COMBINATORIAL OBJECTS FOR q-EXPONENTIAL FUNCTIONS

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ABSTRACT. There are e_q and E_q as q-analogues of exponential function. We introduce a third q-exponential function $\mathcal{E}_q$ which is closely related to the Askey-Wilson operators. We survey some properties of $\mathcal{E}_q$. There is the well-known identity related to the $\mathcal{E}_q$ function and the q-Hermite polynomials $H_n(x|q)$, i.e.

$$(qt^2;q^2)_\infty \mathcal{E}_q(x;t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}t^n}{(q;q)_n} H_n(x|q).$$

We define the combinatorial objects and the map on them which prove the previous identity combinatorially.

1. PRELIMINARIES

For complex number a and q, $|q|<1$, let $(a;q)_n = \prod_{k=0}^{n-1}(1-aq^k)$ for the q-shifted factorials. We use the standard notations for the q-binomial coefficients

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad n \in \mathbb{Z}.$$

and for the q-basic hypergeometric series

$${}_r\Phi_s \left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix} ; q, z \right] = \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}} \right]^{1+s-r} z^n$$

where $(a_1, a_2, \ldots, a_k; q)_n = (a_1;q)_n (a_2;q)_n \cdots (a_k;q)_n$.

2. q-EXPONENTIAL FUNCTIONS

There are two q-analogues e_q and E_q of exponential function as follows:

$$\begin{aligned} e_q(x) &= \sum_{n\geq 0} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty} \\ E_q(x) &= \sum_{n \geq 0} q^{\binom{n}{2}} \frac{x^n}{(q;q)_n} = (-x;q)_\infty. \end{aligned}$$

By calculating easily, $\lim_{q\to 1^-} e_q(x(1-q)) = \lim_{q \to 1^-} E_q(x(1-q)) = \exp(x)$ and $e_q(x) \cdot E_q(-x) = 1$. But two addition theorems for e_q and E_q is false: $e_q(A+B) \neq e_q(A)e_q(B)$ and $E_q(A+B) \neq E_q(A)E_q(B)$.

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Definition 2.1. $\mathcal{E}_q$ is called a *q-exponential function* and sometimes a *curly* for short and defined by

$$\mathcal{E}_q(x,y;\alpha) = \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty}\sum_{n=0}^{\infty}\frac{q^{n^2/4}\alpha^n}{(q;q)_\infty}e^{-\,\mathrm{i}\,n\phi}(-q^{(1-n)/2}e^{\mathrm{i}(\phi+\theta)},-q^{(1-n)/2}e^{-\,\mathrm{i}(\phi-\theta)};q)_n$$

where $x=\cos\theta$, $y=\cos\phi$. Denote $\mathcal{E}_q(x;\alpha)=\mathcal{E}_q(x,0;\alpha)$ (See [2, 5, 10])

Proposition 2.2. *The $\mathcal{E}_q$ function have the addition theorem*

$$\mathcal{E}_q(x,y;\alpha)=\mathcal{E}_q(x;\alpha)\mathcal{E}_q(y;\alpha).$$

This is a q-analogue of $\exp(\alpha(x+y))=\exp(\alpha x)\exp(\alpha y)$. *Thus, it is symmetric as* $\mathcal{E}_q(x,y;\alpha)=\mathcal{E}_q(y,x;\alpha)$

Different proofs of this relation were given in [4, 5]. This q-exponential function satisfies the followings:

(1) $\lim_{q\to1^-}\mathcal{E}_q(x;\frac{1-q}{2}\alpha)=\exp(\alpha x)$
(2) $\mathcal{E}_q(x;\alpha)$ is a q-analogue of $\exp(\alpha x)$ but is not symmetric in x and α.
(3) $\lim_{q\to1^-}\mathcal{E}_q(x,y;\frac{1-q}{2}\alpha)=\exp(\alpha(x+y))$
(4) $\mathcal{E}_q(x,y;\alpha)$ is a q-analogue of $\exp(\alpha(x+y))$, symmetric in x and y.

Theorem 2.3 (Ismail & Zhang, 1994)**.** *The generating function for the continuous q-Hermite polynomials $H_n(x|q)$ is*

$$(qt^2;q^2)_\infty\mathcal{E}_q(x;t)=\sum_{n=0}^{\infty}\frac{q^{n^2/4}t^n}{(q;q)_n}H_n(x|q)$$

which is obtained in [2] *and discussed in* [3, 4, 5]

3. q-Hermite Polynomials

Definition 3.1. The *q-Hermite polynomials* $H_n(x|q)$ is defined by

$$H(x,r)=\sum_{n=0}^{\infty}\frac{H_n(x|q)}{(q;q)_n}r^n=\prod_{k=0}^{\infty}(1-2xrq^k+r^2q^{2k})^{-1}$$

As usual, $H_n(x|q)=H_n(x)=H_n$ for short.

$H(\cos\theta,r)=\dfrac{1}{(re^{\mathrm{i}\,\theta})_\infty(re^{-\,\mathrm{i}\,\theta})_\infty}$ and $H_n(\cos\theta|q)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q e^{\mathrm{i}(2k-n)\theta}$ are found in [1, 3, 8].

Now, we shall construct the objects for q-Hermite polynomials using the recurrence relation. In other words, we shall make the set $\mathcal{H}_n$ such that $H_n(x|q)=\sum_{G\in\mathcal{H}_n}w(G)$.

Proposition 3.2. *The q-Hermite Polynomials $H_n(x)$ satisfies the following recurrence relation:*

$$\begin{cases}H_n(x)=2xH_{n-1}(x)+(q^{n-1}-1)H_{n-2}(x) & \textit{for } n\geq 2\\ H_0(x)=1,\quad H_1(x)=2x\end{cases}\tag{3.1}$$

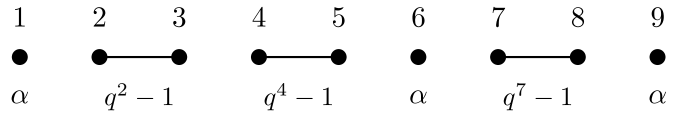


FIGURE 1. Example of $\mathcal{H}_9$

We define the set $\mathcal{H}_n$ of graphs with vertex set $\{1, 2, \dots, n\}$ for q-Hermite polynomials. By the recurrence (3.1) of $H_n(x)$, $\mathcal{H}_n$ is the disjoint union of two sets; the one is the pairs of $\mathcal{H}_{n-1}$ and one vertex n with the weight $\alpha(= 2x)$ and $\mathcal{H}_{n-2}$ and the other one is the pairs of $\mathcal{H}_{n-2}$ and one edge $(n-1, n)$ with the weight $q^{n-1} - 1$. Hence vertices of these graphs are zero or one and edges are adjacent. The weights of the vertex is α with degree zero and 1 withe degree one. The weights of the edges $(i, i+1)$ is $q^i - 1$. In the sequel, as figure 1, the set $\mathcal{H}_n$ of graphs and the weight function w is defined by

$$\begin{aligned}\mathcal{H}_n &= \{G = (V, E) : V = [n], E \subset E_n, \deg(v) \le 1, \forall v \in V\}\\ w(G) &= \prod_{v\in V(G)} \alpha^{1-\deg(v)} \prod_{e\in E(G)} w(e)\end{aligned}$$

where $E_n = \{(i, i+1) : i = 1, \dots, n-1\}$ and $w(e) = q^i - 1$ for $e = (i, i+1)$. Thus, $H_n(x|q) = \sum_{G\in\mathcal{H}_n} w(G)$. As usual, $\mathcal{H} = \bigcup_{n\ge 0} \mathcal{H}_n$.

4. Combinatorial Obejects

Recall the theorem of Ismail and Zhang.

$$(qt^2; q^2)_\infty \mathcal{E}_q(x; t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4} t^n}{(q; q)_n} H_n(x|q)$$

Dividing both sides by $(t^2; q^2)_\infty$ and comparing only the coefficients of t^{2m} and t^{2m+1} with some calculation using the definition of $\mathcal{E}_q$, we obtain two identities

$$\prod_{k=0}^{m-1} ((q^{2k+1} - 1)^2 + \alpha^2 q^{2k+1}) = \sum_{i=0}^{m} \frac{(q; q^2)_m}{(q; q^2)_i} H_{2i}(x|q) q^{i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \tag{4.1}$$

$$\alpha \prod_{k=0}^{m-1} ((q^{2k+2} - 1)^2 + \alpha^2 q^{2k+2}) = \sum_{i=0}^{m} \frac{(q; q^2)_{m+1}}{(q; q^2)_{i+1}} H_{2i+1}(x|q) q^{i^2+i} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \tag{4.2}$$

where $\alpha = 2x$. In this paper, we aim to prove above identities. First of all, let us find a combinatorial proof of the identity (4.1).

Let F_n(or G_n) be the right(or left)-hand side of (4.1) respectively, i.e.

$$\begin{aligned}F_n &= \sum_{i=0}^{m} \frac{(q; q^2)_m}{(q; q^2)_i} H_{2i}(x|q) q^{i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2}\\ G_n &= \prod_{k=0}^{m-1} ((q^{2k+1} - 1)^2 + \alpha^2 q^{2k+1})\end{aligned}$$

We shall prove $F_n = G_n$ combinatorially. Since G_n is a product, it has a simple recurrence

$$G_{n+1} = G_n \times ((q^{2n+1} - 1)^2 + \alpha^2 q^{2n+1}).$$

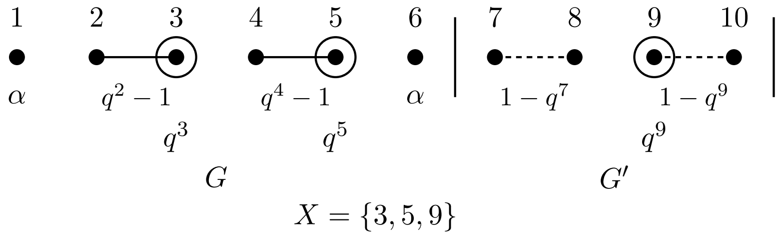


FIGURE 2. Example of $\mathcal{F}_{5,3}$.

If we prove that for all $n \geq 0$,

$$\begin{aligned} F_{n+1} &= F_n \times ((q^{2n+1}-1)^2 + \alpha^2 q^{2n+1}) \\ &= F_n \times ((1-q^{2n+1}) + q^{2n+1}(q^{2n+1}-1) + \alpha^2 q^{2n+1}) \end{aligned}$$

then $F_n = G_n$ for all n, since $F_0 = G_0 = 1$.

We define the set $\mathcal{F}_n$ of graphs with n vertices for F_n. Since F_n is the sum of terms which is product of three pieces, $\mathcal{F}_n$ is the disjoint union of sets $\mathcal{F}_{n,i}$ whose elements are triple (G, G', X). G is the combintorial objects for $H_{2i}(x|q)$, G' for $\dfrac{(q;q^2)_m}{(q;q^2)_i}$ and X for $\begin{bmatrix} m \\ i \end{bmatrix}_{q^2}$. In the sequel, as figure 2, the set $\mathcal{F}_{n,k}$ of graphs is defined by

$$\mathcal{F}_{n,i} = \{F = (G, G', X) : G \in \mathcal{H}_{2i}, G' = (V_{n,i}, E_{n,i}), X \subset O_n, |X| = i\}$$

where

$$\begin{aligned} O_n &= \{1, 3, \ldots, 2n-1\}, \\ V_{n,i} &= \{2i+1, 2i+2, \ldots, 2n\} \text{ and} \\ E_{n,i} &= \{(2i+1, 2i+2), (2i+3, 2i+4), \ldots, (2n-1, 2n)\}. \end{aligned}$$

and a weight function w by

$$\begin{aligned} w(F) &= w(G)w(G')w(X) \\ &= \prod_{v \in V(G)} \alpha^{1-\deg(v)} \prod_{e \in E(G)} w(e) \prod_{e' \in E(G')} w(e') \prod_{i \in X} q^i \end{aligned}$$

where

$$\begin{aligned} w(e) &= q^i - 1 \text{ for } e = (i, i+1) \in E(G) \text{ and} \\ w(e') &= 1 - q^i \text{ for } e' = (i, i+1) \in E(G'). \end{aligned}$$

As usual, $\mathcal{F}_n = \bigcup_{0 \leq i \leq n} \mathcal{F}_{n,i}$.

5. MAIN PROOF

We shall construct a weight-preserving bijection

$$\Psi : \mathcal{F}_{n+1} \to \mathcal{F}_n \times \{a, b, c\}$$

such that

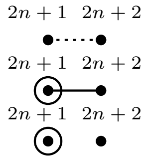
$$w(F) = w(\Psi(F)), \quad \forall F \in \mathcal{F}_{n+1}$$

where

$a =$, $w(a) = 1 - q^{2n+1}$

$b =$, $w(b) = q^{2n+1}(q^{2n+1} - 1)$

$c =$, $w(c) = q^{2n+1}\alpha^2.$

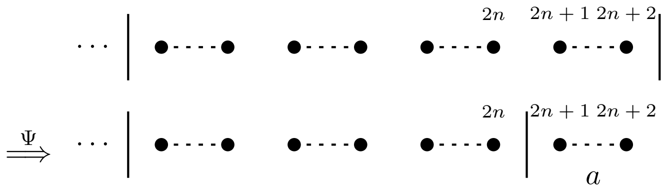


If we make such Ψ map, then we get what we want as follows:

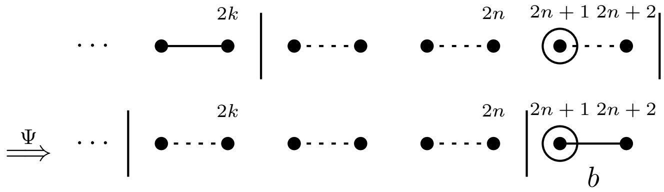
$$\begin{aligned} w(\mathcal{F}_{n+1}) &= w(\mathcal{F}_n) w(\{a, b, c\}) \\ F_{n+1} &= F_n \times ((1 - q^{2n+1}) + q^{2n+1}(q^{2n+1} - 1) + \alpha^2 q^{2n+1}) \\ &= F_n \times ((q^{2n+1} - 1)^2 + \alpha^2 q^{2n+1}) \end{aligned}$$

and $F_0 = G_0 = 1$; thus $F_n = G_n$ for all $n \geq 0$. Now, we make the Ψ map.

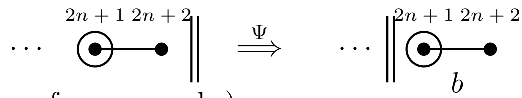
- Case 1: $2n + 1 \notin X$



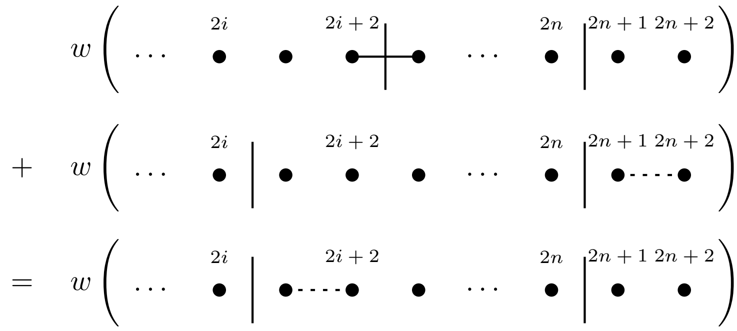
- Case 2: $2n + 1 \in X$ and $\deg(2k) = 1$ where $2k = V(G)$



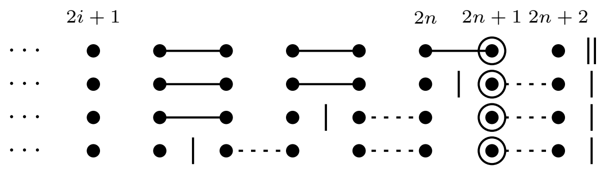
Especially,

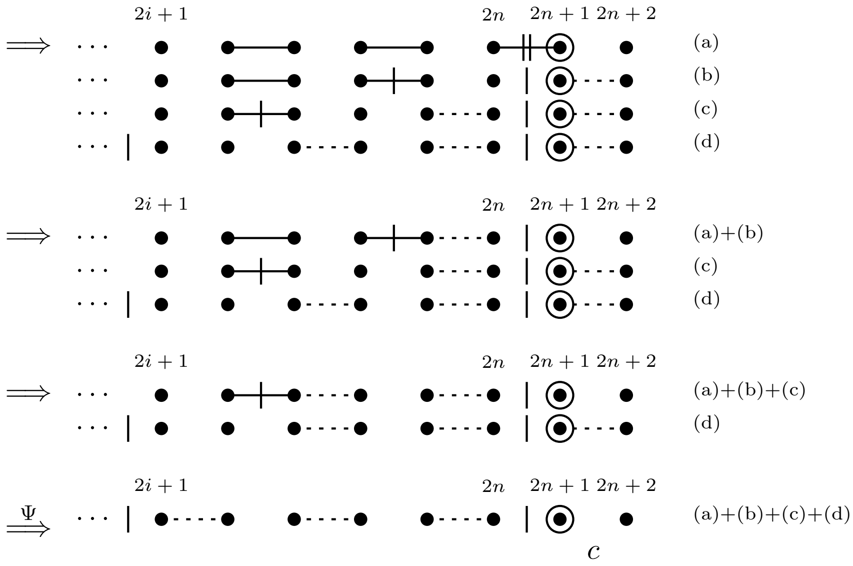


Lemma 5.1 (Weight-sum of some graphs)**.**



- Case 3: $2n + 1 \in X$ and $\deg(2k) = 0$ where $2k = V(G)$ by using Lemma 5.1.





We construct the map Ψ, but it is not a bijection. However, it is a weight-preserving surjective map, in the other sense,

$$\sum_{\Psi(F')=(F,\delta)} w(F') = w(F,\delta)$$

for all $(F,\delta) \in \mathcal{F}_n \times \{a,b,c\}$. Therefore,

$$w(\Psi^{-1}(F,\delta)) = w(F,\delta).$$

So there is no problem using the map ψ. Finally, we get $F_n = G_n$ for all $n \geq 0$ combinatorially.

Theorem 5.2. *The following identity (4.1) has a combinatorial interpretation.*

$$\prod_{k=0}^{m-1}((q^{2k+1}-1)^2 + \alpha^2 q^{2k+1}) = \sum_{i=0}^{m} \frac{(q;q^2)_m}{(q;q^2)_i} H_{2i}(x|q) q^{i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2}$$

where $\alpha = 2x$.

6. Further Study

The other equation (4.2) could be proved similarly combinatorially with $\alpha = 2x$.

$$\alpha \prod_{k=0}^{m-1}((q^{2k+2}-1)^2 + \alpha^2 q^{2k+2}) = \sum_{i=0}^{m} \frac{(q;q^2)_{m+1}}{(q;q^2)_{i+1}} H_{2i+1}(x|q) q^{i^2+i} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2}.$$

If so, how to prove the identity of Ismail and Zhang combinatorially?

$$(qt^2;q^2)_\infty \mathcal{E}_q(x;t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4} t^n}{(q;q)_n} H_n(x|q)$$

Can we find more combinatorial properties of $\mathcal{E}_q(x;t)$ and then interpret them combinatorially?

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