

Pure Math 30:

Exponential and Logarithmic Functions

$$y = \log_2 \left(\frac{A}{B} \right)$$

Lesson 2

Laws of Logarithms

Pure Math
30:

EXPLAINED!

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Logarithms Lesson 2

Part II - Logarithmic to Exponential Form

CONVERTING FROM LOGARITHMIC TO EXPONENTIAL FORM:

Example 1: Convert $\log_2 x = y$
to exponential form:

$$\log_2 x \leftrightarrow y$$

$$2^y = x$$

Example 2: Given $3 = \log_5 x$
solve for x.

$$\log_5 x = 3$$

$$\log_5 x \leftrightarrow 3$$

$$5^3 = x$$

$$x = 125$$

Put the 3
on the other
side so you
can do the
seven rule.

Example 3: Convert $3\log b = a$
to exponential form

$$3\log b = a$$

$$\log_{10} b = \frac{a}{3}$$

$$10^{\frac{a}{3}} = b$$

Whenever you
have a log
written without
a base,
it actually has
a base of 10.

Example 4: Solve for x in

$$\log_3 (2x) = y$$

$$\log_3 2x = y$$

$$3^y = 2x$$

$$x = \frac{3^y}{2}$$

7 Rule

THE "SEVEN" RULE IS AN
EASY WAY OF REMEMBERING
THE CONVERSION. JUST DRAW A
SEVEN AS SHOWN BELOW, AND IT WILL
GIVE THE EXPONENTIAL FORM!

$$\log_a b = c$$

$$\log_a b \leftrightarrow c$$

$$a^c = b$$

QUESTIONS:

Convert each of the following
logarithms to exponential form:

1) $\log_3 a = b$

2) $5 = \log_m n$

3) $\log_4 y = x$

4) $3 = \log_2 b$

Solve for x in each of the following:

5) $\log_2 (x-1) = 3$

6) $\log_3 (2x+4) = 2$

ANSWERS:

1) $3^b = a$ 2) $m^5 = n$ 3) $4^x = y$

4) $b = 8$ 5) $x = 9$ 6) $x = 2.5$

Logarithms Lesson 2

Part II - Exponential to Logarithmic Form

CONVERTING FROM EXPONENTIAL TO LOGARITHMIC FORM

Example 1: Convert $y = x^2$ to logarithmic form

First write out the logarithm with the base:

$$\log_x \square = \square$$

Now fill in the rest so the “seven” rule will give you back what you started with.

$$\log_x y = 2$$

Example 2: Convert $a = 10x^4$ to logarithmic form

First get the x^4 by itself.

$$x^4 = \frac{a}{10}$$

Now set up the logarithm

$$\log_x \square = \square$$

Then fill it in so the “seven” rule works.

$$\log_x \frac{a}{10} = 4$$

Example 3: Convert $\sqrt{y} = 3x$ to logarithmic form

Rewrite as: $y^{\frac{1}{2}} = 3x$

$$\log_y 3x = \frac{1}{2}$$

Example 4: Convert $10^{x-y} = \frac{a}{b}$

to logarithmic form:

$$\log_{10} \square = \square$$

$$\log_{10} \frac{a}{b} = x - y$$

Example 5: The logarithmic form of $a = \frac{b^m}{b^n}$ is:

First simplify using exponent rules:

$$a = b^{m-n}$$

$$\log_b \square = \square$$

$$\log_b a = m - n$$

**A Base is
Always a Base**

REMEMBER “A BASE IS ALWAYS A BASE” WHEN DOING THIS CONVERSION.

IF YOU HAVE $A^C = B$, THE A , BEING THE BASE, WILL ALSO BE THE BASE OF YOUR LOGARITHM! PLACE THIS FIRST.

$$\log_a \square = \square$$

NOW FILL IN THE REST SO WHEN THE SEVEN RULE IS APPLIED, YOU GET BACK THE EXPONENTIAL FUNCTION

$$\log_a b = c$$

QUESTIONS:

Convert each of the following from exponential to logarithmic form

1) $y = x^3$

2) $4 = 3a^2$

3) $m^5 = n$

4) $2x^6 = 8$

5) $b = \frac{a^5}{a^3}$

6) $2\sqrt{x} = 5$

ANSWERS:

1) $\log_x y = 3$ 2) $\log_a \frac{4}{3} = 2$ 3) $\log_m n = 5$

4) $\log_x 4 = 6$ 5) $\log_a b = 2$ 6) $\log_x \left(\frac{5}{2} \right) = \frac{1}{2}$

Logarithms Lesson 2

Part III - Change of Base

CHANGE OF BASE:

Example 1: Evaluate $\log_2 3$

$$\log_2 3 = \frac{\log 3}{\log 2} = 1.585$$

Example 2: Evaluate $\log_2 \frac{2}{3}$

$$\log_2 \frac{2}{3} = \frac{\log \frac{2}{3}}{\log 2} = -0.585$$

Example 3: Evaluate $\log 5$

$$\log 5 = 0.699$$

Already
base 10.
Change of
base not
needed.

Example 4: Expand $\log_{2x}(y+z)$

$$\log_{2x}(y+z) = \frac{\log(y+z)}{\log 2x}$$

We can't
expand $\log(y+z)$ any
further since logs are not
distributive!

$$\log(y+z) \neq \log y + \log z$$

Example 5: Express $\frac{\log 4}{\log 7}$

as a single logarithm

$$\frac{\log 4}{\log 7} = \log_7 4$$

Change of Base

THE ONLY LOGS YOU CAN DO IN
YOUR CALCULATOR ARE BASE 10 LOGS.
CHANGE OF BASE LETS YOU DO ANY
LOGARITHM IN YOUR CALCULATOR!

$$\log_a b = \frac{\log_{10} b}{\log_{10} a}$$

BY WRITING A LOG AS A FRACTION,
YOU AUTOMATICALLY CONVERT IT TO
BASE 10 LOGS, SO NOW YOU CAN
TYPE IT INTO YOUR CALCULATOR.

Example 6: Express $(\log_a x)(\log_x b)$
as a single logarithm

$$(\log_a x)(\log_x b) = \left(\frac{\log x}{\log a} \right) \left(\frac{\log b}{\log x} \right) = \frac{\log b}{\log a} = \log_a b$$

Example 7: Evaluate the
expression: $3^{\log_2 8}$

$$3^{\log_2 8} = 3^3 = 27$$

Logarithms Lesson 2

Part III - Change of Base

QUESTIONS:

Evaluate using change of base :

1) $\log_4 5 =$

2) $\log_5 \frac{8}{9} =$

3) $\log 3 =$

Expand each of the following

4) $\log_{4x} (y - 2z)$

5) $\log_{(a+b)} (x + y)$

Express each as a single logarithm

6) $\frac{\log 5}{\log 8}$

7) $\frac{\log (a - 2b)}{\log c}$

8) $(\log_m n)(\log_n m)$

9) $(\log_a b)(\log_b c)(\log_c d)(\log_d x)$

10) Evaluate : $5^{\log_2 3}$

ANSWERS:

1) 1.16

2) - 0.073

3) 0.48

4) $\frac{\log(y - 2z)}{\log 4x}$

5) $\frac{\log(x + y)}{\log(a + b)}$

6) $\log_8 5$

7) $\log_c (a - 2b)$

8) 1

9) $\log_a x$

10) 12.82

Logarithms Lesson 2

Part IV - Multiplication Law

MULTIPLICATION LAW OF LOGARITHMS

Example 1: Expand $\log(xy)$

$$\log(xy) = \log x + \log y$$

Example 2: Expand $\log(3a \cdot 2b)$

$$\begin{aligned}\log(3a \cdot 2b) \\ &= \log(3a) + \log(2b) \\ &= \log 3 + \log a + \log 2 + \log b\end{aligned}$$

Example 3: Expand: $\log(x + y)$

$$\log(x + y) = \log(x + y) \quad \text{REMEMBER: A LOG CAN'T BE MULTIPLIED THROUGH THE BRACKETS!}$$

Example 4: Condense

$$\log 3 + \log 4$$

$$\log 3 + \log 4 = \log(3 \cdot 4) = \log 12$$

Example 5: Condense

$$\log(x + 1) + \log(x - 2)$$

$$\begin{aligned}\log(x + 1) + \log(x - 2) \\ &= \log(x + 1)(x - 2) \\ &= \log[x^2 - x - 2]\end{aligned}$$

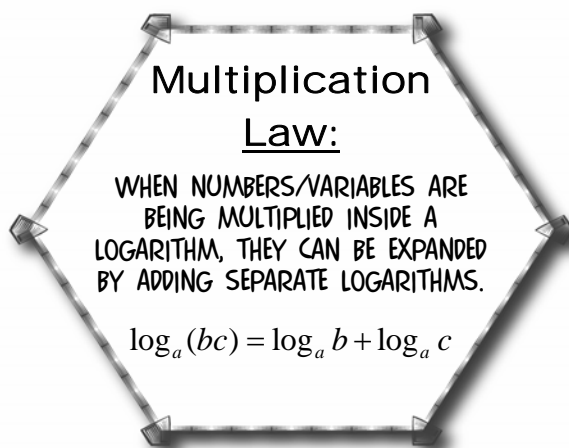
Example 6: Condense

$$\log x + \log y$$

$$\begin{aligned}\log x + \log y \\ &= \log(xy) \\ &= \log(xy)\end{aligned}$$

Example 7: Solve for y in the equation: $2 = \log_a x + \log_a y$

$$\begin{aligned}2 &= \log_a x + \log_a y \\ 2 &= \log_a(xy) \\ a^2 &= xy \\ y &= \frac{a^2}{x}\end{aligned}$$



Logarithms Lesson 2

Part IV - Multiplication Law

QUESTIONS:

Expand each of the following

1) $\log(abc)$

2) $2\log(4x)$

3) $3\log(x + y)$

Condense each of the following

4) $\log 2 + \log 6$

5) $\log(x + 3) + \log x$

6) $a\log(xy) + a\log(xz)$

7) $\log(2x + 1) + \log(3x - 2)$

Solve for x

8) $4 = \log_b x + \log_b y$

9) $7 = \log_m x + \log_m x$

ANSWERS:

1) $\log a + \log b + \log c$

2) $2\log 4 + 2\log x$

3) $3\log(x + y)$

4) $\log 12$

5) $\log(x^2 + 3x)$

6) $a\log(x^2 yz)$

7) $\log(6x^2 - x - 2)$

8) $x = \frac{b^4}{y}$

9) $x = \sqrt{m^7}$

Logarithms Lesson 2

Part B - Division Law

DIVISION LAW OF LOGARITHMS

Example 1: Expand $\log\left(\frac{x}{y}\right)$

$$\log\left(\frac{x}{y}\right) = \log x - \log y$$

Example 2: Expand $\log(3a - 2b)$

$$\log(3a - 2b) = \log(3a - 2b) \quad \text{*DIVISION RULE DOES NOT APPLY HERE.}$$

Example 3: Expand $-5\log\left(\frac{x}{3}\right)$

$$\begin{aligned} & -5\log\left(\frac{x}{3}\right) \\ &= -5[\log x - \log 3] \\ &= -5\log x + 5\log 3 \end{aligned}$$

Example 4: Expand: $\log(x - y)$

$$\log(x - y) = \log(x - y) \quad \text{*DIVISION RULE DOES NOT APPLY HERE.}$$

Example 5: Condense $\log 12 - \log 4$

$$\begin{aligned} & \log 12 - \log 4 \\ &= \log\left(\frac{12}{4}\right) \\ &= \log 3 \end{aligned}$$

Example 6: Condense $\log(x - 1) - \log(x + 2)$

$$\begin{aligned} & \log(x - 1) - \log(x + 2) \\ &= \log\left(\frac{x - 1}{x + 2}\right) \end{aligned}$$

Division Law:

WHEN NUMBERS/VARIABLES ARE BEING DIVIDED INSIDE A LOGARITHM, THEY CAN BE EXPANDED BY SUBTRACTING SEPARATE LOGARITHMS.

$$\log_a\left(\frac{b}{c}\right) = \log_a b - \log_a c$$

Example 7: Condense $a\log x - a\log y$

$$a\log x - a\log y$$

$$a(\log x - \log y)$$

$$a\log\frac{x}{y}$$

Logarithms Lesson 2

Part B - Division Law

QUESTIONS:

Expand the following

1) $\log \frac{a}{b}$

2) $-3\log\left(\frac{a}{2}\right)$

3) $\log(x - y)$

4) $\log(\sqrt{x} - 2)$

Condense the following

5) $\log 16 - \log 8$

6) $\log(x + 2) - \log(x - 1)$

7) $\frac{\log x}{\log y}$

8) $3\log 27 - 3\log 3$

9) $\log(8a^2b^4) - \log(4ab^2)$

10) $\log(2a^3b^{-2}) - \log(8a^{-5}b^6)$

ANSWERS:

1) $\log a - \log b$

2) $-3\log a + 3\log 2$

3) $\log(x - y)$

4) $\log(\sqrt{x} - 2)$

5) $\log 2$

6) $\log\left(\frac{x+2}{x-1}\right)$

7) $\log_y x$ (Change of Base!)

8) $3\log 9$

9) $\log(2ab^2)$

10) $\log\left(\frac{a^8}{4b^8}\right)$

Logarithms Lesson 2

Part III - Power Law

POWER LAW OF LOGARITHMS

Example 1: Simplify $\log x^2$

$$\log x^2 = 2\log x$$

Example 2: Simplify $\log x^2 + \log x^4$

$$\log x^2 + \log x^4 = 2\log x + 4\log x = 6\log x$$

Example 3: Expand $(\log x)^2$

$$(\log x)^2 = (\log x)^2$$

Power law does not apply when the entire log is raised to an exponent.

Example 4: Condense $3\log(xy)$

$$3\log(xy)$$

$$= \log(xy)^3$$

$$= \log(x^3y^3)$$

Example 5: Condense $2\log(x-1)$

$$2\log(x-1)$$

$$= \log(x-1)^2$$

You can also write as: $\log(x^2 - 2x + 1)$

Example 6: Condense: $4\log a - x$

$$4\log a - x = \log a^4 - x$$

Power Law:

WHEN THERE IS AN EXPONENT INSIDE A LOGARITHM, IT CAN BE TAKEN OUT IN FRONT OF THE LOGARITHM.

$$\log_a b^c = c \log_a b$$

Logarithms Lesson 2

Part VI - Power Law

QUESTIONS:

1) Expand : $\log a^3$

2) Expand : $\log a^3 + \log a^7$

3) Expand : $(\log a)^3$

4) Condense : $5\log(ab)$

5) Condense : $2\log(a - b)$

6) Condense : $3\log\left(\frac{a}{b}\right) - 7$

7) Simplify : $(2\log 10)^2$

8) Solve for x : $\log_3 x^2 = 6$

ANSWERS:

1) $3\log a$

2) $10\log a$

3) $(\log a)^3$

4) $\log a^5 b^5$

5) $\log (a-b)^2$

6) $\log\left(\frac{a}{b}\right)^3 - 7$

7) 4

8) $x = 27$

Logarithms Lesson 2

Part VIII ~ Other Laws

OTHER LAWS OF LOGARITHMS

Example 1: Evaluate $\log_{3x} 3x$

$$\log_{3x} 3x = 1$$

Example 2: Evaluate $\log_{3x} 0$

$$\log_{3x} 0 = \text{Undefined}$$

Example 3: Evaluate $\log_{3x} 1$

$$\log_{3x} 1 = 0$$

Example 4: Evaluate $\log_{3x} (-3)$

$$\log_{3x} (-3) = \text{Undefined}$$

Example 5: Evaluate $\log_3 3^4$

$$\log_3 3^4 = 4\log_3 3 = 4(1) = 4$$

Example 6: Evaluate $\log_{x-1} (x-1)^2$

$$\log_{x-1} (x-1)^2 = 2\log_{x-1} (x-1) = 2(1) = 2$$

Example 7: Evaluate $3^{\log_3 x}$

$$3^{\log_3 x} = x$$

Example 8: Evaluate $4 \cdot 2^{\log_2 6}$

$$4 \cdot 2^{\log_2 6} = 4 \cdot 6 = 24$$

Example 9: Simplify the expression $\log_5 25^k$

$$\begin{aligned} \log_5 25^k &= \log_5 (5^2)^k \\ &= \log_5 5^{2k} \\ &= 2k\log_5 5 \\ &= 2k(1) \\ &= 2k \end{aligned}$$

Other Laws:

- 1) $\log_a x$ is undefined for $x \leq 0$.
- 2) $\log_a 1 = 0$
- 3) $\log_a a = 1$
- 4) $a^{\log_a x} = x$
- 5) $\log_a a^x = x$

Example 10: Simplify the expression $\log_a (\sqrt{a})^x$

$$\begin{aligned} \log_a (\sqrt{a})^x &= \log_a \left(a^{\frac{1}{2}} \right)^x \\ &= \log_a a^{\frac{x}{2}} \\ &= \frac{x}{2} \log_a a \\ &= \frac{x}{2} (1) \\ &= \frac{x}{2} \end{aligned}$$

Logarithms Lesson 2

Part VIII - Other Laws

QUESTIONS:

1) Evaluate : $\log_{4xy} 4xy$

7) Simplify : $\log_3 81^k$

2) Evaluate : $\log_3 1$

8) Simplify : $\log_{\frac{1}{2}} \left(\sqrt{\frac{1}{2}} \right)^{3k}$

3) Evaluate : $\log_b 0$

9) Solve for x : $\log_5 125^x = 9$

4) Evaluate : $-2\log_{(a+b)}(a+b)$

10) Solve for x : $\log_4 256^{2x} = 32$

5) Evaluate : $\log_{(a-b)}(a-b)^{2x}$

6) Evaluate : $5^{\log_5 3x}$

ANSWERS:

1) 1

2) 0

3) *Undefined*

4) - 2

5) $2x$

6) $3x$

7) $4k$

8) $\frac{3k}{2}$

9) $x = 3$

10) $x = 4$

Logarithms Lesson 2

Part VIII - Diploma Style

DIPLOMA STYLE LOGARITHM QUESTIONS

Example 1: Given that $\log_3 4 = x$,
evaluate $\log_3 16$

$$\log_3 16$$

$$\log_3 (4 \cdot 4)$$

$$\log_3 4 + \log_3 4$$

$$x + x$$

$$2x$$

Example 2: If $\log_m a = 3$

and $\log_m b = 4$, evaluate $\left(\log_m \left(\frac{1}{ab} \right) \right)$

$$\left(\log_m \left(\frac{1}{ab} \right) \right)$$

$$\log_m (1) - \log_m (ab)$$

$$0 - [\log_m a + \log_m b]$$

$$-[3 + 4]$$

$$-7$$

Example 3: If $x = \frac{y}{z^2}$, determine
an expression for $\log x$

$$\log x$$

$$\log \left(\frac{y}{z^2} \right)$$

$$\log y - \log z^2$$

$$\log y - 2 \log z$$

Example 4: If $\log x = 3$,
evaluate $\log 10x^2$

$$\log 10x^2$$

$$\log 10 + \log x^2$$

$$1 + 2 \log x$$

$$1 + 2(3)$$

$$7$$

Example 5: If $\log_2 A = B$,
then $\log_4 A = ?$

$$\log_2 A = B \rightarrow A = 2^B$$

$$\log_4 A = \log_4 2^B = B \log_4 2 = B \left(\frac{1}{2} \right) = \frac{B}{2}$$

Example 6: If $\log_a b = 0.92$,
then the value of $\log_a \left(\frac{a}{b} \right)$ is:

$$\log_a \left(\frac{a}{b} \right) = \log_a a - \log_a b$$

$$= 1 - 0.92$$

$$= 0.08$$

Example 7: If $\log_3 x = 20$, then
the value of $\log_3 \left(\frac{1}{3} x \right)$ is:

$$\log_3 \left(\frac{1}{3} x \right) = \log_3 \left(\frac{x}{3} \right)$$

$$= \log_3 x - \log_3 3$$

$$= 20 - 1$$

$$= 19$$

Logarithms Lesson 2

Part VIII - Diploma Style

Example 8: If $\log x = 3.2$ and

$\log y = -0.9$, then $\frac{x}{y} =$

$$\log_{10} x = 3.2 \rightarrow x = 10^{3.2} \rightarrow x = 1584.89$$

$$\log_{10} y = -0.9 \rightarrow y = 10^{-0.9} \rightarrow y = 0.1259$$

$$\frac{x}{y} = \frac{1584.89}{0.1259} = 12589.25$$

Example 9: If $\log_m 9 = 2$ and

$\log_8 n = 2$, then $\log_2(mn) = ?$

$$\log_m 9 = 2 \rightarrow m^2 = 9 \rightarrow m = 3$$

$$\log_8 n = 2 \rightarrow 8^2 = n \rightarrow n = 64$$

$$\log_2(mn) = \log_2(3 \cdot 64) = \log_2 192 = 7.58$$

Example 10: If $x = y^2z$, then
find an expression for $\log z$

$$x = y^2z \rightarrow z = \frac{x}{y^2}$$

$$\log z = \log \left(\frac{x}{y^2} \right)$$

$$= \log x - \log y^2$$

$$= \log x - 2\log y$$

Example 11: If $\log_b A = M$,

then $\log_b \frac{1}{A^2} = ?$

$$\log_b \frac{1}{A^2} = \log_b 1 - \log_b A^2$$

$$= 0 - 2\log_b A$$

$$= -2M$$

Example 12: $\log_3(27a) = ?$

$$\log_3(27a)$$

$$= \log_3 27 + \log_3 a$$

$$= 3 + \log_3 a$$

Example 13: If $10^a = 4$, then

$10^{1+2a} = ?$

$$10^{1+2a} = 10 \cdot 10^a \cdot 10^a$$

$$= 10 \cdot 4 \cdot 4$$

$$= 160$$

Logarithms Lesson 2

Part VIII - Diploma Style

QUESTIONS:

1) Given that $\log_2 3 = a$, evaluate $\log_2 81$

2) If $\log_a b = 4$, evaluate $\log_a \left(\frac{a}{b} \right)$

3) If $ab = 5$, evaluate $\log_5 a^2 b^2$

4) If $a = \frac{b^3}{c^2}$, expand $\log a$

5) Given $\log x = 2$, evaluate $\log 100x^4$

6) If $\log_{36} x = A$, then $\log_6 x = ?$

7) If $\log a = 0.6$, and $\log b = 0.4$, evaluate $\log \left(\frac{a^2}{b} \right)$

8) If $\log_4 x = 15$, then $\log_4 \left(\frac{4}{x^{-2}} \right) =$

9) If $\log x = 4.3$ and $\log y = -0.8$, then $\frac{y}{x} = ?$

10) If $\log_a 32 = 5$, and $\log_4 b = 2.5$, then $\log_4 (ab) = ?$

11) If $a = bc^2$, find an expression for $\log b$

12) If $\log_a x = b$, then $\log_a \left(\frac{a}{x^3} \right) =$

13) If $5^x = b$, evaluate $25^x \cdot 125^{3x}$

14) If $3^x = a$, evaluate $\log_3 a^4$

15) Given $\frac{y^2}{xz^3} = 3$, evaluate $\log_3 \left(\frac{1}{x} \right) + 2\log_3 y - \log_3 z^3$

ANSWERS:

1) $4a$

2) -3

3) 2

4) $3 \log b - 2 \log c$

5) 10

6) $2A$

7) 0.8

8) 31

9) $10^{-5.1}$

10) 3

11) $\log a - 2 \log c$

12) $1 - 3b$

13) b^{11}

14) $4x$

15) 1