Combinatorial Objects for q-exponential Functions

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Outline

1. Preliminaries
   - Notation
2. Motivation
   - q-Exponential Functions
   - q-Hermite Polynomials
3. Main Result
   - Interpretation
   - Main Proof

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q-analogue of n!

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\begin{aligned}
n &= 1 + 1 + 1 + \cdots + 1 \\
[n]_q &= 1 + q + q^2 + \cdots + q^{n-1}
\end{aligned}
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n! &= 1 \cdot 2 \cdots n \\
[n]_q! &= [1]_q [2]_q \cdots [n]_q
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q-Shifted Factorial

$$(a;q)_n = (a)_n = (1-a)(1-aq)\cdots(1-aq^{n-1}) = \prod_{k=0}^{n-1}(1-aq^k)$$

$$(a_1, a_2, \ldots, a_r; q)_n = (a_1;q)_n(a_2;q)_n\cdots(a_r;q)_n$$

$$(q;q)_n = (q)_n = [n]_q!(1-q)^n$$

$$(a;q)_\infty = (a)_\infty = (1-a)(1-aq)\cdots(1-aq^k)\cdots = \prod_{k=0}^{\infty}(1-aq^k)$$

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q-Binomial Coefficients

$$\binom{n}{k} = \frac{n(n-1)\cdots(n-k+1)}{k!} = \frac{n!}{k!(n-k)!}, \quad n \in \mathbb{Z}$$

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}, \quad n \in \mathbb{Z}$$

$$= \frac{(q)_n}{(q)_k (q)_{n-k}}, \quad n \in \mathbb{Z}$$

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q-Basic Hypergeometric Series

$$
{}_r\Phi_s\left[\begin{matrix} a_1, a_2, \ldots, a_r \\ b_1, b_2, \ldots, b_s \end{matrix}; q, z\right]
= \sum_{n=0}^{\infty} \frac{(a_1, a_2, \ldots, a_r; q)_n}{(q, b_1, b_2, \ldots, b_s; q)_n} \left[(-1)^n q^{\binom{n}{2}}\right]^{1+s-r} z^n
$$

Outline

Exponential Function

$$\exp(x) = \sum_{n\geqslant 0} \frac{x^n}{n!}$$

$$e_q(x) = \sum_{n\geqslant 0} \frac{x^n}{(q;q)_n} = \frac{1}{(x;q)_\infty}$$

$$E_q(x) = \sum_{n\geqslant 0} q^{\binom{n}{2}} \frac{x^n}{(q;q)_n} = (-x;q)_\infty.$$

$$\lim_{q\to 1^-} e_q(x(1-q)) = \lim_{q\to 1^-} E_q(x(1-q)) = \exp(x)$$

$$e_q(x)\cdot E_q(-x) = 1$$

But addition theorem for e_q and E_q is false.

$$\begin{aligned} \exp(A+B) &= \exp(A)\exp(B) \\ e_q(A+B) &\neq e_q(A)e_q(B) \\ E_q(A+B) &\neq E_q(A)E_q(B) \end{aligned}$$

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Exponential Function

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q-analogue of $\exp(\alpha(x+y))$

Definition (q-exponential function)

1. $\mathcal{E}_q$ is called a *q-exponential function* and sometimes a *curly* for short.

$$\mathcal{E}_q(x,y;\alpha) \equiv \frac{(\alpha^2;q^2)_\infty}{(q\alpha^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}\alpha^n}{(q;q)_\infty} \times e^{-\mathrm{i}n\phi}(-q^{(1-n)/2}e^{\mathrm{i}(\phi+\theta)}, -q^{(1-n)/2}e^{-\mathrm{i}(\phi-\theta)};q)_n$$

where $x = \cos\theta$, $y = \cos\phi$.

2. $\mathcal{E}_q(x;\alpha) \equiv \mathcal{E}_q(x,0;\alpha)$

Interpretation of q-Exponential Function

Note.

- $\lim_{q\to 1^-} \mathcal{E}_q(x; \frac{1-q}{2}\alpha) = \exp(\alpha x)$
- $\mathcal{E}_q(x;\alpha)$ is a q-analogue of $\exp(\alpha x)$ but is not symmetric in x and α.
- $\lim_{q\to 1^-} \mathcal{E}_q(x, y; \frac{1-q}{2}\alpha) = \exp(\alpha(x+y))$
- $\mathcal{E}_q(x, y;\alpha)$ is a q-analogue of $\exp(\alpha(x+y))$, symmetric in x and y.

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Interpretation of q-Exponential Function

Note.

- $\lim_{q \to 1^-} \mathcal{E}_q(x; \frac{1-q}{2}\alpha) = \exp(\alpha x)$
- $\mathcal{E}_q(x; \alpha)$ is a q-analogue of $\exp(\alpha x)$ but is not symmetric in x and α.
- $\lim_{q \to 1^-} \mathcal{E}_q(x, y; \frac{1-q}{2}\alpha) = \exp(\alpha(x + y))$
- $\mathcal{E}_q(x, y; \alpha)$ is a q-analogue of $\exp(\alpha(x + y))$, symmetric in x and y.

q-analogue of the Addition Theorem

Proposition (Addition Theorem for $\mathcal{E}_q$)

1. $\mathcal{E}_q(x, y; \alpha) = \mathcal{E}_q(y, x; \alpha)$
2. $\mathcal{E}_q(x, y; \alpha) = \mathcal{E}_q(x; \alpha)\mathcal{E}_q(y; \alpha)$

Note. This is a q-analogue of the addition theorem

$$\exp(\alpha(x+y)) = \exp(\alpha x)\exp(\alpha y).$$

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q-analogue of $\exp(\alpha x + \beta y)$

Definition (q-exponential function)

$$\mathcal{E}_q(x, y; \alpha, \beta) \equiv \frac{(\beta^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}\beta^n}{(q;q)_\infty}$$
$$\times\, e^{-in\phi}\left(-q^{(1-n)/2}e^{i(\phi+\theta)}\alpha/\beta, -q^{(1-n)/2}e^{-i(\phi-\theta)}\alpha/\beta; q\right)_n$$
$$\times\, {}_2\Phi_2\left[\begin{matrix} q^{-n}, \alpha^2/\beta^2 \\ -q^{(1-n)/2}e^{i(\phi+\theta)}\alpha/\beta, -q^{(1-n)/2}e^{-i(\phi-\theta)}\alpha/\beta \end{matrix}; q, qe^{2i\phi}\right]$$

where $x = \cos\theta$, $y = \cos\phi$.

Note. $\lim_{q\to 1^-} \mathcal{E}_q(x, y; \frac{1-q}{2}\alpha, \frac{1-q}{2}\beta) = \exp(\alpha x + \beta y)$

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q-analogue of the Addition Theorem

Proposition (Addition Theorem for $\mathcal{E}_q$)

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Properties of q-Exponential Function

$\mathcal{E}_q(x, y; \alpha, \beta)$ satisfies the following simple properties:

1. $\mathcal{E}_q(x, y; \alpha, \alpha) = \mathcal{E}_q(x, y; \alpha)$
2. $\mathcal{E}_q(x, y; \alpha, \beta) = \mathcal{E}_q(y, x; \beta, \alpha)$
3. $\mathcal{E}_q(x, 0; \alpha, \beta) = \mathcal{E}_q(x; \alpha)$
4. $\mathcal{E}_q(0, y; \alpha, \beta) = \mathcal{E}_q(y; \beta)$
5. $\mathcal{E}_q(x, y; \alpha, \beta)\mathcal{E}_q(z, w; \gamma, \delta) = \mathcal{E}_q(x, z; \alpha, \gamma)\mathcal{E}_q(y, w; \beta, \delta)$
   is a q-analogue of
   $\exp(\alpha x + \beta y)\exp(\gamma z + \delta w) = \exp(\alpha x + \gamma z)\exp(\beta y + \delta w)$

Questions

1. How to interpret and understand $\mathcal{E}_q$ combinatorially?

$$\mathcal{E}_q(x, y; \alpha) = \frac{(\alpha^2; q^2)_\infty}{(q\alpha^2; q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4}\alpha^n}{(q; q)_\infty} \times e^{-\mathrm{i}n\phi}(-q^{(1-n)/2}e^{\mathrm{i}(\phi+\theta)}, -q^{(1-n)/2}e^{-\mathrm{i}(\phi-\theta)}; q)_n$$

where $x = \cos\theta$, $y = \cos\phi$.

Questions

Theorem (Ismail & Zhang, 1994)

$$(qt^2;q^2)_\infty \mathcal{E}_q(x;t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4}t^n}{(q;q)_n} H_n(x|q)$$

2. How to prove the previous identity combinatorially?
3. What is the $H_n(x|q)$?

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   - q-Exponential Functions
   - q-Hermite Polynomials
3. Main Result
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   - Main Proof

Hermite Polynomials

$$\sum_{n=0}^{\infty} H_n(x)\frac{t^n}{n!} = \exp(xt - \frac{1}{2}t^2)$$

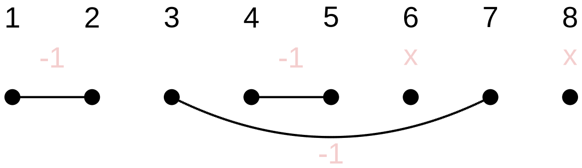
$$\Rightarrow H_n(x) = \sum_{0\leqslant 2k\leqslant n} \frac{n!(-1)^k}{2^k k!(n-2k)!}x^{n-2k}$$

$$\Rightarrow H_n(x) = \sum_{\sigma\in Inv_n} x^{\mathrm{fix}(\sigma)}(-1)^{\mathrm{cyc}_2(\sigma)}$$

where Inv_n is the set of involutions on $[n] = \{1, 2, \ldots, n\}$.

Example of Objects for Hermite Polynomials

Example. $n = 8$



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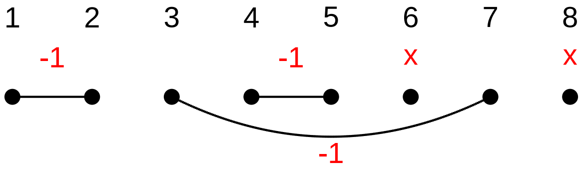

q-Hermite Polynomials

Definition (q-Hermite Polynomials)

$$H(x,r) = \sum_{n=0}^{\infty} \frac{H_n(x|q)}{(q;q)_n} r^n = \prod_{k=0}^{\infty} (1 - 2xrq^k + r^2 q^{2k})^{-1}$$

$H_n(x|q)$ is called a *(continuous) q-Hermite Polynomials*. As usual, $H_n(x|q) = H_n(x) = H_n$.

Note.

1. $H(\cos\theta, r) = \dfrac{1}{(re^{\mathrm{i}\theta})_\infty (re^{-\mathrm{i}\theta})_\infty}$
2. $H_n(\cos\theta|q) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q e^{\mathrm{i}(2k-n)\theta}$

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Recurrence Relation

Question. How to construct the objects for q-hermite polynomials? In other words, is there $\mathcal{H}_n$ such that

$$H_n(x|q) = \sum_{G \in \mathcal{H}_n} w(G).$$

Proposition

The q-Hermite Polynomials $H_n(x)$ satisfies the following:

$$\begin{cases} H_n(x) = 2xH_{n-1}(x) + (q^{n-1} - 1)H_{n-2}(x) & \text{for } n \geqslant 2 \\ H_0(x) = 1, \quad H_1(x) = 2x \end{cases}$$

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Proof of Proposition

By the definition,

$$\begin{aligned}
H(x,r) &= \prod_{k=0}^{\infty} \frac{1}{1-2xrq^k+r^2q^{2k}} \\
&= \frac{1}{1-2xr+r^2} \prod_{k=1}^{\infty} \frac{1}{1-2xrq^k+r^2q^{2k}} \\
&= \frac{1}{1-2xr+r^2} H(x,rq) \\
H(x,rq) &= (1-2xr+r^2)H(x,r)
\end{aligned}$$

By the comparison of coefficients of $r^n/(q)_{n-1}$,

$$\begin{aligned}
\frac{q^n H_n(x|q)}{1-q^n} &= \frac{H_n(x|q)}{1-q^n} - 2xH_{n-1}(x|q) + (1-q^{n-1})H_{n-2}(x) \\
H_n(x|q) &= 2xH_{n-1}(x|q) + (q^{n-1}-1)H_{n-2}(x) \quad \text{for } n \geqslant 2.
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Examples

Let $\alpha = 2x$ in the following.

$$\begin{aligned}
H_0 &= 1 \\
H_1 &= \alpha \\
H_2 &= \alpha + q - 1 \\
H_3 &= \alpha(\alpha + q - 1) + (q^2 - 1)\alpha \\
&= (\alpha^2 - \alpha) + q\alpha + q^2\alpha \\
H_4 &= \alpha(\alpha^2 - \alpha + q\alpha + q^2\alpha) + (q^3 - 1)(\alpha + q - 1) \\
&= (\alpha^3 - 2\alpha^2 - \alpha + 1) + q(\alpha^2 - 1) + q^2\alpha^2 + q^3(\alpha - 1) + q^4
\end{aligned}$$

and so on.

Combinatorial Objects for q-Hermite Polynomials I

- Recall the recurrence of $H_n(x)$.

$$\begin{cases} H_n(x) = 2xH_{n-1}(x) + (q^{n-1} - 1)H_{n-2}(x) & \text{for } n \geqslant 2 \\ H_0(x) = 1, \quad H_1(x) = 2x \end{cases}$$

- Define the set of graphs.

$$\mathcal{H} = \bigcup_{n \geqslant 0} \mathcal{H}_n$$

$$\mathcal{H}_n = \{G = (V, E) : V = [n], E \subset E_n, \deg(v) \leqslant 1, \forall v \in V\}$$

where $E_n = \{(i, i+1) : i = 1, \ldots, n-1\}$.

Combinatorial Objects for q-Hermite Polynomials II

- We assign a weight to the graph.

$$w(G) = w(V)w(E) = \prod_{v \in V(G)} w(v) \prod_{e \in E(G)} w(e)$$

where $w(e) = q^i - 1$ for $e = (i, i+1)$ and

$$w(v) = \begin{cases} \alpha(= 2x) & \text{if } \deg(v) = 0, \\ 0 & \text{if } \deg(v) = 1. \end{cases}$$

- Thus, $H_n(x|q) = \sum_{G \in \mathcal{H}_n} w(G)$.

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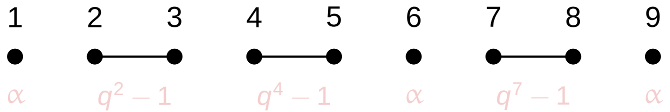
Exmaple

Example. $n = 9$



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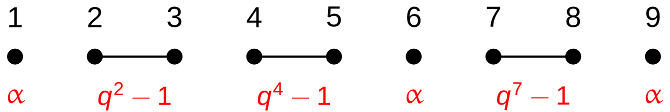


Outline

Calculation I

Recall the theorem of Ismail and Zhang.

$$(qt^2;q^2)_\infty \mathcal{E}_q(x;t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4} t^n}{(q;q)_n} H_n(x|q)$$

Dividing both sides by $(t^2;q^2)_\infty$ and then separating even and odd powers, we obtain

$$\begin{cases} \sum_{m=0}^{\infty} \frac{t^{2m}}{(q;q)_{2m}} \prod_{k=0}^{m-1} ((q^{2k+1}-1)^2 + 4x^2 q^{2k+1}) = \frac{1}{(t^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2} t^{2m}}{(q;q)_{2m}} H_{2m}(x|q) \\ 2x \sum_{m=0}^{\infty} \frac{t^{2m+1}}{(q;q)_{2m+1}} \prod_{k=0}^{m-1} ((q^{2k+2}-1)^2 + 4x^2 q^{2k+2}) = \frac{1}{(t^2;q^2)_\infty} \sum_{m=0}^{\infty} \frac{q^{m^2+m} t^{2m+1}}{(q;q)_{2m+1}} H_{2m+1}(x|q) \end{cases}$$

We think about only the coefficients of t^{2m} and t^{2m+1}.

$$\begin{cases} \prod_{k=0}^{m-1} ((q^{2k+1}-1)^2 + 4x^2 q^{2k+1}) = \sum_{i=0}^{m} \frac{(q;q^2)_m}{(q;q^2)_i} H_{2i}(x|q) q^{i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \\ 2x \prod_{k=0}^{m-1} ((q^{2k+2}-1)^2 + 4x^2 q^{2k+2}) = \sum_{i=0}^{m} \frac{(q;q^2)_{m+1}}{(q;q^2)_{i+1}} H_{2i+1}(x|q) q^{i^2+i} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \end{cases}$$

Calculation II

Let $\alpha = 2x$.

$$\prod_{k=0}^{m-1}((q^{2k+1}-1)^2+\alpha^2 q^{2k+1}) = \sum_{i=0}^{m} \frac{(q;q^2)_m}{(q;q^2)_i} H_{2i}(x|q) q^{i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \quad (1)$$

$$\alpha \prod_{k=0}^{m-1}((q^{2k+2}-1)^2+\alpha^2 q^{2k+2}) = \sum_{i=0}^{m} \frac{(q;q^2)_{m+1}}{(q;q^2)_{i+1}} H_{2i+1}(x|q) q^{i^2+i} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2} \quad (2)$$

Objectives

Let F_n(or G_n) be the right(or left)-hand side of (1) respectively, i.e.

$$F_n = \sum_{i=0}^{m} \frac{(q;q^2)_m}{(q;q^2)_i} H_{2i}(x|q) q^{i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2}$$

$$G_n = \prod_{k=0}^{m-1} ((q^{2k+1}-1)^2 + \alpha^2 q^{2k+1})$$

We will prove $F_n = G_n$ combinatorially.

Methods

Since G_n is a product, it has a simple recurrence.

$$G_{n+1} = G_n \times \left((q^{2n+1} - 1)^2 + \alpha^2 q^{2n+1}\right)$$

If we prove that for all $n \geqslant 0$,

$$\begin{aligned} F_{n+1} &= F_n \times \left((q^{2n+1} - 1)^2 + \alpha^2 q^{2n+1}\right) \\ &= F_n \times \left((1 - q^{2n+1}) + q^{2n+1}(q^{2n+1} - 1) + \alpha^2 q^{2n+1}\right) \end{aligned}$$

then $F_n = G_n$ for all n, since $F_0 = G_0 = 1$.

Combinatorial Objects for F_n

- Define the set of graphs.

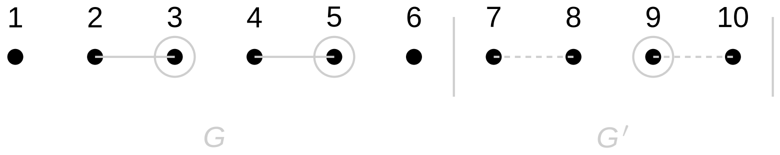
$$\begin{aligned}
\mathcal{F}_n &= \bigcup_{k \geqslant 0} \mathcal{F}_{n,k} \\
\mathcal{F}_{n,k} &= \{F = (G, G', X) : \\
&\quad G \in \mathcal{H}_{2k}, G' = (V_{n,k}, E_{n,k}), X \subset O_n, |X| = k\}
\end{aligned}$$

where

$$\begin{aligned}
O_n &= \{1, 3, \ldots, 2n-1\}, \\
V_{n,k} &= \{2k+1, 2k+2, \ldots, 2n\} \text{ and} \\
E_{n,k} &= \{(2k+1, 2k+2), (2k+3, 2k+4), \ldots, (2n-1, 2n)\}.
\end{aligned}$$

Example of Objects for F_n

Example. $n = 10$ and $k = 3$



$X = \{3, 5, 9\}$

Example of Objects for F_n

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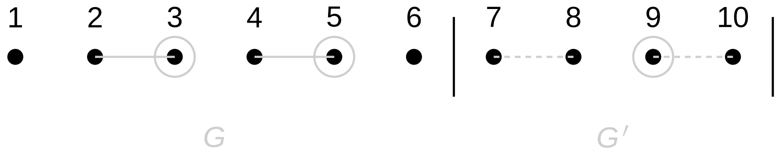

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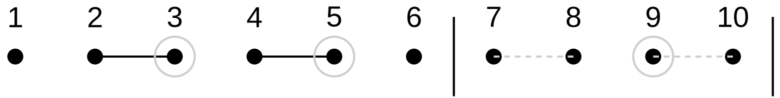

G G'

$X = \{3, 5, 9\}$

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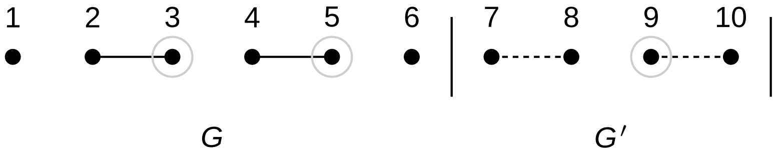

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Example of Objects for F_n

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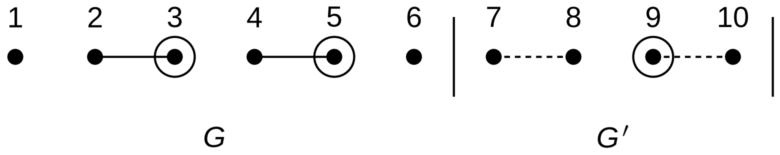

$$X = \{3, 5, 9\}$$

Weights of Objects for F_n I

- We assign a weight to the graph.

$$w(F) = w(G)w(G')w(X) = \prod_{v \in V(G)} w(v) \prod_{e \in E(G)} w(e) \prod_{e' \in E(G')} w(e') \prod_{i \in X} w(i)$$

where

$$\begin{aligned} w(v) &= \begin{cases} \alpha(=2x) & \text{if } \deg_G(v) = 0, \\ 0 & \text{if } \deg_G(v) = 1, \end{cases} \\ w(e) &= q^i - 1 \text{ for } e = (i, i+1) \in E(G) \text{ and} \\ w(e') &= 1 - q^i \text{ for } e' = (i, i+1) \in E(G'), \\ w(i) &= q^i \text{ for } i \in X. \end{aligned}$$

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Weights of Objects for F_n II

- We evaluate the weight sum for $\mathcal{F}_{n,k}$.

$$\begin{aligned}
\mathcal{F}_{n,k} &= \mathcal{H}_{2k} \times \mathcal{G}_{n,k} \times X_{n,k} \\
\sum_{G \in \mathcal{H}_{2k}} w(G) &= \mathcal{H}_{2k}(x|q) \\
\sum_{G' \in \mathcal{G}_{n,k}} w(G') &= (1-q^{2k+1})(1-q^{2k+3}) \cdots (1-q^{2n-1}) \\
&= (q^{2k+1};q^2)_{n-k} = \frac{(q;q^2)_n}{(q;q^2)_k} \\
\sum_{X \in X_{n,k}} w(X) &= q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}
\end{aligned}$$

Weights of Objects for F_n III

$$\Rightarrow \sum_{F\in\mathcal{F}_{n,k}} w(F) = \sum_{G\in\mathcal{H}_{2k}} w(G) \sum_{G'\in\mathcal{G}_{n,k}} w(G') \sum_{X\in X_{n,k}} w(X)$$
$$= \mathcal{H}_{2k}(x|q)\frac{(q;q^2)_n}{(q;q^2)_k} q^{k^2} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$$
$$= F_n$$

Therefore, $F_n = \sum_{F\in\mathcal{F}_n} w(F)$.

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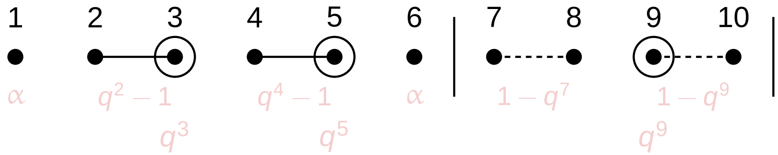
Example of Objects for F_n

Example. $n = 10$ and $k = 3$

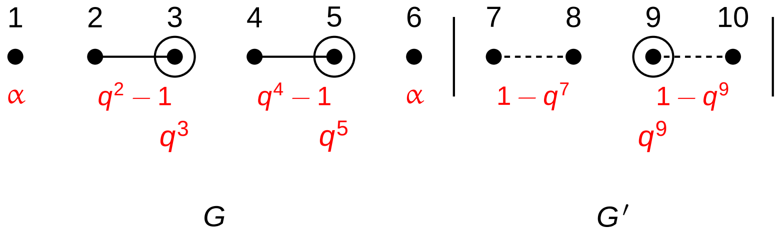


G G'

$X = \{3, 5, 9\}$

Example of Objects for F_n

Example. $n = 10$ and $k = 3$



$X = \{3, 5, 9\}$

Outline

1. Preliminaries
   - Notation
2. Motivation
   - q-Exponential Functions
   - q-Hermite Polynomials
3. Main Result
   - Interpretation
   - Main Proof

Construct the Map Ψ I

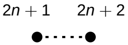
- We will construct a weight-preserving bijection

$$\Psi : \mathfrak{F}_{n+1} \to \mathfrak{F}_n \times \{a, b, c\}$$

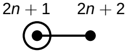
such that

$$w(F) = w(\Psi(F)), \quad \forall F \in \mathfrak{F}_{n+1}$$

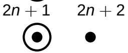
where

$a =$ 

$, \quad w(a) = 1 - q^{2n+1}$

$b =$ 

$, \quad w(b) = q^{2n+1}(q^{2n+1} - 1)$

$c =$ 

$, \quad w(c) = q^{2n+1}\alpha^2$

Construct the Map Ψ II

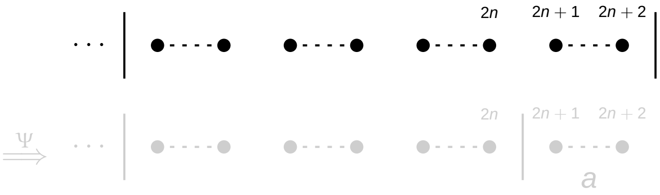
- If such Ψ exists, then we get a result that we want.

$$\begin{aligned} w(\mathfrak{F}_{n+1}) &= w(\mathfrak{F}_n)w(\{a,b,c\}) \\ F_{n+1} &= F_n \times ((1-q^{2n+1}) + q^{2n+1}(q^{2n+1}-1) + \alpha^2 q^{2n+1}) \\ &= F_n \times ((q^{2n+1}-1)^2 + \alpha^2 q^{2n+1}) \end{aligned}$$

Therefore, $F_n = G_n$ for all $n \geqslant 0$, since $F_0 = G_0 = 1$.

Case 1

$2n + 1 \notin X$



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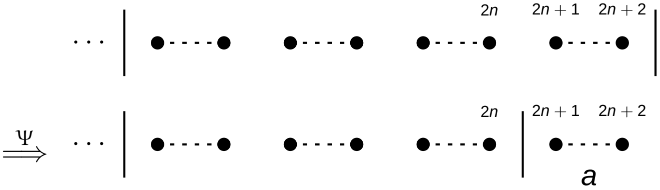
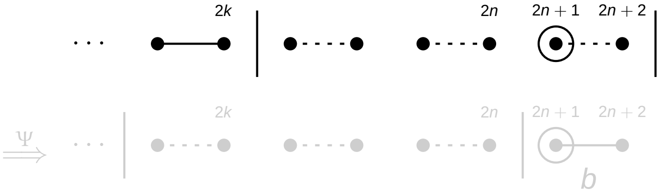


Case 2

$2n+1 \in X$ and $\deg(2k) = 1$ where $2k = V(G)$



Especially,

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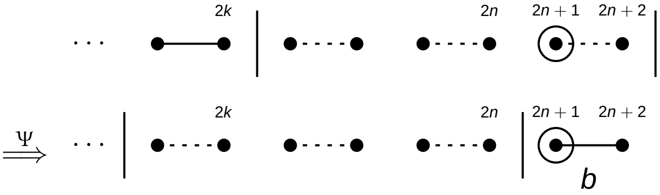

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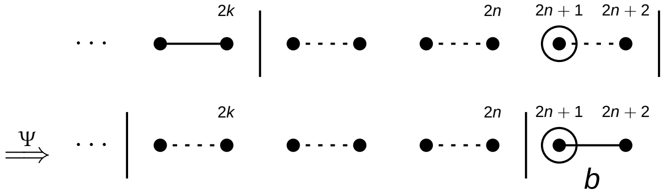

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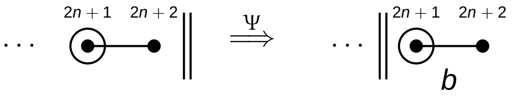
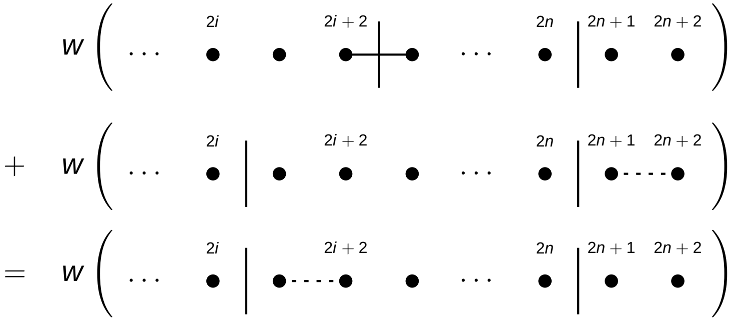


Lemma

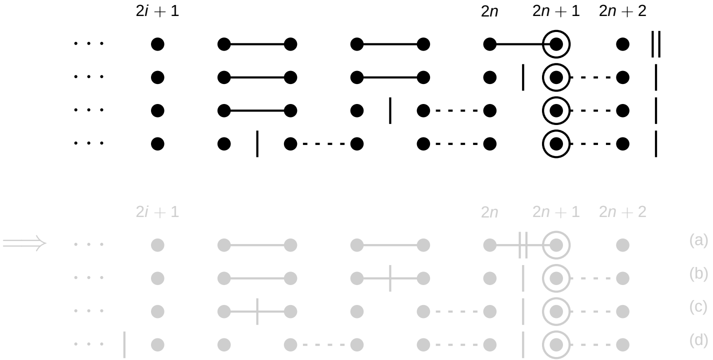
$$\begin{aligned}
&\begin{bmatrix} n \\ i+1 \end{bmatrix}_{q^2} q^{(i+1)^2}(q^{2i+2}-1) + \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} q^{i^2}(1-q^{2n+1}) \\
&= \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} q^{(i+1)^2}(q^{2n-2i}-1) + \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} q^{i^2}(1-q^{2n+1}) \\
&= \begin{bmatrix} n \\ i \end{bmatrix}_{q^2} q^{i^2}(1-q^{2i+1})
\end{aligned}$$

Graphical Representation of Lemma



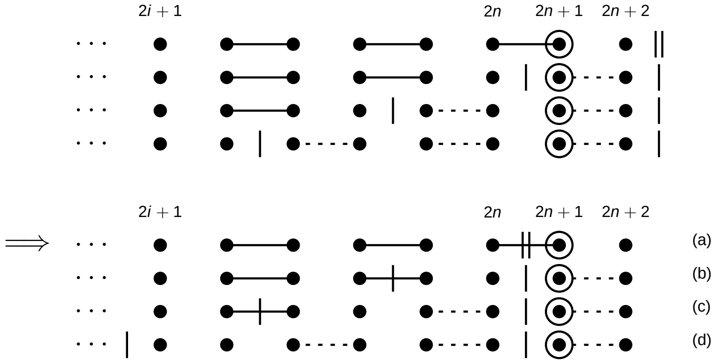
Case 3

$2n+1 \in X$ and $\deg(2k) = 0$ where $2k = V(G)$

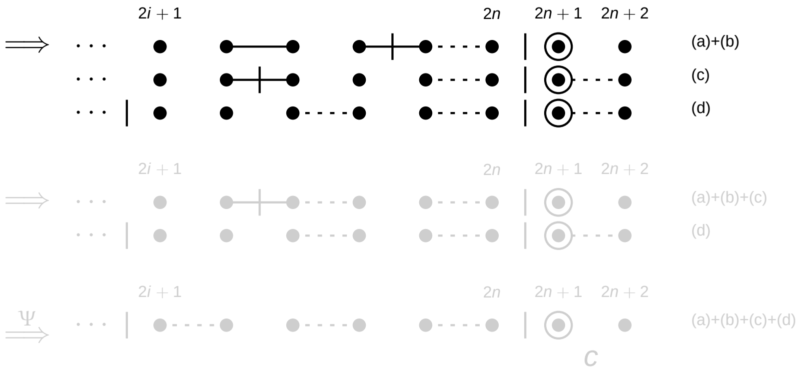


Case 3

$2n+1 \in X$ and $\deg(2k) = 0$ where $2k = V(G)$



Case 3



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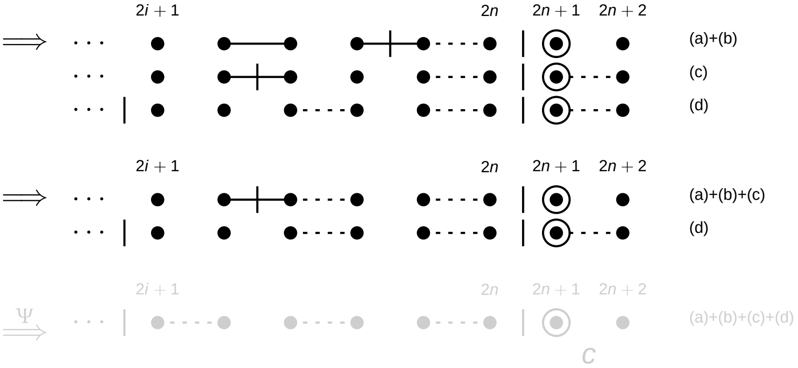

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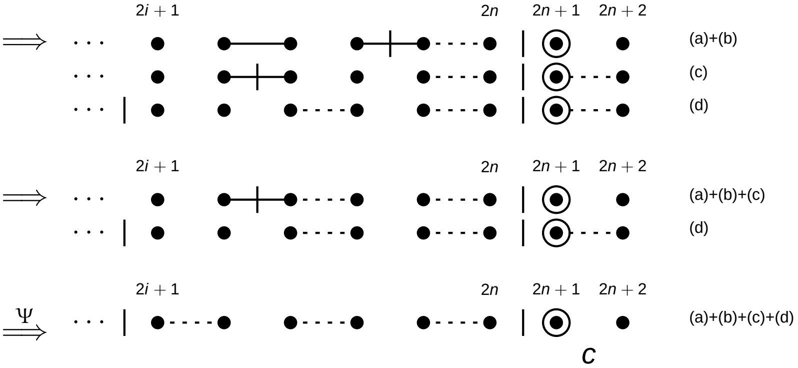

Conclusions

- We construct map Ψ, but it is not a bijection. However, it is a weight-preserving surjective map, in the other sense,

$$\sum_{\Psi(F')=(F,\delta)} w(F') = w(F,\delta)$$

for all $(F,\delta) \in \mathfrak{F}_n \times \{a,b,c\}$.

- Therefore,

$$w(\Psi^{-1}(F,\delta)) = w(F,\delta).$$

So there is no problem using the map ψ.

- Finally, we get $F_n = G_n$ for all $n \geqslant 0$.

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Our Results

Theorem

The following identity has a combinatorial interpretation.

$$\prod_{k=0}^{m-1}((q^{2k+1}-1)^2+\alpha^2 q^{2k+1}) = \sum_{i=0}^{m} \frac{(q;q^2)_m}{(q;q^2)_i} H_{2i}(x|q) q^{i^2} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2}$$

where $\alpha = 2x$.

Further Study I

1. Can the other equation (2) be proved similarly combinatorially?

$$\alpha \prod_{k=0}^{m-1}((q^{2k+2}-1)^2+\alpha^2 q^{2k+2}) = \sum_{i=0}^{m} \frac{(q;q^2)_{m+1}}{(q;q^2)_{i+1}} H_{2i+1}(x|q) q^{i^2+i} \begin{bmatrix} m \\ i \end{bmatrix}_{q^2}$$

where $\alpha = 2x$.

2. How to prove the identity combinatorially?

$$(qt^2;q^2)_\infty \mathcal{E}_q(x;t) = \sum_{n=0}^{\infty} \frac{q^{n^2/4} t^n}{(q;q)_n} H_n(x|q)$$

3. Can we find more combinatorial properties of $\mathcal{E}_q(x;t)$ and then interpret them combinatorially?

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Further Study II

4. Does there exist combinatorial objects for $\mathcal{E}_q(x;t)$ and $\mathcal{E}_q(x,y;t)$?

$$\mathcal{E}_q(x,y;t) = \frac{(t^2;q^2)_\infty}{(qt^2;q^2)_\infty} \sum_{n=0}^{\infty} \frac{q^{n^2/4} t^n}{(q;q)_\infty} \times e^{-\mathrm{i}n\phi} (-q^{(1-n)/2} e^{\mathrm{i}(\phi+\theta)}, -q^{(1-n)/2} e^{-\mathrm{i}(\phi-\theta)}; q)_n$$

where $x = \cos\theta$, $y = \cos\phi$.

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Thank you for listening!

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