Pre - Calculus Math 40S:

TRIGONOMETRY II

$$(\cos x - 1)(\tan x - 1) = 0$$

LESSON THREE

Sum & Difference Identities

Pre - Calculus Math 40S EXPLAINED!

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Part I Expanding Sum & Difference

Sum & Difference Identities: ***The following formulas are used to expand trigonometric functions that have addition & subtraction in brackets.***

$\sin(A+B) \neq \sin A + \sin B$, *so we must use these rules whenever we want to expand.*

$$\sin(A + B) = \sin A\cos B + \cos A\sin B$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B$$
$$\cos(A + B) = \cos A\cos B - \sin A\sin B$$
$$\cos(A - B) = \cos A\cos B + \sin A\sin B$$

Example 1: Expand $\sin(60° - 45°)$

$$\begin{aligned}
&\sin(60^\circ - 45^\circ) \\
&= \sin 60^\circ \cos 45^\circ - \cos 60^\circ \sin 45^\circ \\
&= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) \\
&= \frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4} \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

Example 2: Expand $\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

$$\begin{aligned}
&\cos\left(\frac{\pi}{4} + \frac{\pi}{6}\right) \\
&= \cos\frac{\pi}{4}\cos\frac{\pi}{6} - \sin\frac{\pi}{4}\sin\frac{\pi}{6} \\
&= \left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right) \\
&= \left(\frac{\sqrt{6}}{4}\right) - \left(\frac{\sqrt{2}}{4}\right) \\
&= \frac{\sqrt{6}-\sqrt{2}}{4}
\end{aligned}$$

Part I Expanding Sum & Difference

Find the exact value by expanding each of the following:

1) $\sin(45° + 60°)$

5) $\sin\left(\frac{\pi}{2} - \frac{\pi}{3}\right)$

2) $\cos(45° - 30°)$

6) $\sin\left(\frac{\pi}{4} + \frac{\pi}{6}\right)$

3) $\sin(60° - 135°)$

7) $\cos\left(0 - \frac{3\pi}{4}\right)$

4) $\cos(150° + 45°)$

8) $\cos\left(\frac{\pi}{2} + \frac{\pi}{3}\right)$

1. $\sin(45^\circ + 60^\circ)$

$=\sin45^\circ\cos60^\circ+\cos45^\circ\sin60^\circ$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)$

$=\frac{\sqrt{2}}{4}+\frac{\sqrt{6}}{4}$

$=\frac{\sqrt{2}+\sqrt{6}}{4}$

2. $\cos(45^\circ - 30^\circ)$

$=\cos45^\circ\cos30^\circ+\sin45^\circ\sin30^\circ$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$

$=\frac{\sqrt{6}+\sqrt{2}}{4}$

3. $\sin(60^\circ - 135^\circ)$

$=\sin60^\circ\cos135^\circ - \cos60^\circ\sin135^\circ$

$=\left(\frac{\sqrt{3}}{2}\right)\left(-\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\frac{-\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$

$=\frac{-\sqrt{6}-\sqrt{2}}{4}$

4. $\cos(150^\circ + 45^\circ)$

$=\cos150^\circ\cos45^\circ - \sin150^\circ\sin45^\circ$

$=\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$=\frac{-\sqrt{6}}{4}-\frac{\sqrt{2}}{4}$

$=\frac{-\sqrt{6}-\sqrt{2}}{4}$

5. $\sin(\frac{\pi}{2} - \frac{\pi}{3})$

$=\sin\frac{\pi}{2}\cos\frac{\pi}{3} - \cos\frac{\pi}{2}\sin\frac{\pi}{3}$

$=(1)\left(\frac{1}{2}\right)-(0)\left(\frac{\sqrt{3}}{2}\right)$

$=\frac{1}{2}$

6. $\sin(\frac{\pi}{4} + \frac{\pi}{6})$

$=\sin\frac{\pi}{4}\cos\frac{\pi}{6} + \cos\frac{\pi}{4}\sin\frac{\pi}{6}$

$=\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$=\frac{\sqrt{6}}{4}+\frac{\sqrt{2}}{4}$

$=\frac{\sqrt{6}+\sqrt{2}}{4}$

7. $\cos(0-\frac{3\pi}{4})$

$=\cos0\cos\frac{3\pi}{4}+\sin0\sin\frac{3\pi}{4}$

$=(1)\left(-\frac{\sqrt{2}}{2}\right)+(0)\left(\frac{\sqrt{2}}{2}\right)$

$=-\frac{\sqrt{2}}{2}$

8. $\cos(\frac{\pi}{2}+\frac{\pi}{3})$

$=\cos\frac{\pi}{2}\cos\frac{\pi}{3}-\sin\frac{\pi}{2}\sin\frac{\pi}{3}$

$=(0)\left(\frac{1}{2}\right)-(1)\left(\frac{\sqrt{3}}{2}\right)$

$=-\frac{\sqrt{3}}{2}$

Part II Condensing Sum & Difference

Given an expanded form, we can work backwards to find a single trigonometric expression that can be easily solved using the unit circle.

Example 1: Express $\sin 85^\circ \cos 5^\circ + \cos 85^\circ \sin 5^\circ$ as a single trigonometric expression and solve.

We know $\sin(A+B) = \sin A\cos B + \cos A\sin B$

Looking at the question and matching to the sum formula, we can see that:
$A = 85^\circ$ and $B = 5^\circ$
Now plug A & B into the *left* side of the formula:

$\sin(A + B)$
$= \sin(85^\circ + 5^\circ)$
$= \sin(90^\circ)$
$= 1$

Example 2: Express $\dfrac{1}{\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)}$ as a single trigonometric expression and solve.

We know: $\sin(A - B) = \sin A\cos B - \cos A\sin B$

We have from the denominator of the question: $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \cos\left(\frac{\pi}{3}\right)\sin\left(\frac{\pi}{6}\right)$

Comparing the two, $A = \frac{\pi}{3}$ & $B = \frac{\pi}{6}$
Plug A & B into $\sin(A - B)$
$= \sin\left(\frac{\pi}{3} - \frac{\pi}{6}\right)$
$= \sin\left(\frac{2\pi}{6} - \frac{\pi}{6}\right)$
$= \sin\left(\frac{\pi}{6}\right)$

Subtract radians by finding a common denominator. (Or think in terms of degrees.)

We now know the denominator is $\sin\left(\frac{\pi}{6}\right)$.

Thus, we have $\dfrac{1}{\sin\left(\frac{\pi}{6}\right)} = \csc\left(\frac{\pi}{6}\right)$. Solving, $\dfrac{1}{\sin\left(\frac{\pi}{6}\right)} = \dfrac{1}{\frac{1}{2}} = 2$

For each of the following, express as a single trigonometric expression and solve using the unit circle.

1) $\cos 60^\circ \cos 15^\circ + \sin 60^\circ \sin 15^\circ$

2) $\cos^2 \frac{\pi}{6} - \sin^2 \frac{\pi}{6}$

3) $\dfrac{1}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{3}\right)}$

4) $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{3}\right)$

5) $\dfrac{1}{\cos(-15^\circ)\cos(30^\circ)+\sin(-15^\circ)\sin(30^\circ)}$

6) $\sin\left(\dfrac{\pi}{3}\right)\cos\left(\dfrac{\pi}{6}\right)-\sin\left(\dfrac{\pi}{6}\right)\cos\left(\dfrac{\pi}{3}\right)$

7) $\dfrac{1}{\cos\left(\dfrac{\pi}{2}\right)\cos\left(\dfrac{\pi}{4}\right)+\sin\left(\dfrac{\pi}{2}\right)\sin\left(\dfrac{\pi}{4}\right)}$

8) $\cos\left(\dfrac{2\pi}{3}\right)\cos\left(\dfrac{\pi}{2}\right)+\sin\left(\dfrac{2\pi}{3}\right)\sin\left(\dfrac{\pi}{2}\right)$

1. $\cos 60^\circ \cos 15^\circ + \sin 60^\circ \sin 15^\circ$

$A = 60^\circ$ & $B = 15^\circ$

$\cos(60^\circ - 15^\circ)$

$\cos(45^\circ)$

$= \frac{\sqrt{2}}{2}$

2. $\cos\frac{\pi}{6}\cos\frac{\pi}{6} - \sin\frac{\pi}{6}\sin\frac{\pi}{6}$

$A = \frac{\pi}{6}$ & $B = \frac{\pi}{6}$

$\cos(\frac{\pi}{6} + \frac{\pi}{6})$

$= \cos(\frac{2\pi}{6})$

$= \cos(\frac{\pi}{3})$

$= \frac{1}{2}$

3. $\frac{1}{\sin\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{3}\right) - \cos\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{3}\right)}$

$A = \frac{\pi}{2}$ & $B = \frac{\pi}{3}$

$\frac{1}{\sin(\frac{\pi}{2} - \frac{\pi}{3})}$

$= \csc(\frac{\pi}{2} - \frac{\pi}{3})$

$= \csc(\frac{3\pi}{6} - \frac{2\pi}{6})$ Getting A Common Denominator

$= \csc(\frac{\pi}{6})$

$= 2$

4. $\sin\left(\frac{5\pi}{12}\right)\cos\left(\frac{\pi}{3}\right) + \cos\left(\frac{5\pi}{12}\right)\sin\left(\frac{\pi}{3}\right)$

$A = \frac{5\pi}{12}$ & $B = \frac{\pi}{3}$

$\sin(\frac{5\pi}{12} + \frac{\pi}{3})$

$= \sin(\frac{5\pi}{12} + \frac{4\pi}{12})$

$= \sin(\frac{9\pi}{12})$

$= \sin(\frac{3\pi}{4})$

$= \frac{\sqrt{2}}{2}$

5. $\frac{1}{\cos(-15^\circ)\cos(30^\circ) + \sin(-15^\circ)\sin(30^\circ)}$

$A = -15^\circ$ & $B = 30^\circ$

$\frac{1}{\cos(-15^\circ - 30^\circ)}$

$= \frac{1}{\cos\left(-45^\circ\right)}$

$= \frac{1}{\frac{\sqrt{2}}{2}}$

$= \frac{2}{\sqrt{2}}$

$= \frac{2}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}}$

$= \sqrt{2}$

6. $\sin\left(\frac{\pi}{3}\right)\cos\left(\frac{\pi}{6}\right) - \sin\left(\frac{\pi}{6}\right)\cos\left(\frac{\pi}{3}\right)$

$A = \frac{\pi}{3}$ & $B = \frac{\pi}{6}$

$\sin(\frac{\pi}{3} - \frac{\pi}{6})$

$= \sin(\frac{2\pi}{6} - \frac{\pi}{6})$

$= \sin\frac{\pi}{6}$

$= \frac{1}{2}$

7. $\frac{1}{\cos\left(\frac{\pi}{2}\right)\cos\left(\frac{\pi}{4}\right) + \sin\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{4}\right)}$

$A = \frac{\pi}{2}$ & $B = \frac{\pi}{4}$

$\frac{1}{\cos(\frac{\pi}{2} - \frac{\pi}{4})}$

$= \sec(\frac{\pi}{2} - \frac{\pi}{4})$

$= \sec(\frac{2\pi}{4} - \frac{\pi}{4})$

$= \sec(\frac{\pi}{4})$

$= \frac{2}{\sqrt{2}}$

$= \sqrt{2}$

8. $\cos\left(\frac{2\pi}{3}\right)\cos\left(\frac{\pi}{2}\right) + \sin\left(\frac{2\pi}{3}\right)\sin\left(\frac{\pi}{2}\right)$

$A = \frac{2\pi}{3}$ & $B = \frac{\pi}{2}$

$\cos(\frac{2\pi}{3} - \frac{\pi}{2})$

$= \cos(\frac{4\pi}{6} - \frac{3\pi}{6})$

$= \cos\frac{\pi}{6}$

$= \frac{\sqrt{3}}{2}$

The sum & difference formulas are useful in determining the exact values of sine & cosine for angles <u>not</u> on the unit circle.

<u>Example 1:</u> Find the exact value of $\sin 15°$

First, think of how you can get 15° by using angles on the unit circle:
15° = 60° - 45°
15° = 45° - 30°
15° = 135° - 120°
15° = -30° + 45°

As you can see, there are many possibilities, you can choose any of them and still get the right answer.

We'll use the top one, 15° = 60° - 45°

$$\begin{aligned}&\sin 15^\circ\\ &=\sin(60^\circ-45^\circ)\\ &=\sin 60^\circ\cos 45^\circ-\cos 60^\circ\sin 45^\circ\\ &=\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)-\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)\\ &=\frac{\sqrt{6}-\sqrt{2}}{4}\end{aligned}$$

<u>Example 2:</u> Find the exact value of $\sec\left(-\frac{5\pi}{12}\right)$

It's easiest to think in terms of degrees, so convert: $-\frac{5\pi}{12}\times\frac{180^\circ}{\pi}=-75^\circ$ (-75° = -45° - 30°)

$$\begin{aligned}&\sec(-75^\circ)\\ &=\sec(-45^\circ-30^\circ)\\ &=\frac{1}{\cos(-45^\circ-30^\circ)}\\ &=\frac{1}{\cos(-45^\circ)\cos(30^\circ)+\sin(-45^\circ)\sin(30^\circ)}\\ &=\frac{1}{\left(\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right)+\left(\frac{-\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)}\\ &=\frac{1}{\frac{\sqrt{6}}{4}-\frac{\sqrt{2}}{4}}\\ &=\frac{1}{\frac{\sqrt{6}-\sqrt{2}}{4}}\\ &=\frac{4}{\sqrt{6}-\sqrt{2}}\end{aligned}$$

Now rationalize the denominator of your answer.

$$\begin{aligned}&\frac{4}{\sqrt{6}-\sqrt{2}}\\ &=\frac{4}{\sqrt{6}-\sqrt{2}}\times\frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}\\ &=\frac{4(\sqrt{6}+\sqrt{2})}{6-2}\\ &=\frac{4(\sqrt{6}+\sqrt{2})}{4}\\ &=\sqrt{6}+\sqrt{2}\end{aligned}$$

Trigonometry II - Lesson 3

Part III Non Unit Circle Angles

Find the exact value of each of the following:

Note that there are multiple ways of getting to the correct answer. Rationalize the denominator when necessary.

1) $\cos(-15^\circ)$ **2)** $\sec(105^\circ)$ **3)** $\csc(-105^\circ)$

4) $\sin(-\frac{5\pi}{12})$ **5)** $\csc(165^\circ)$ **6)** $\sec(\frac{13\pi}{12})$

1. $\cos(30^\circ - 45^\circ)$

$= \cos 30^\circ \cos 45^\circ + \sin 30^\circ \sin 45^\circ$

$= \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)$

$= \frac{\sqrt{6}}{4} + \frac{\sqrt{2}}{4}$

$= \frac{\sqrt{6}+\sqrt{2}}{4}$

2. $\sec(60^\circ + 45^\circ)$

$= \frac{1}{\cos(60^\circ + 45^\circ)}$

$= \frac{1}{\cos 60^\circ \cos 45^\circ - \sin 60^\circ \sin 45^\circ}$

$= \frac{1}{\left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{\frac{\sqrt{2}}{4} + \frac{\sqrt{6}}{4}}$

$= \frac{1}{\frac{\sqrt{2}+\sqrt{6}}{4}}$

$= \frac{4}{\sqrt{2}+\sqrt{6}}$

$= \frac{4}{\sqrt{2}+\sqrt{6}} \times \frac{\sqrt{2}-\sqrt{6}}{\sqrt{2}-\sqrt{6}}$

$= \frac{4(\sqrt{2}-\sqrt{6})}{2-6}$

$= \frac{4(\sqrt{2}-\sqrt{6})}{-4}$

$= -1(\sqrt{2}-\sqrt{6})$

$= \sqrt{6}-\sqrt{2}$

3. $\csc(-60^\circ - 45^\circ)$

$= \frac{1}{\sin(-60^\circ - 45^\circ)}$

$= \frac{1}{\sin(-60^\circ)\cos 45^\circ - \cos(-60^\circ)\sin 45^\circ}$

$= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$

$= \frac{1}{\frac{-\sqrt{6}-\sqrt{2}}{4}}$

$= \frac{4}{-\sqrt{6}-\sqrt{2}}$

$= \frac{4}{-\sqrt{6}-\sqrt{2}} \times \frac{-\sqrt{6}+\sqrt{2}}{-\sqrt{6}+\sqrt{2}}$

$= \frac{4(-\sqrt{6}+\sqrt{2})}{6-2}$

$= \frac{4(-\sqrt{6}+\sqrt{2})}{4}$

$= -\sqrt{6}+\sqrt{2}$

4. $\sin(-\frac{5\pi}{12})$

$= \sin(-75^\circ)$

$= \sin(-45^\circ - 30^\circ)$

$= \sin(-45^\circ)\cos 30^\circ - \cos(-45^\circ)\sin 30^\circ$

$= \left(-\frac{\sqrt{2}}{2}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{\sqrt{2}}{2}\right)\left(\frac{1}{2}\right)$

$= -\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}$

$= \frac{-(\sqrt{6}+\sqrt{2})}{4}$

5. $\csc(165^\circ)$

$= \frac{1}{\sin(165^\circ)}$

$= \frac{1}{\sin(120^\circ + 45^\circ)}$

$= \frac{1}{\sin(120^\circ)\cos 45^\circ + \cos(120^\circ)\sin 45^\circ}$

$= \frac{1}{\left(\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) + \left(-\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$

$= \frac{1}{\frac{\sqrt{6}-\sqrt{2}}{4}}$

$= \frac{4}{\sqrt{6}-\sqrt{2}}$

$= \frac{4}{\sqrt{6}-\sqrt{2}} \times \frac{\sqrt{6}+\sqrt{2}}{\sqrt{6}+\sqrt{2}}$

$= \frac{4(\sqrt{6}+\sqrt{2})}{4}$

$= \sqrt{6}+\sqrt{2}$

6. $\sec\left(\frac{13\pi}{12}\right)$

$= \sec(195^\circ)$

$= \sec(150^\circ + 45^\circ)$

$= \frac{1}{\cos(150^\circ + 45^\circ)}$

$= \frac{1}{\cos 150^\circ \cos 45^\circ - \sin 150^\circ \sin 45^\circ}$

$= \frac{1}{\left(-\frac{\sqrt{3}}{2}\right)\left(\frac{\sqrt{2}}{2}\right) - \left(\frac{1}{2}\right)\left(\frac{\sqrt{2}}{2}\right)}$

$= \frac{1}{-\frac{\sqrt{6}}{4} - \frac{\sqrt{2}}{4}}$

$= \frac{1}{\frac{-\sqrt{6}-\sqrt{2}}{4}}$

$= -\frac{4}{(\sqrt{6}+\sqrt{2})}$

$= -\frac{4}{(\sqrt{6}+\sqrt{2})} \times \frac{\sqrt{6}-\sqrt{2}}{\sqrt{6}-\sqrt{2}}$

$= -\frac{4(\sqrt{6}-\sqrt{2})}{4}$

$= -(\sqrt{6}-\sqrt{2})$

The following examples illustrate some basic proofs you can do with the sum & difference identities:

Example 1: Prove that $\cos(\frac{\pi}{2} - x) = \sin x$

We start with the formula: $\cos(A-B) = \cos A\cos B + \sin A\sin B$

Plug in $A = \frac{\pi}{2}$ & $B = x$

$$\cos(\frac{\pi}{2} - x) = \cos A\cos B + \sin A\sin B$$
$$= \cos\frac{\pi}{2}\cos x + \sin\frac{\pi}{2}\sin x$$
$$= (0)\cos x + (1)\sin x$$
$$= \sin x$$

Example 2: Prove that $\csc(\pi + x) = -\csc x$

$$\csc(\pi + x)$$
$$= \frac{1}{\sin(\pi + x)}$$
$$= \frac{1}{\sin\pi\cos x + \cos\pi\sin x}$$
$$= \frac{1}{(0)\cos x + (-1)\sin x}$$
$$= \frac{1}{-\sin x}$$
$$= -\csc x$$

Prove each of the following:

1) $\cos(\frac{3\pi}{2} - x) = -\sin x$

5) $\sin(\frac{\pi}{2} - x) =$

2) $\sin(270^\circ - x) = -\cos x$

6) $\csc(\frac{\pi}{2} + x) =$

3) $\cos(\frac{\pi}{2} + x) = -\sin x$

7) $\sec(\pi + x)$

4) $\cos(\pi - x) =$

8) $\csc(\pi - x)$

1. $\cos(\frac{3\pi}{2}-x)$

$=\cos\frac{3\pi}{2}\cos x+\sin\frac{3\pi}{2}\sin x$

$=(0)\cos x+(-1)\sin x$

$=-\sin x$

2. $\sin(270^\circ-x)$

$=\sin 270^\circ\cos x-\cos 270^\circ\sin x$

$=(-1)\cos x-(0)\sin x$

$=-\cos x$

3. $\cos(\frac{\pi}{2}+x)$

$=\cos\frac{\pi}{2}\cos x-\sin\frac{\pi}{2}\sin x$

$=(0)\cos x-(1)\sin x$

$=-\sin x$

4. $\cos(\pi-x)$

$=\cos\pi\cos x+\sin\pi\sin x$

$=(-1)\cos x+(0)\sin x$

$=-\cos x$

5. $\sin(\frac{\pi}{2}-x)$

$=\sin\frac{\pi}{2}\cos x-\cos\frac{\pi}{2}\sin x$

$=(1)\cos x-(0)\sin x$

$=\cos x$

6. $\csc(\frac{\pi}{2}+x)$

$=\dfrac{1}{\sin(\frac{\pi}{2}+x)}$

$=\dfrac{1}{\sin\frac{\pi}{2}\cos x+\cos\frac{\pi}{2}\sin x}$

$=\dfrac{1}{(1)\cos x+(0)\sin x}$

$=\dfrac{1}{\cos x}$

$=\sec x$

7. $\sec(\pi+x)$

$=\dfrac{1}{\cos(\pi+x)}$

$=\dfrac{1}{\cos\pi\cos x-\sin\pi\sin x}$

$=\dfrac{1}{(-1)\cos x-(0)\sin x}$

$=\dfrac{1}{-\cos x}$

$=-\sec x$

8. $\csc(\pi-x)$

$=\dfrac{1}{\sin(\pi-x)}$

$=\dfrac{1}{\sin\pi\cos x-\cos\pi\sin x}$

$=\dfrac{1}{(0)\cos x-(-1)\sin x}$

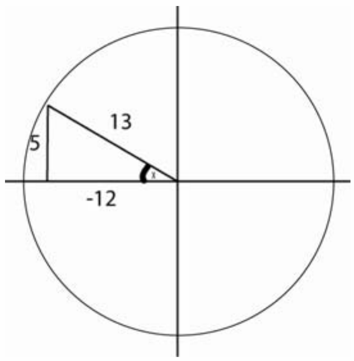
$=\dfrac{1}{\sin x}$

$=\csc x$

In some questions, you will be given incomplete information, and must use triangles to find all the trig ratios required by the sum & difference formulas.

Example 1: Given $\tan x = -\frac{5}{12}$ (In quadrant II) and $\tan y = \frac{3}{5}$ (In quadrant III), find the exact value of $\sec(x+y)$

Draw a triangle corresponding to $\tan x = -\frac{5}{12}$ and find the unknown side.

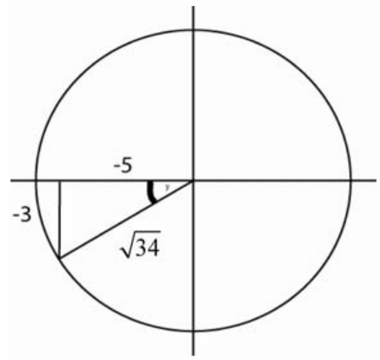


$$a^2 + b^2 = c^2$$
$$(-12)^2 + (5)^2 = c^2$$
$$169 = c^2$$
$$c = 13$$

Now find $\sin x$ & $\cos x$.

$$\sin x = \frac{5}{13}$$
$$\cos x = \frac{-12}{13}$$

Draw a triangle corresponding to $\tan y = \frac{3}{5}$ and find the unknown side.



$$a^2 + b^2 = c^2$$
$$(-5)^2 + (-3)^2 = c^2$$
$$34 = c^2$$
$$c = \sqrt{34}$$

Now find $\sin y$ & $\cos y$

$$\sin y = \frac{-3}{\sqrt{34}}$$
$$\cos y = \frac{-5}{\sqrt{34}}$$

$$\sec(x+y) = \frac{1}{\cos(x+y)}$$
$$= \frac{1}{\cos x \cos y - \sin x \sin y}$$
$$= \frac{1}{\left(\frac{-12}{13}\right)\left(\frac{-5}{\sqrt{34}}\right) - \left(\frac{5}{13}\right)\left(\frac{-3}{\sqrt{34}}\right)}$$
$$= \frac{1}{\frac{60}{13\sqrt{34}} + \frac{15}{13\sqrt{34}}}$$
$$= \frac{1}{\frac{75}{13\sqrt{34}}}$$
$$= \frac{13\sqrt{34}}{75}$$ **Final Answer**

PART V TRIANGLE QUESTIONS

1)

$\sin x = -\frac{1}{2}$ (Quadrant III)

$\tan y = \frac{3}{4}$ (Quadrant I)

$\boxed{\text{Find } \cos(x-y)}$

2)

$\cos x = \frac{12}{13}$ (Quadrant IV)

$\csc y = -\frac{5}{2}$ (Quadrant III)

$\boxed{\text{Find } \csc(x+y)}$

3)

$\sec x = \frac{7}{5}$ (Quadrant I)

$\cot y = -\frac{3}{4}$ (Quadrant II)

$\boxed{\text{Find } \sin(x-y)}$

4)

$\tan x = \frac{-6}{7}$ (Quadrant IV)

$\sin y = -\frac{2}{5}$ (Quadrant IV)

Find $\sec(x+y)$

5)

$\cot x = -5$ $(\cos x < 0)$

$\tan y = \frac{3}{4}$ $(\sin y > 0)$

Find $\csc(x-y)$

6)

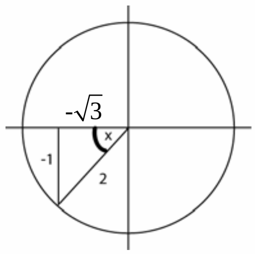
$\sec x = -\frac{8}{7}$ $(\sin x > 0)$

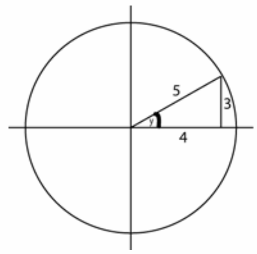
$\tan y = -3$ $(\sin y < 0)$

Find $\sec(x-y)$

TRIGONOMETRY II - LESSON 3
PART V TRIANGLE QUESTIONS

1) First draw out each triangle, then use Pythagoras to find the unknown side:





Now find the sine & cosine trig ratios for each triangle:

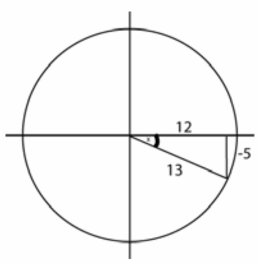
$$\sin x = \frac{-1}{2} \qquad \sin y = \frac{3}{5}$$

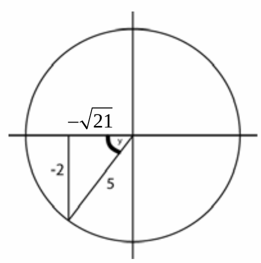
$$\cos x = \frac{-\sqrt{3}}{2} \qquad \cos y = \frac{4}{5}$$

Use these to evaluate $\cos(x-y)$

$$\cos(x-y)$$
$$= \cos x \cos y + \sin x \sin y$$
$$= \left(\frac{-\sqrt{3}}{2}\right)\left(\frac{4}{5}\right) + \left(\frac{-1}{2}\right)\left(\frac{3}{5}\right)$$
$$= \frac{-4\sqrt{3}}{10} - \frac{3}{10}$$
$$= \frac{-4\sqrt{3}-3}{10}$$

2) First draw out each triangle, then use Pythagoras to find the unknown side:





Now state the sine cosine trig ratios for each triangle:

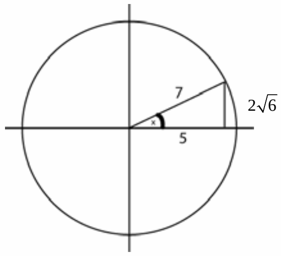
$$\sin x = \frac{-5}{13} \qquad \sin y = \frac{-2}{5}$$

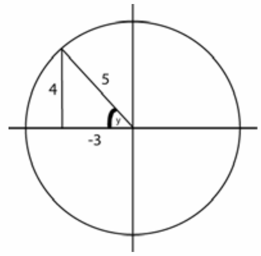
$$\cos x = \frac{12}{13} \qquad \cos y = \frac{-\sqrt{21}}{5}$$

Use these to evaluate $\csc(x+y)$

$$\csc(x+y)$$
$$= \frac{1}{\sin x \cos y + \cos x \sin y}$$
$$= \frac{1}{\left(\frac{-5}{13}\right)\left(\frac{-\sqrt{21}}{5}\right) + \left(\frac{12}{13}\right)\left(\frac{-2}{5}\right)}$$
$$= \frac{1}{\left(\frac{-5\sqrt{21}}{65}\right) + \left(\frac{-24}{65}\right)}$$
$$= \frac{1}{\frac{-5\sqrt{21}-24}{65}}$$
$$= \frac{65}{-5\sqrt{21}-24}$$

3) First draw out each triangle, and use Pythagoras to find the unknown side:





Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{2\sqrt{6}}{7} \qquad \sin y = \frac{4}{5}$$

$$\cos x = \frac{5}{7} \qquad \cos y = \frac{-3}{5}$$

Use these to evaluate $\sin(x-y)$

$\sin(x-y)$

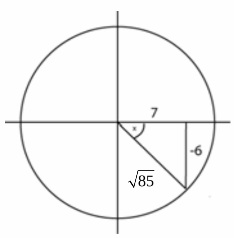
$= \sin x \cos y - \cos x \sin y$

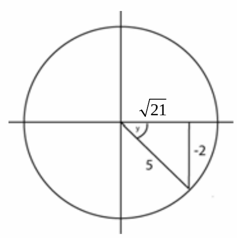
$= \left(\frac{2\sqrt{6}}{7}\right)\left(\frac{-3}{5}\right) - \left(\frac{5}{7}\right)\left(\frac{4}{5}\right)$

$= \frac{-6\sqrt{6}}{35} - \frac{20}{35}$

$= \frac{-6\sqrt{6}-20}{35}$

4) First draw out each triangle, and use Pythagoras to find the unknown side:





Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{-6}{\sqrt{85}} \qquad \sin y = \frac{-2}{5}$$

$$\cos x = \frac{7}{\sqrt{85}} \qquad \cos y = \frac{\sqrt{21}}{5}$$

Use these to evaluate $\sec(x+y)$

$\sec(x+y)$

$= \frac{1}{\cos x \cos y - \sin x \sin y}$

$= \frac{1}{\left(\frac{7}{\sqrt{85}}\right)\left(\frac{\sqrt{21}}{5}\right) - \left(\frac{-6}{\sqrt{85}}\right)\left(\frac{-2}{5}\right)}$

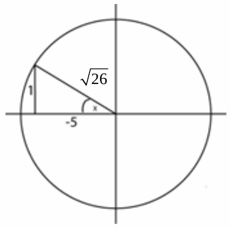
$= \frac{1}{\left(\frac{7\sqrt{21}}{5\sqrt{85}}\right) - \left(\frac{12}{5\sqrt{85}}\right)}$

$= \frac{1}{\frac{7\sqrt{21}-12}{5\sqrt{85}}}$

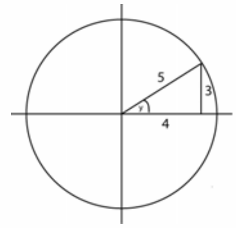
$= \frac{5\sqrt{85}}{7\sqrt{21}-12}$

5) First draw out each triangle, and use Pythagoras to find the unknown side:

$\cot x < 0$ in quadrants II & IV
$\cos x < 0$ in quadrants II & III
Overlap in II



$\tan y > 0$ in quadrants I & III
$\sin y > 0$ in quadrants I & II
Overlap in I



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{1}{\sqrt{26}} \qquad \sin y = \frac{3}{5}$$

$$\cos x = \frac{-5}{\sqrt{26}} \qquad \cos y = \frac{4}{5}$$

Use these to evaluate $\csc(x-y)$

$\csc(x-y)$

$$= \frac{1}{\sin x \cos y - \cos x \sin y}$$

$$= \frac{1}{\left(\frac{1}{\sqrt{26}}\right)\left(\frac{4}{5}\right) - \left(\frac{-5}{\sqrt{26}}\right)\left(\frac{3}{5}\right)}$$

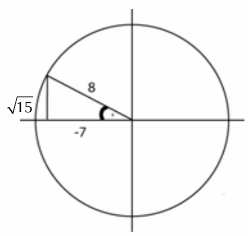
$$= \frac{1}{\left(\frac{4}{5\sqrt{26}}\right) - \left(\frac{-15}{5\sqrt{26}}\right)}$$

$$= \frac{1}{\frac{19}{5\sqrt{26}}}$$

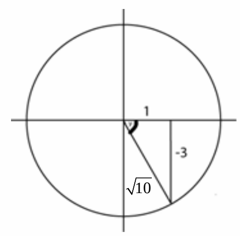
$$= \frac{5\sqrt{26}}{19}$$

6) First draw out each triangle, and use Pythagoras to find the unknown side:

$\sec x < 0$ in quadrants II & III
$\sin x > 0$ in quadrants I & II
Overlap in II



$\tan y < 0$ in quadrants II & IV
$\sin y < 0$ in quadrants III & IV
Overlap in IV



Now find the sine & cosine trig ratios for each triangle:

$$\sin x = \frac{\sqrt{15}}{8} \qquad \sin y = \frac{-3}{\sqrt{10}}$$

$$\cos x = \frac{-7}{8} \qquad \cos y = \frac{1}{\sqrt{10}}$$

Use these to evaluate $\sec(x-y)$

$\sec(x-y)$

$$= \frac{1}{\cos x \cos y + \sin x \sin y}$$

$$= \frac{1}{\left(\frac{-7}{8}\right)\left(\frac{1}{\sqrt{10}}\right) + \left(\frac{\sqrt{15}}{8}\right)\left(\frac{-3}{\sqrt{10}}\right)}$$

$$= \frac{1}{\left(\frac{-7}{8\sqrt{10}}\right) + \left(\frac{-3\sqrt{15}}{8\sqrt{10}}\right)}$$

$$= \frac{1}{\frac{-7-3\sqrt{15}}{8\sqrt{10}}}$$

$$= \frac{8\sqrt{10}}{-7-3\sqrt{15}}$$

PART VI DOUBLE ANGLE IDENTITIES

The following double angle identities are frequently used:

$$\sin 2A = 2\sin A\cos A$$

$$\cos 2A = \cos^2 A - \sin^2 A$$

$$\cos 2A = 2\cos^2 A - 1$$

$$\cos 2A = 1 - 2\sin^2 A$$

Example 1: Expand sin 60° using $\sin 2A = 2\sin A\cos A$:

$$\sin 2A = 2\sin A\cos A$$
$$\sin 60^\circ = 2\sin 30^\circ \cos 30^\circ$$

What goes here is
always half of the
angle on the left side.

Example 2: Expand cos 90° using $\cos 2A = 2\cos^2 A - 1$

$$\cos 2A = 2\cos^2 A - 1$$
$$\cos 90^\circ = 2\cos^2 45^\circ - 1$$

Example 3: Expand $\cos \frac{2\pi}{3}$ using $\cos 2A = \cos^2 A - \sin^2 A$

$$\cos 2A = \cos^2 A - \sin^2 A$$
$$\cos\frac{2\pi}{3} = \cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}$$

Example 4: Expand cos 8x using $\cos 2A = 1 - 2\sin^2 A$

$$\cos 2A = 1 - 2\sin^2 A$$
$$\cos 8x = 1 - 2\sin^2 4x$$

In the following examples, you must work backwards to condense the identity:

Example 5: Condense: $1 - 2\sin^2 150°$

$\cos 2A = 1 - 2\sin^2 A$
$\cos 300^\circ = 1 - 2\sin^2 150^\circ$

What goes on the left side is double the angle on the right.

Example 6: Condense: $2\sin 3x\cos 3x$

$\sin 2A = 2\sin A\cos A$
$\sin 6x = 2\sin 3x\cos 3x$

Example 7: Condense: $\cos^2\pi - \sin^2\pi$

$\cos 2A = \cos^2 A - \sin^2 A$
$\cos 2\pi = \cos^2 \pi - \sin^2 \pi$

Questions:

1) Expand $\sin 180^\circ$ using $\sin 2A = 2\sin A\cos A$

2) Expand $\cos\frac{\pi}{3}$ using $\cos 2A = 1 - 2\sin^2 A$

3) Expand $\cos 16x$ using $\cos 2A = \cos^2 A - \sin^2 A$

4) Expand $\cos\frac{\pi}{2}$ using $\cos 2A = 2\cos^2 A - 1$

5) Condense: $2\cos^2\frac{2\pi}{3} - 1$

6) Condense: $\cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6}$

7) Condense: $\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

1) $\sin 180^\circ$

$\sin 2A = 2\sin A\cos A$

$\sin 180^\circ = 2\sin 90^\circ \cos 90^\circ$

2) $\cos\frac{\pi}{3}$

$\cos 2A = 1 - 2\sin^2 A$

$\cos\frac{\pi}{3} = 1 - 2\sin^2\frac{\pi}{6}$

3) $\cos 16x$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos 16x = \cos^2 8x - \sin^2 8x$

4) $\cos\frac{\pi}{2}$

$\cos 2A = 2\cos^2 A - 1$

$\cos\frac{\pi}{2} = 2\cos^2\frac{\pi}{4} - 1$

5) $2\cos^2\frac{2\pi}{3} - 1$

$\cos 2A = 2\cos^2 A - 1$

$\cos\frac{4\pi}{3} = 2\cos^2\frac{2\pi}{3} - 1$

6) $\cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6}$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos\frac{\pi}{3} = \cos^2\frac{\pi}{6} - \sin^2\frac{\pi}{6}$

7) $\cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

$\cos 2A = \cos^2 A - \sin^2 A$

$\cos x = \cos^2\frac{1}{2}x - \sin^2\frac{1}{2}x$

In the following examples, the double-angle identities will be used in completing proofs.

Example 1: Prove $\sin 2x + \cos x = \cos x(2\sin x + 1)$

$$\sin 2x + \cos x$$
$$= 2\sin x\cos x + \cos x$$
$$= \cos x(2\sin x + 1)$$

Example 2: Prove: $\cos 2x = \cos^2 x - \sin^2 x$ Hint: Write $\cos 2x$ as $\cos(x + x)$

$$\cos(x+x) = \cos x\cos x - \sin x\sin x$$
$$\cos(x+x) = \cos^2 x - \sin^2 x$$

Questions:

1) $\cos 2x + \cos x = (2\cos x - 1)(\cos x + 1)$

2) $2\cos 2x - \sin x + 1 = -(4\sin x - 3)(\sin x + 1)$

3) $\cos 2x = 2\cos^2 x - 1$

4) $\cos 2x = 1 - 2\sin^2 x$

5) $\frac{1+\cos 2x}{\sin 2x}=\cot x$

6) $\frac{2}{1+\cos 2x}=\sec^2 x$

7) $\sin 2x=2\sin x\cos x$

8) $\cos 2x-1+2\sin x=2\sin x(1-\sin x)$

9) $(\sin x+\cos x)^2=1+\sin 2x$

10) $\sin(x-y)\sin(x+y)=\cos^2 y-\cos^2 x$

TRIGONOMETRY II - LESSON 3

PART VII DOUBLE ANGLE PROOFS

1) $\cos 2x + \cos x$

$= 2\cos^2 x - 1 + \cos x$

$= 2\cos^2 x + \cos x - 1$

$= (2\cos x - 1)(\cos x + 1)$

2) $2\cos 2x - \sin x + 1$

$= 2(1 - 2\sin^2 x) - \sin x + 1$

$= 2 - 4\sin^2 x - \sin x + 1$

$= -4\sin^2 x - \sin x + 3$

$= -(4\sin^2 x + \sin x - 3)$

$= -(4\sin x - 3)(\sin x + 1)$

3) $\cos 2x = \cos(x + x)$

$\cos(x + x) = \cos x \cos x - \sin x \sin x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = \cos^2 x - (1 - \cos^2 x)$

$\cos 2x = \cos^2 x - 1 + \cos^2 x$

$\cos 2x = 2\cos^2 x - 1$

4) $\cos 2x = 1 - 2\sin^2 x$

$\cos(x + x) = \cos x \cos x - \sin x \sin x$

$\cos 2x = \cos^2 x - \sin^2 x$

$\cos 2x = (1 - \sin^2 x) - \sin^2 x$

$\cos 2x = 1 - \sin^2 x - \sin^2 x$

$\cos 2x = 1 - 2\sin^2 x$

5) $\dfrac{1 + \cos 2x}{\sin 2x}$

$= \dfrac{1 + (2\cos^2 x - 1)}{2\sin x \cos x}$

$= \dfrac{2\cos^2 x}{2\sin x \cos x}$

6) $\dfrac{2}{1 + \cos 2x} = \sec^2 x$

$= \dfrac{2}{1 + (2\cos^2 x - 1)}$

$= \dfrac{2}{2\cos^2 x}$

$= \dfrac{1}{\cos^2 x}$

$= \sec^2 x$

7) $\sin 2x = \sin(x + x)$

$\sin(x + x) = \sin x \cos x + \cos x \sin x$

$= 2\sin x \cos x$

8) $\cos 2x - 1 + 2\sin x$

$= (1 - 2\sin^2 x) - 1 + 2\sin x$

$= -2\sin^2 x + 2\sin x$

$= 2\sin x(-\sin x + 1)$

$= 2\sin x(1 - \sin x)$

9) $(\sin x + \cos x)^2$

$= \sin^2 x + 2\sin x \cos x + \cos^2 x$

$= \sin^2 x + \cos^2 x + 2\sin x \cos x$

$= 1 + 2\sin x \cos x$

10)

$\sin(x - y)\sin(x + y) = \cos^2 y - \cos^2 x$

$= [\sin x \cos y - \cos x \sin y][\sin x \cos y + \cos x \sin y]$

$= \left(\sin x \cos y\right)^2 + \sin x \cos y \cos x \sin y - \cos x \sin y \sin x \cos y - (\cos x \sin y)^2$

$= \sin^2 x \cos^2 y - \cos^2 x \sin^2 y$

$= \sin^2 x \cos^2 y - (1 - \sin^2 x)(1 - \cos^2 y)$

$= \sin^2 x \cos^2 y - [1 - \cos^2 y - \sin^2 x + \sin^2 x \cos^2 y]$

$= \sin^2 x \cos^2 y - 1 + \cos^2 y + \sin^2 x - \sin^2 x \cos^2 y$

$= \cos^2 y + \sin^2 x - 1$

$= \cos^2 y - \left(1 - \sin^2 x\right)$

$= \cos^2 y - \cos^2 x$

Tangent Identities: *The following identities may be used in the same manner as the sine and cosine identities.*

$$\tan(A-B)=\frac{\tan A-\tan B}{1+\tan A\tan B}$$
$$\tan(A+B)=\frac{\tan A+\tan B}{1-\tan A\tan B}$$
$$\tan 2A=\frac{2\tan A}{1-\tan^2 A}$$

Example 1: Determine the exact value of $\tan(15°)$

Rewrite as $\tan(45°-30°)$ and use the formula $\tan(A-B)=\dfrac{\tan A-\tan B}{1+\tan A\tan B}$

$$\tan\left(45^o-30^0\right)=\frac{\tan 45^o-\tan 30^0}{1+\tan 45^o\tan 30^0}$$

$$=\frac{1-\frac{1}{\sqrt{3}}}{1+(1)\left(\frac{1}{\sqrt{3}}\right)}=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}$$

$$=\frac{\frac{\sqrt{3}}{\sqrt{3}}-\frac{1}{\sqrt{3}}}{\frac{\sqrt{3}}{\sqrt{3}}+\frac{1}{\sqrt{3}}}$$ Get a common denominator for top & bottom

$$=\frac{\frac{\sqrt{3}-1}{\sqrt{3}}}{\frac{\sqrt{3}+1}{\sqrt{3}}}$$

$$=\frac{\sqrt{3}-1}{\sqrt{3}}\times\frac{\sqrt{3}}{\sqrt{3}+1}$$ Divide fractions by multiplying the reciprocal

$$=\mathbf{\frac{\sqrt{3}-1}{\sqrt{3}+1}}$$

Evaluate the required tan ratios before doing the calculation on the left.

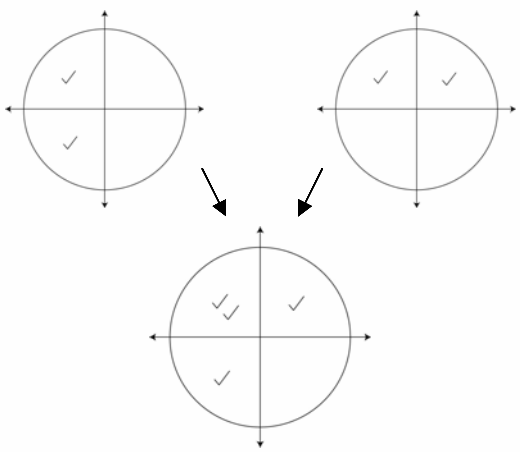
$$\tan 30^o=\frac{\sin 30^o}{\cos 30^o}=\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}}=\frac{1}{2}\times\frac{2}{\sqrt{3}}=\frac{1}{\sqrt{3}}$$

$$\tan 45^o=\frac{\sin 45^o}{\cos 45^o}=\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}}=1$$

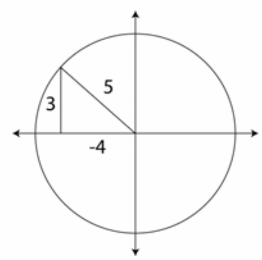
Example 2: If $\cos\theta = -\frac{4}{5}$ and $\csc\theta > 0$, determine the exact value of $\tan 2\theta$

First determine which quadrant the angle is found in:

$\cos\theta < 0$ $\csc\theta > 0$



Now fill in the triangle and use Pythagoras to determine the unknown side:



$$a^2 + b^2 = c^2$$
$$(-4)^2 + b^2 = 5^2$$
$$16 + b^2 = 25$$
$$b^2 = 9$$
$$b = 3$$

From the triangle,

$$\sin\theta = \frac{3}{5}$$
$$\cos\theta = -\frac{4}{5}$$
$$\tan\theta = -\frac{3}{4}$$

Use the formula: $\tan 2\theta = \dfrac{2\tan\theta}{1-\tan^2\theta}$

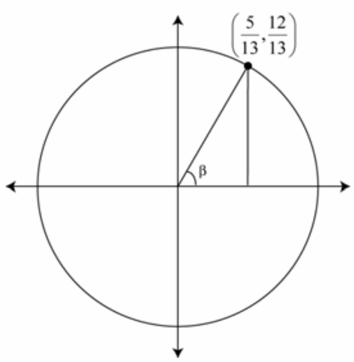
$$\tan 2\theta = \frac{2\left(-\frac{3}{4}\right)}{1-\left(-\frac{3}{4}\right)^2}$$
$$= \frac{-\frac{6}{4}}{1-\frac{9}{16}}$$
$$= \frac{-\frac{6}{4}}{\frac{16}{16}-\frac{9}{16}}$$
$$= \frac{-\frac{6}{4}}{\frac{7}{16}}$$
$$= -\frac{6}{4} \times \frac{16}{7}$$
$$= -\mathbf{\frac{24}{7}}$$

Questions:

1) Determine the exact value of $\tan(75°)$

2) Determine the exact value of $\tan(-15°)$

3) Given the point shown on the circle, determine the value of $\tan 2\beta$



Answers:

1) $\frac{\sqrt{3}+1}{\sqrt{3}-1}$

2) $\frac{1-\sqrt{3}}{1+\sqrt{3}}$

3) $-\frac{120}{119}$