathrm{sec}$. At what rate is the height of the water in the tank dropping when the water is 0.7 m deep?

Solution:



$\frac{r}{h} = \frac{2}{4}$ so $r = \frac{h}{2}$. Thus

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{h}{2}\right)^2 h = \frac{1}{12}\pi h^3.$$

The equation $V = \frac{1}{12}\pi h^3$ gives a snapshot of the problem at one instant of time. To show what happens as time moves forward, take $\frac{d}{dt}$ of both sides of the equation.

$\frac{dV}{dt} = \frac{1}{4}\pi h^2\frac{dh}{dt}$ Given: $h = 0.7$, $r = \frac{0.7}{2} = 0.35$, $\frac{dV}{dt} = -0.15$.

$-0.15 = \frac{1}{4}\pi(0.7)^2\frac{dh}{dt}$, so $\frac{dh}{dt} = \frac{-0.15}{\pi(0.7)^2}\cdot 4 = -0.39\ \mathrm{m/sec}$.

The water level is dropping at about 0.39 meters per second.

Local Maximum, Local Max
See Relative Maximum, Relative Max

Local Minimum, Local Min
See Relative Minimum, Relative Min

Remainder of a Series

The difference between the nth partial sum and the sum of a series.

Example: Consider $\sum_{n=1}^{\infty} \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$. The first few partial sums and remainders are:

$$S_1 = \frac{1}{2} \qquad R_1 = \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$
$$S_2 = \frac{1}{2} + \frac{1}{4} \qquad R_2 = \frac{1}{8} + \frac{1}{16} + \frac{1}{32} + \cdots$$
$$S_3 = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} \qquad R_3 = \frac{1}{16} + \frac{1}{32} + \cdots$$

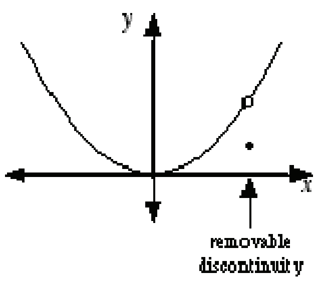
The nth partial sum S_n and nth remainder R_n are thus

$$S_n = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots + \frac{1}{2^n} \qquad R_n = \frac{1}{2^{n+1}} + \frac{1}{2^{n+2}} + \frac{1}{2^{n+3}} + \cdots$$

Removable Discontinuity
Hole

A hole in a graph. That is, a discontinuity that can be "repaired" by filling in a single point. In other words, a removable discontinuity is a point at which a graph is not connected but can be made connected by filling in a single point.

Formally, a removable discontinuity is one at which the limit of the function exists but does not equal the value of the function at that point; this may be because the function does not exist at that point.



Riemann Sum

An approximation of the definite integral $\int_a^b f(x)dx$. This is accomplished in a three-step procedure.

1. Partition the interval $[a, b]$. Select a number from each sub-interval. Let c_i be the number from the ith interval.

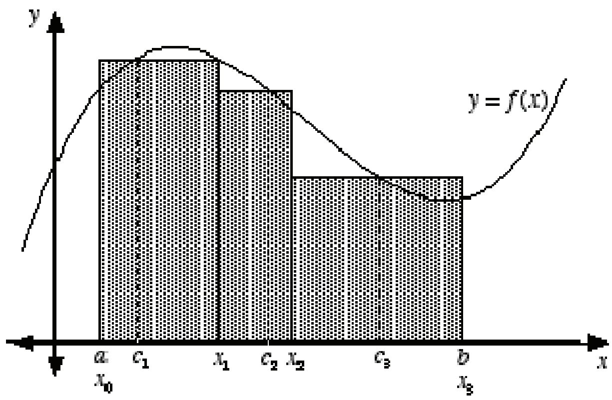
2. Draw a rectangle over each sub-interval. The width of each rectangle is the width of the sub-interval. The height of each rectangle is $f(c_i)$.

3. Add up the areas of all the rectangles. This is the Riemann sum.

$$f(c_1)(x_1-x_0)+f(c_2)(x_2-x_1)+f(c_3)(x_3-x_2)+\cdots+f(c_n)(x_n-x_{n-1})=\sum_{k=1}^{n} f(c_k)(x_k-x_{k-1})$$

Note: The smaller the norm of the partition, the better the approximation of $\int_a^b f(x)dx$. In fact,

$$\int_a^b f(x)dx = \lim_{\text{norm}\to 0}\sum_{k=1}^{n} f(c_k)(x_k-x_{k-1}).$$



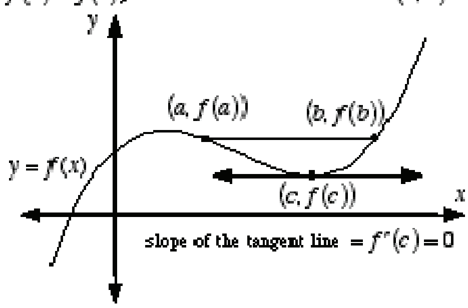
The shaded area is an approximation of the area below $y = f(x)$.

$$\int_a^b f(x)dx = f(c_1)(x_1-x_0)+f(c_2)(x_2-x_1)+f(c_3)(x_3-x_2)$$

Rolle's Theorem

A theorem of calculus that ensures the existence of a critical point between any two points on a "nice" function that have the same *y*-value.

Rolle's Theorem: If function f is continuous on $[a, b]$ and differentiable on (a, b), and if $f(a)=f(b)$, then there exists a number c in (a, b) such that $f'(c)=0$.



Root Test

A convergence test used when series terms contain nth powers.

Compute $r = \lim_{n\to\infty} \sqrt[n]{|a_n|}$.

1. If $r < 1$, the series converges absolutely.
2. If $r > 1$ (including $r = \infty$), the series diverges.
3. Otherwise ($r = 1$ or the limit does not exist) the test is inconclusive .

Sandwich Theorem
Squeeze Theorem
Pinching Theorem

A theorem which allows the computation of the limit of an expression by trapping the expression between two other expressions which have limits that are easier to compute.

Pinching Theorem

If $g(x) \le f(x) \le h(x)$ for all x in a deleted neighborhood of a and $\lim_{x\to a} g(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} f(x) = L$. This includes the possibility that $L = \infty$ or $L = -\infty$.

Example: Prove $\lim_{x\to\infty} \frac{\sin x}{x} = 0$.

Note the following:

$$-1 \le \sin x \le 1$$

$$-\frac{1}{x} \le \frac{\sin x}{x} \le \frac{1}{x}$$

$$\lim_{x\to\infty}\left(-\frac{1}{x}\right) = 0 \qquad \lim_{x\to\infty}\frac{1}{x} = 0$$

Thus $\lim_{x\to\infty} \frac{\sin x}{x} = 0$ by the pinching theorem.

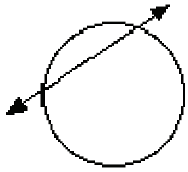
Scalar

Any real number, or any quantity that can be measured using a single real number. Temperature, length, and mass are all scalars. A scalar is said to have magnitude but no direction. A quantity with both direction and magnitude, such as force or velocity, is called a vector.

Secant Line

A line which passes through at least two points of a curve. Note: If the two points are close together, the secant line is nearly the same as a tangent line.

Secant line of a circle



Second Derivative

The derivative of a derivative. Usually written $f''(x)$, $\frac{d^2y}{dx^2}$, or y''.

Example: Consider $f(x) = 5x^3 - x^2 + 13x - 8$.

First derivative: $f'(x) = 15x^2 - 2x + 13$
Second derivative: $f''(x) = 30x - 2$

Second Derivative Test

A method for determining whether a critical point is a relative minimum or maximum.

Second Derivative Test:

1. If the second derivative is positive at a critical point, then the point is a relative (or absolute) minimum.
2. If the second derivative is negative at a critical point, then the point is a relative (or absolute) maximum.
3. If the second derivative is 0 or undefined, the test is inconclusive. The inflection point could be a maximum, minimum, or neither. If this happens, use the first derivative test.

Example: Consider $y = x^3 - 3x + 4$.

Note that $y' = 3x^2 - 3 = 3(x-1)(x+1)$ and $y'' = 6x$. The critical points are at $(1, 2)$ and $(-1, 6)$.

At $(1, 2)$ we find $y'' = 6(1) = 6$, so $(1, 2)$ is a relative minimum.
At $(-1, 6)$ we find $y'' = 6(-1) = -6$, so $(-1, 6)$ is a relative maximum.

Second Order Critical Point

A point on the graph of a function at which the second derivative is either 0 or undefined. A second order critical point may or may not be an inflection point.

Note: The phrase *second order critical point* is NOT in common usage among mathematicians or in textbooks. Nevertheless, it is a useful name for a type of point which otherwise has no name.

Second Order Differential Equation

An ordinary differential equation of order 2. That is, a differential equation in which the highest derivative is a second derivative.

Example: $$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = e^{x}$$

Separable Differential Equation

A first order ordinary differential equation which can be solved by separating all occurrences of the two variables on either side of the equal sign and then integrating.

Example: $$\frac{dy}{dx} = \frac{x^2}{y}$$

Solution:
$$\begin{aligned} \frac{dy}{dx} &= \frac{x^2}{y} \\ y\,dy &= x^2 dx \\ \int y\,dy &= \int x^2 dx \\ \frac{1}{2}y^2 &= \frac{1}{3}x^3 + C \\ y^2 &= \frac{2}{3}x^3 + C \\ y &= \pm\sqrt{\frac{2}{3}x^3 + C} \end{aligned}$$

Sequence

A list of numbers set apart by commas, such as 1, 3, 5, 7, . . .

Sequence of Partial Sums

The sequence of nth partial sums of a series.

Example: For the series $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots + \frac{1}{n^2} + \cdots$ we have

1st partial sum $S_1 = 1$

2nd partial sum $S_2 = 1 + \frac{1}{4}$

3rd partial sum $S_3 = 1 + \frac{1}{4} + \frac{1}{9}$

4th partial sum $S_4 = 1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16}$

etc.

Series

The sum of the terms of a sequence. For example, the series for the sequence 1, 3, 5, 7, 9, . . . , 131, 133 is the sum 1 + 3 + 5 + 7 + 9 + . . . + 131 + 133.

Series Rules

Algebra rules for convergent series are given below.

If $\sum a_n$ and $\sum b_n$ are convergent series, then

1. $\sum ca_n$ is convergent and $\sum ca_n = c\sum a_n$.
2. $\sum(a_n + b_n)$ is convergent and $\sum(a_n + b_n) = \sum a_n + \sum b_n$.
3. $\sum(a_n - b_n)$ is convergent and $\sum(a_n - b_n) = \sum a_n - \sum b_n$.

Shell Method

See Cylindrical Shell Method

Sigma Notation
Continued Sum

A notation using the Greek letter sigma (Σ) that allows a long sum to be written compactly.

Example: $\sum_{k=3}^{100} k^2 = 3^2 + 4^2 + 5^2 + 6^2 + \cdots + 100^2$

Simple Closed Curve

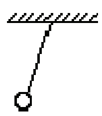
A connected curve that does not cross itself and ends at the same point where it begins. Examples are circles, ellipses, and polygons.

Note: Despite the name "curve", a simple closed curve does not actually have to curve.

Simple Harmonic Motion
SHM

Any kind of periodic motion that can be modeled using a sinusoid. That is, motion that can be approximately or exactly described using a sine or cosine function. Examples include the swinging back and forth of a pendulum and the bobbing up and down of a mass hanging from a spring.

pendulum



Simpson's Rule

A method for approximating a definite integral $\int_a^b f(x)dx$ using parabolic approximations of *f*. The parabolas are drawn as shown below.

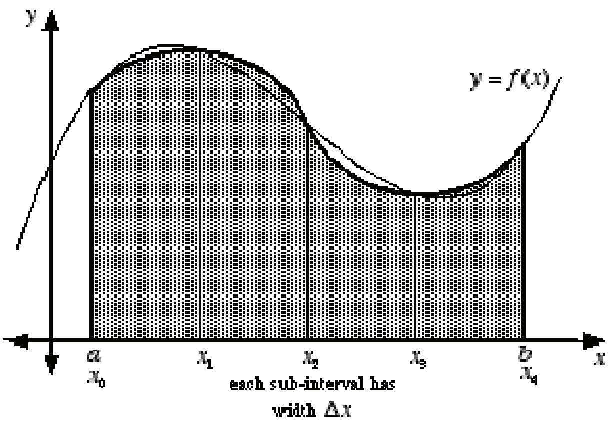
To use Simpson's rule follow these two steps:

1. Partition $[a,b]$ into the set $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ so that there are n sub-intervals of equal width, where n is an even number.

2. The approximate value of the integral $\int_a^b f(x)dx$ is given below. The width Δx of each sub-interval is given by $\Delta x = \dfrac{b-a}{n}$.

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+\cdots+2f(x_{n-2})+4f(x_{n-1})+f(x_n)\right]$$

The formula is obtained by finding the sum of the areas under parabolic approximations of the graph as illustrated. Larger values of n result in better approximations. Note: Simpson's rule generally gives a better approximation than the trapezoid rule.



The shaded area bounded by the parabolas (the thicker curves) is approximately equal to the area bounded by $y = f(x)$.

$$\int_a^b f(x)dx \approx \frac{\Delta x}{3}\left[f(x_0)+4f(x_1)+2f(x_2)+4f(x_3)+f(x_4)\right]$$

Slope of a Curve

A number which is used to indicate the steepness of a curve at a particular point. The slope of a curve at a point is defined to be the slope of the tangent line. Thus the slope of a curve at a point is found using the derivative.

Example: Find the slope of $y = x^2$ at $(3,9)$.

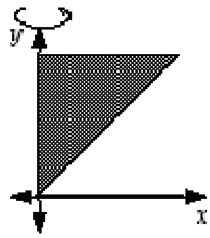
Since $y' = 2x$, the slope of $y = x^2$ at $(3,9)$ is $y'(3) = 2(3) = 6$.

Solid
Geometric Solid
Solid Geometric Figure

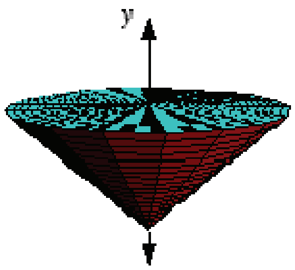
The collective term for all bounded three-dimensional geometric figures. This includes polyhedra, pyramids, prisms, cylinders, cones, spheres, ellipsoids, etc.

Solid of Revolution

A solid that is obtained by rotating a plane figure in space about an axis coplanar to the figure. The axis may not intersect the figure.



The shaded region is revolved about the y-axis.



This cone, a solid of revolution, is the result.

Solve Analytically

Use algebraic and/or numeric methods as the main technique for solving a math problem. Usually when a problem is solved analytically, no graphing calculator is used.

Solve Graphically

Use graphs and/or pictures as the main technique for solving a math problem. When a problem is solved graphically, graphing calculators are commonly used.

Speed

Distance covered per unit of time. Speed is a nonnegative scalar. For motion in one dimension, such as on a number line, speed is the absolute value of velocity. For motion in two or three dimensions, speed is the magnitude of the velocity vector.

Squeeze Theorem
See Sandwich Theorem

Step Discontinuity
See Jump Discontinuity

Substitution Method
See Integration by Substitution

Surface

A geometric figure in three dimensions excluding interior points, if any.

Surface Area of a Surface of Revolution

The formulas below give the surface area of a surface of revolution. The axis of rotation must be either the x-axis or the y-axis. The curve being rotated can be defined using rectangular, polar, or parametric equations.

Rotation about the x-axis: Surface Area $= \int_a^b 2\pi y \, ds$

Rotation about the y-axis: Surface Area $= \int_a^b 2\pi x \, ds$

In either formula, ds can be found by any of the following:

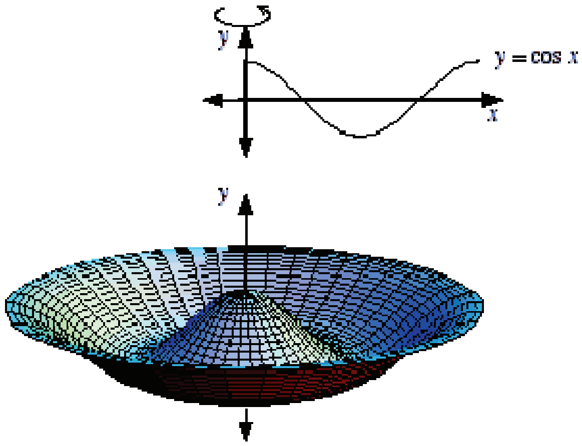
1. In rectangular form, use $ds = \sqrt{1 + \left(\frac{dy}{dx}\right)^2} \, dx$ or $ds = \sqrt{1 + \left(\frac{dx}{dy}\right)^2} \, dy$

2. In parametric form, use $ds = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \, dt$

3. In polar form, use $ds = \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} \, d\theta$

Surface of Revolution

A surface that is obtained by rotating a plane curve in space about an axis coplanar to the curve.

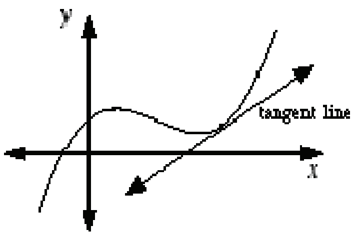


This surface of revolution is obtained by revolving the graph of $y = \cos x$ about the y-axis.

Tangent Line

A line that touches a curve at a point without crossing over. Formally, it is a line which intersects a differentiable curve at a point where the slope of the curve equals the slope of the line.

Note: A line tangent to a circle is perpendicular to the radius to the point of tangency.



Taylor Polynomial
nth Degree Taylor Polynomial

An approximation of a function using terms from the function's Taylor series. An nth degree Taylor polynomial uses all the Taylor series terms up to and including the term using the nth derivative.

nth degree Taylor polynomial

$$\sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Taylor Series

The power series in $x - a$ for a function f. Note: If $a = 0$ the series is called a Maclaurin series.

Taylor series:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$
$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$$

Taylor Series Remainder

A quantity that measures how accurately a Taylor polynomial estimates the sum of a Taylor series.

Taylor series: $f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n + \cdots$

Taylor polynomial: $f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$

Remainder: $R_n(x) = \frac{f^{(n+1)}(a)}{(n+1)!}(x-a)^{n+1} + \frac{f^{(n+2)}(a)}{(n+2)!}(x-a)^{n+2} + \frac{f^{(n+3)}(a)}{(n+3)!}(x-a)^{n+3} + \cdots$

Derivative form of remainder: $R_n(x) = \frac{f^{(n+1)}(z)}{(n+1)!}(x-a)^{n+1}$ where z is a number between a and x.

Integral form of remainder: $R_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)(x-t)^n \, dt$

Pappus's Theorem
Theorem of Pappus

A method for finding the volume of a solid of revolution. The volume equals the product of the area of the region being rotated times the distance traveled by the centroid of the region in one rotation.

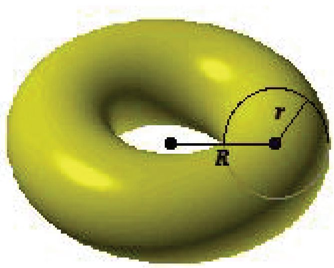
Torus

A doughnut shape. Formally, a torus is a surface of revolution obtained by revolving (in three dimensional space) a circle about a line which does not intersect the circle.

Torus

Volume $= 2\pi^2 Rr^2$

Surface Area $= 4\pi^2 Rr$



Trapezoid Rule

A method for approximating a definite integral $\int_a^b f(x)\,dx$ using linear approximations of *f*. The trapezoids are drawn as shown below. The bases are vertical lines.

To use the trapezoid rule follow these two steps:

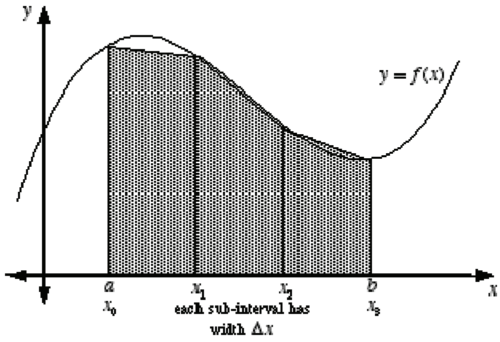
1. Partition $[a, b]$ into the set $\{x_0, x_1, x_2, x_3, \ldots, x_n\}$ so that there are n sub-intervals of equal width.

2. The integral $\int_a^b f(x)\,dx$ is estimated by adding the areas of all the trapezoids as illustrated below.

The width Δx of each sub-interval is given by $\Delta x = \dfrac{b-a}{n}$.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + 2f(x_3) + \cdots + 2f(x_{n-1}) + f(x_n)\right]$$

Note: The larger the value of n, the better the approximation. Also, Simpson's rule generally gives a better approximation than the trapezoid rule.



The area of the trapezoids (shaded) approximately equals the area bounded by $y = f(x)$.

$$\int_a^b f(x)\,dx \approx \frac{\Delta x}{2}\left[f(x_0) + 2f(x_1) + 2f(x_2) + f(x_3)\right].$$

Trig Substitution

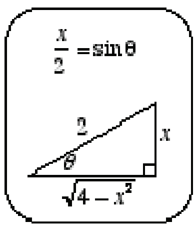
A method for computing integrals often used when the integrand contains expressions of the form $a^2 - x^2$, $a^2 + x^2$, or $x^2 - a^2$.

Expression	Substitution
$a^2 - x^2$	$x = a\sin\theta$
$a^2 + x^2$	$x = a\tan\theta$
$x^2 - a^2$	$x = a\sec\theta$

Example: $\displaystyle\int \frac{x^2}{\sqrt{4-x^2}}\,dx$ $\qquad x = 2\sin\theta,\quad \sqrt{4-x^2} = \sqrt{4-4\sin^2\theta} = 2\cos\theta$

$dx = 2\cos\theta\,d\theta$

$$\begin{aligned}
\int \frac{x^2}{\sqrt{4-x^2}}\,dx &= \int \frac{4\sin^2\theta}{2\cos\theta}\,2\cos\theta\,d\theta \\
&= 4\int \sin^2\theta\,d\theta \\
&= 4\int \frac{1-\cos 2\theta}{2}\,d\theta \\
&= 2\left(\theta - \frac{1}{2}\sin 2\theta\right) + C \\
&= 2\theta - 2\sin\theta\cos\theta + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) - 2\left(\frac{x}{2}\right)\left(\frac{\sqrt{4-x^2}}{2}\right) + C \\
&= 2\sin^{-1}\left(\frac{x}{2}\right) - \frac{1}{2}x\sqrt{4-x^2} + C
\end{aligned}$$



u-Substitution

See Integration by Substitution

Uniform

All the same or all in the same manner; constant.

Vector Calculus

See Multivariable Calculus

Velocity

The rate of change of the position of an object. For motion in one dimension, such as along the number line, velocity is a scalar. For motion in two dimensions or through three-dimensional space, velocity is a vector.

Volume

The total amount of space enclosed in a solid.

For the following tables,
h = height of solid s = slant height P = perimeter or circumference of the base
l = length of solid B = area of the base r = radius of sphere
w = width of solid R = radius of the base a = length of an edge

Figure	Volume	Lateral Surface Area	Area of the Base(s)	Total Surface Area
Box (also called rectangular parallelepiped, right rectangular prism)	lwh	2lh + 2wh	2lw	2lw + 2lh + 2wh
Prism	Bh	Ph	2B	Ph + 2B
Pyramid	$\frac{1}{3}Bh$	–	B	–
Right Pyramid	$\frac{1}{3}Bh$	$\frac{1}{2}Ps$	B	$B+\frac{1}{2}Ps$
Cylinder	Bh	–	2B	–
Right Cylinder	Bh	Ph	2B	Ph + 2B
Right Circular Cylinder	$\pi R^2 h$	$2\pi Rh$	$2\pi R^2$	$2\pi Rh + 2\pi R^2$
Cone	$\frac{1}{3}Bh$	–	B	–
Right Circular Cone	$\frac{1}{3}\pi R^2 h$	πRs or $\pi R\sqrt{R^2+h^2}$	πR^2	$\pi Rs + \pi R^2$ or $\pi R\sqrt{R^2+h^2}+\pi R^2$

Figure	Volume	Total Surface Area
Sphere	$\frac{4}{3}\pi r^3$	$4\pi r^2$
Regular Tetrahedron	$\frac{\sqrt{2}}{12}a^3$	$a^2\sqrt{3}$
Cube (regular hexahedron)	a^3	$6a^2$
Regular Octahedron	$\frac{\sqrt{2}}{3}a^3$	$2a^2\sqrt{3}$
Regular Dodecahedron	$\frac{15+7\sqrt{5}}{4}a^3$	$3a^2\sqrt{25+10\sqrt{5}}$
Regular Icosahedron	$\frac{5(3+\sqrt{5})}{12}a^3$	$5a^2\sqrt{3}$

Volume by Parallel Cross Sections

See **Parallel Cross Sections**

Washer

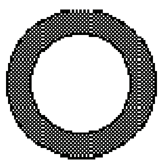
Annulus

The region between two concentric circles which have different radii.

Annulus

R = radius of larger circle
r = radius of smaller circle

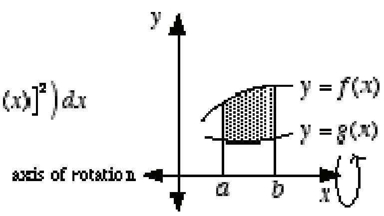
$\text{Area} = \pi(R^2 - r^2)$



Washer Method

A technique for finding the volume of a solid of revolution. The washer method is a generalized version of the disk method. Both the washer and disk methods are specific cases of volume by parallel cross-sections.

$$\text{Volume} = \int_a^b \pi\left([f(x)]^2 - [g(x)]^2\right)dx$$



Work

The physics term for the amount of energy required to move an object over a given path subject to a given force.

Work $= \int_a^b f(x)\,dx$, where $f(x) =$ the force acting in the direction of or opposite to movement.

or $\int_a^b \mathbf{f}(\mathbf{x}) \cdot \mathbf{dx}$, where $\mathbf{f}(\mathbf{x}) =$ the vector function for force acting at the vector-valued point $\mathbf{x}$.