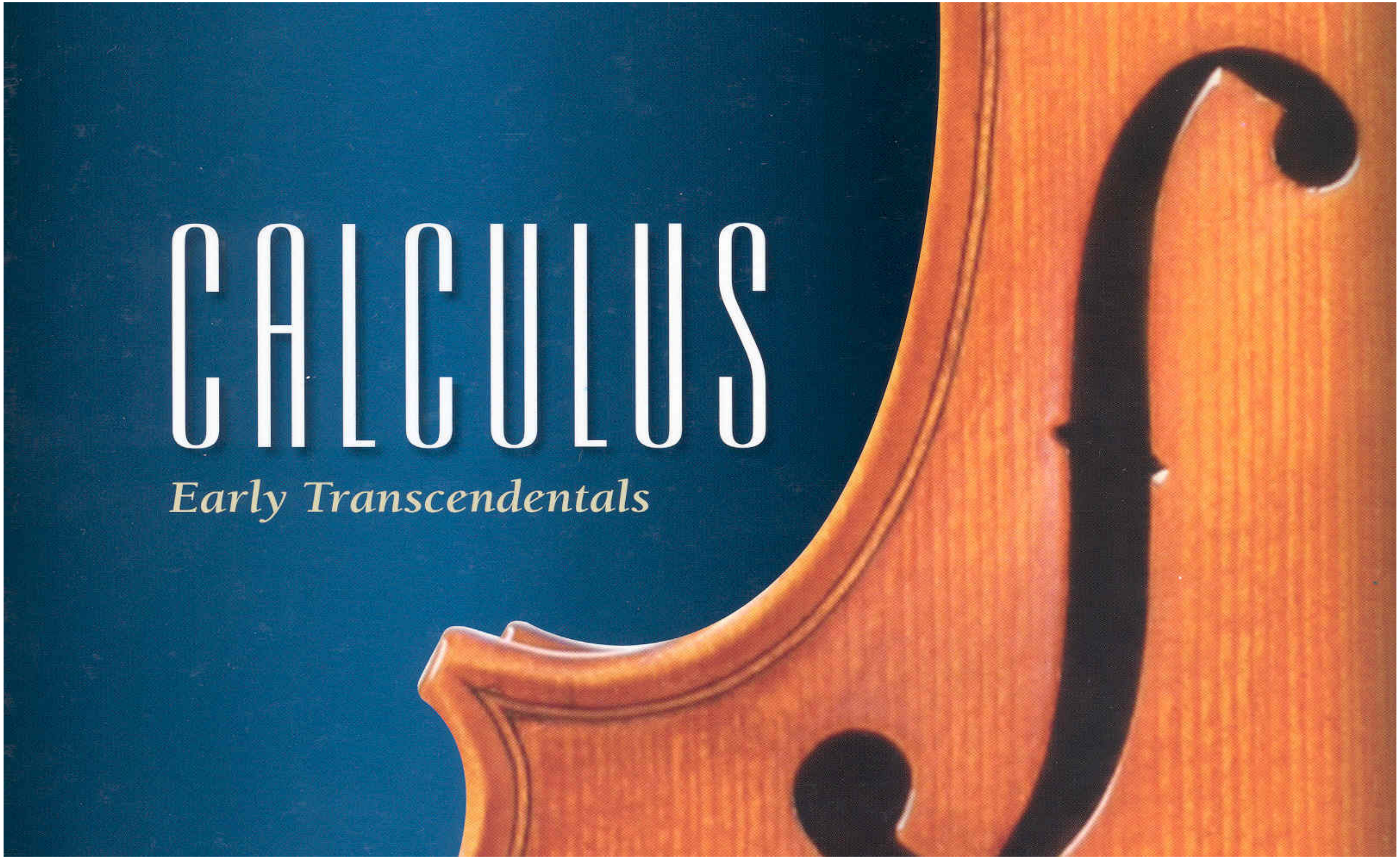


Tutorial Lab 2

11–30 Evaluate the limit, if it exists.

20. $\lim_{h \to 0} \frac{(2+h)^3 - 8}{h}$

(sol) $$\lim_{h \to 0} \frac{(2+h)^3-8}{h} = \lim_{h \to 0} \frac{\left(8+12h+6h^2+h^3\right)-8}{h} = \lim_{h \to 0} \frac{12h+6h^2+h^3}{h}$$

$$= \lim_{h \to 0} \left(12+6h+h^2\right) = 12+0+0 = 12$$

22. $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h}$

(sol) $\lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} = \lim_{h\to 0} \frac{\sqrt{1+h}-1}{h} \cdot \frac{\sqrt{1+h}+1}{\sqrt{1+h}+1} = \lim_{h\to 0} \frac{(1+h)-1}{h\left(\sqrt{1+h}+1\right)}$

$$= \lim_{h\to 0} \frac{h}{h\left(\sqrt{1+h}+1\right)}$$

$$= \lim_{h\to 0} \frac{1}{\sqrt{1+h}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{2}$$

26. $\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2 + t} \right)$

(sol) $\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2+t} \right) = \lim_{t \to 0} \frac{(t^2+t)-t}{t(t^2+t)} = \lim_{t \to 0} \frac{t^2}{t \cdot t(t+1)} = \lim_{t \to 0} \frac{1}{t+1} = \frac{1}{0+1} = 1$

29. $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right)$

(sol) $\lim_{t \to 0} \left(\frac{1}{t\sqrt{1+t}} - \frac{1}{t} \right) = \lim_{t \to 0} \frac{1-\sqrt{1+t}}{t\sqrt{1+t}} = \lim_{t \to 0} \frac{(1-\sqrt{1+t})(1+\sqrt{1+t})}{t\sqrt{t+1}(1+\sqrt{1+t})}$

$= \lim_{t \to 0} \frac{-t}{t\sqrt{1+t}(1+\sqrt{1+t})} = \lim_{t \to 0} \frac{-1}{\sqrt{1+t}(1+\sqrt{1+t})} = \frac{-1}{\sqrt{1+0}(1+\sqrt{1+0})} = -\frac{1}{2}$

47. Let $F(x) = \dfrac{x^2 - 1}{|x - 1|}$.

(a) Find

(i) $\lim_{x \to 1^+} F(x)$ (ii) $\lim_{x \to 1^-} F(x)$

(sol) **(a)**

(i) $$\lim_{x\to 1^+} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^+} \frac{x^2-1}{x-1} = \lim_{x\to 1^+} (x+1) = 2$$

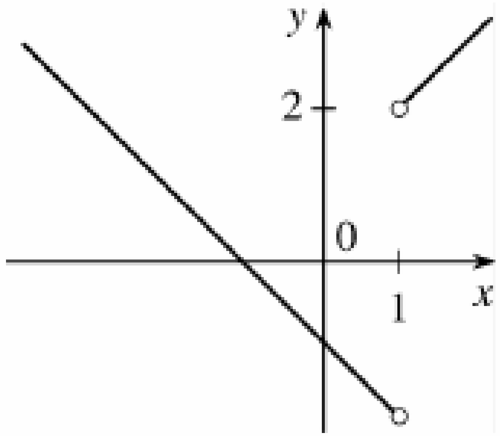
(ii) $$\lim_{x\to 1^-} \frac{x^2-1}{|x-1|} = \lim_{x\to 1^-} \frac{x^2-1}{-(x-1)} = \lim_{x\to 1^-} -(x+1) = -2$$

(b) Does $\lim_{x \to 1} F(x)$ exist?

(c) Sketch the graph of F.

(sol) **(b)** No, $\lim_{x \to 1} F(x)$ does not exist since $\lim_{x \to 1^+} F(x) \neq \lim_{x \to 1^-} F(x)$.

(c)

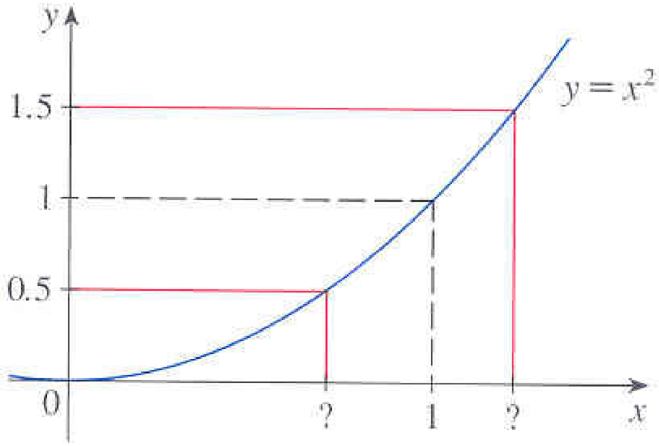


55. If $\lim_{x \to 1} \dfrac{f(x) - 8}{x - 1} = 10$, find $\lim_{x \to 1} f(x)$.

(sol)

4. Use the given graph of $f(x) = x^2$ to find a number δ such that

$$\text{if} \quad |x - 1| < \delta \quad \text{then} \quad |x^2 - 1| < \tfrac{1}{2}$$



(sol) The left hand question mark is the positive solution of $x^2 = \frac{1}{2}$, that is, $x = \frac{1}{\sqrt{2}}$, and the right hand question mark is the positive solution of $x^2 = \frac{3}{2}$, that is, $x = \sqrt{\frac{3}{2}}$.

On the left side, we need $|x - 1| < \left|\frac{1}{\sqrt{2}} - 1\right| \approx 0.292$.

On the right side, we need $|x - 1| < \left|\sqrt{\frac{3}{2}} - 1\right| \approx 0.224$.

The more restrictive of these two conditions must apply, so we choose $\delta = 0.224$ (or any smaller positive number).

7. For the limit

$$\lim_{x \to 1} (4 + x - 3x^3) = 2$$

illustrate Definition 2 by finding values of δ that correspond to $\varepsilon = 1$ and $\varepsilon = 0.1$.

(sol) For $\epsilon = 1$, the definition of a limit requires that we find δ such that

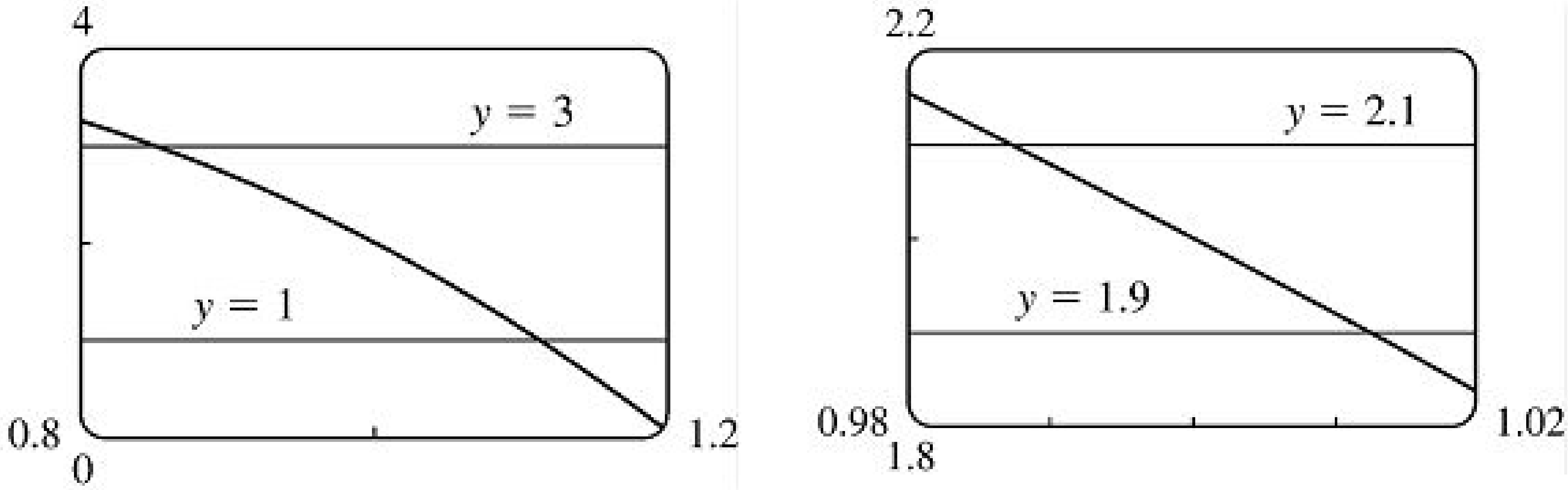
$|(4 + x - 3x^3 - 2)| < 1 \Leftrightarrow 1 < 4 + x - 3x^3 < 3$ whenever $0 < |x - 1| < \delta$.

If we plot the graphs of $y = 1$, $y = 4 + x - 3x^3$ and $y = 3$ on the same screen, we see that we need $0.86 \le x \le 1.11$. So since $|1 - 0.86| = 0.14$ and $|1 - 1.11| = 0.11$, we choose $\delta = 0.11$ (or any smaller positive number). For $\epsilon = 0.1$, we must find δ such that

$|(4 + x - 3x^3 - 2)| < 0.1 \Leftrightarrow 1.9 < 4 + x - 3x^3 < 2.1$ whenever $0 < |x - 1| < \delta$.

From the graph, we see that we need $0.988 \le x \le 1.012$.

So since $|1-0.988| = 0.012$ and $|1-1.012| = 0.012$, we choose $\delta = 0.012$ (or any smaller positive number) for the inequality to hold.



36. Prove that $\lim_{x\to 2}\frac{1}{x}=\frac{1}{2}$.

(pf) 1. Guessing a value for δ .

Let $\epsilon>0$ be given.

We have to find a number $\delta>0$ such that $|\frac{1}{x}-\frac{1}{2}|<\epsilon$

whenever $0<|x-2|<\delta$. But $|\frac{1}{x}-\frac{1}{2}|=|\frac{2-x}{2x}|=|\frac{x-2}{2x}|<\epsilon$.

We find a positive constant C such that

$\frac{1}{|2x|}<C\Rightarrow\frac{|x-2|}{|2x|}<C|x-2|$ and we can make $C|x-2|<\epsilon$

by taking $|x-2|<\frac{\epsilon}{C}=\delta$.

We restrict to lie in the interval $|x-2|<1 \Rightarrow 1<x<3$

so $1>\frac{1}{x}>\frac{1}{3}\Rightarrow\frac{1}{6}<\frac{1}{2x}<\frac{1}{2}\Rightarrow\frac{1}{|2x|}<\frac{1}{2}$. So $C=\frac{1}{2}$ is suitable.

Thus, we should choose $\delta=\min\{1,2\epsilon\}$.

2. Showing that δ works Given $\epsilon>0$ we let $\delta=\min\{1,2\epsilon\}$.

If $0<|x-2|<\delta$, then (as in part 1). Also $|x-2|<2\epsilon$,

so $\left|\frac{1}{x}-\frac{1}{2}\right|=\frac{|x-2|}{|2x|}<\frac{1}{2}2\epsilon=\epsilon|x-2|<1\Rightarrow 1<x<3\Rightarrow\frac{1}{|2x|}<\frac{1}{2}$.

This shows that $\lim_{x\to 2}(1/x)=\frac{1}{2}$

13–14 Use the definition of continuity and the properties of limits to show that the function is continuous on the given interval.

13. $f(x) = \dfrac{2x+3}{x-2}, \quad (2, \infty)$

(pf) For $a>2$, we have $\lim_{x\to a} f(x) = \lim_{x\to a} \dfrac{2x+3}{x-2} = \dfrac{\lim_{x\to a}(2x+3)}{\lim_{x\to a}(x-2)}$ [Limit Law 5]

$= \dfrac{2\lim_{x\to a} x + \lim_{x\to a} 3}{\lim_{x\to a} x - \lim_{x\to a} 2}$ [1, 2, and 3] $= \dfrac{2a+3}{a-2}$ [7 and 8] $= f(a)$.

Thus, f is continuous at $x=a$ for every a in $(2,\infty)$;

that is, f is continuous on $(2,\infty)$.

31–34 Use continuity to evaluate the limit.

33. $\lim_{x \to 1} e^{x^2 - x}$

(sol) Because $x^2 - x$ is continuous on R,

the composite function $f(x) = e^{x^2 - x}$ is continuous on R,

so $\lim_{x \to 1} f(x) = f(1) = e^{1-1} = e^0 = 1.$

55. Prove that f is continuous at a if and only if

$$\lim_{h \to 0} f(a + h) = f(a)$$

(pf) ($\Rightarrow$) If f is continuous at a, then by Theorem 8 with $g(h)=a+h$,

we have $\lim_{h\to 0} f(a+h)=f\left(\lim_{h\to 0} (a+h)\right)=f(a)$.

($\Leftarrow$) Let $\varepsilon>0$. Since $\lim_{h\to 0} f(a+h)=f(a)$, there exists $\delta>0$ such that

$$0<|h|<\delta \Rightarrow |f(a+h)-f(a)|<\varepsilon \ .$$

So if $0<|x-a|<\delta$, then $|f(x)-f(a)|=|f(a+(x-a))-f(a)|<\varepsilon$.

Thus, $\lim_{x\to a} f(x)=f(a)$ and so f is continuous at a.

59. For what values of x is f continuous?

$$f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$$

(sol) $f(x) = \begin{cases} 0 & \text{if } x \text{ is rational} \\ 1 & \text{if } x \text{ is irrational} \end{cases}$ is continuous nowhere.

For, given any number a and any $\delta > 0$, the interval $(a-\delta, a+\delta)$ contains both infinitely many rational and infinitely many irrational numbers.

Since $f(a)=0$ or 1 , there are infinitely many numbers x with $0 < |x-a| < \delta$ and $|f(x)-f(a)| = 1$.

Thus, $\lim_{x \to a} f(x) \neq f(a)$. [In fact $\lim_{x \to a} f(x)$ does not even exist.]

5–8 Find an equation of the tangent line to the curve at the given point.

7. $y = \sqrt{x}, \quad (1, 1)$

(sol) $m = \lim_{x \to 1} \frac{\sqrt{x} - \sqrt{1}}{x - 1} = \lim_{x \to 1} \frac{x - 1}{(x-1)(\sqrt{x}+1)}$

$= \lim_{x \to 1} \frac{1}{\sqrt{x}+1} = \frac{1}{2}$

The equation of tangent line is $y - 1 = \frac{1}{2}(x - 1)$

$\therefore \; y = \frac{1}{2}x + \frac{1}{2}$

9. (a) Find the slope of the tangent to the curve $y = 3 + 4x^2 - 2x^3$ at the point where $x = a$.

(b) Find equations of the tangent lines at the points (1, 5) and (2, 3).

(c) Graph the curve and both tangents on a common screen.

(sol) (a) $m = \lim_{x \to a} \dfrac{(3+4x^2-2x^3)-(3+4a^2-2a^3)}{x-a} = \lim_{x \to a} \dfrac{4(x^2-a^2)-2(x^3-a^3)}{x-a}$

$$= \lim_{x \to a} \frac{4(x-a)(x+a)-2(x-a)(x^2+ax+a^2)}{x-a}$$

$$= \lim_{x \to a} (x-a) \cdot \frac{4(x+a)-2(x^2+ax+a^2)}{x-a}$$

$$= \lim_{x \to a} 4(x+a)-2(x^2+ax+a^2) = 8a-6a^3$$

(b) At (1,5) : $m = 8 \cdot 1 - 6(1)^2 = 2$,

so an equation of the tangent line is

$y - 5 = 2(x - 1) \Leftrightarrow y = 2x + 3$

At (2,3) : $m = 8 \cdot 2 - 6(2)^2 = -8$

so an equation of the tangent line is

$y - 3 = -8(x - 2) \Leftrightarrow y = -8x + 19$

25–30 Find $f'(a)$.

28. $f(x) = \dfrac{x^2+1}{x-2}$

(sol) $$f'(a) = \lim_{h\to 0}\frac{f(a+h)-f(a)}{h} = \lim_{h\to 0}\frac{\frac{(a+h)^2+1}{(a+h)-2}-\frac{a^2+1}{a-2}}{h}$$

$$= \lim_{h\to 0}\frac{(a^2+2ah+h^2+1))(a-2)-(a^2+1)(a+h-2)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0}\frac{(a^3-2a^2+2a^2h-4ah+ah^2-2h^2+a-2)-(a^3+a^2h-2a^2+a+h-2)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0}\frac{a^2h-4ah+ah^2-2h^2-h}{h(a+h-2)(a-2)} = \lim_{h\to 0}\frac{h(a^2-4a+ah-2h-1)}{h(a+h-2)(a-2)}$$

$$= \lim_{h\to 0}\frac{a^2-4a+ah-2h-1}{(a+h-2)(a-2)} = \frac{a^2-4a-1}{(a-2)^2}$$

51–52 Determine whether $f'(0)$ exists.

51. $f(x) = \begin{cases} x \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(sol) Since $f(x)=x\sin(1/x)$ when $x\neq 0$ and $f(0)=0$, we have

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h\sin(1/h)-0}{h}=\lim_{h\to 0}(\sin(1/h)) .$$

This limit does not exist since $\sin(1/h)$ takes the values -1 and 1

on any interval containing 0.

(Compare with Example 4 in Section 2.2.)

52. $f(x) = \begin{cases} x^2 \sin \dfrac{1}{x} & \text{if } x \neq 0 \\ 0 & \text{if } x = 0 \end{cases}$

(sol) Since $f(x)=x^2 sin(1/x)$ when $x\neq 0$ and $f(0)=0$, we have

$$f'(0)=\lim_{h\to 0}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0}\frac{h^2\sin\ (1/h)-0}{h}=\lim_{h\to 0}(h\sin(1/h)).$$

Since $-1\leq sin\frac{1}{h}\leq 1$, we have

$$-|h|\leq |h|\,sin\frac{1}{h}\leq |h| \Rightarrow -|h|\leq h sin\frac{1}{h}\leq |h|.$$

Because $\lim_{h\to 0}(-|h|)=0$ and $\lim_{h\to 0}|h|=0$, we know that

$\lim_{h\to 0}(h\, sin\frac{1}{h})=0$ by the Squeeze Theorem. Thus, $f'(0)=0$.

19–29 Find the derivative of the function using the definition of derivative. State the domain of the function and the domain of its derivative.

27. $G(t) = \dfrac{4t}{t+1}$

(sol) $$G'(t) = \lim_{h\to 0}\frac{G(t+h)-G(t)}{h} = \lim_{h\to 0}\frac{\dfrac{4(t+h)}{(t+h)+1}-\dfrac{4t}{t+1}}{h} = \lim_{h\to 0}\frac{\dfrac{4(t+h)(t+1)-4t(t+h+1)}{(t+h+1)(t+1)}}{h}$$

$$= \lim_{h\to 0}\frac{(4t^2+4ht+4t+4h)-(4t^2+4ht+4t)}{h(t+h+1)(t+1)}$$

$$= \lim_{h\to 0}\frac{4h}{h(t+h+1)(t+1)} = \lim_{h\to 0}\frac{4}{(t+h+1)(t+1)} = \frac{4}{(t+1)^2}$$

Domain of G = domain of $G' = (-\infty,-1)\cup(-1,\infty)$.

29. $f(x) = x^4$

(sol)

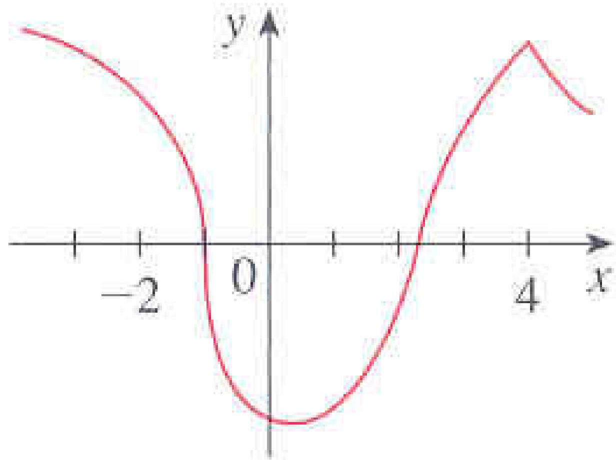
$$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^4 - x^4}{h} = \lim_{h\to 0}\frac{(x^4+4x^3h+6x^2h^2+4xh^3+h^4)-x^4}{h}$$

$$= \lim_{h\to 0}\frac{4x^3h+6x^2h^2+4xh^3+h^4}{h} = \lim_{h\to 0}(4x^3+6x^2h+4xh^2+h^3) = 4x^3$$

Domain of f = domain of $f' = R$.

35–38 The graph of f is given. State, with reasons, the numbers at which f is not differentiable.

37.



(sol) f is not differentiable at $x = 4$,

because the graph has a corner there.

f is not differentiable at $x = -1$,

because the graph has vertical tangents at that point.

45–46 Use the definition of a derivative to find $f'(x)$ and $f''(x)$. Then graph f, f', and f'' on a common screen and check to see if your answers are reasonable.

45. $f(x) = 1 + 4x - x^2$ **46.** $f(x) = 1/x$

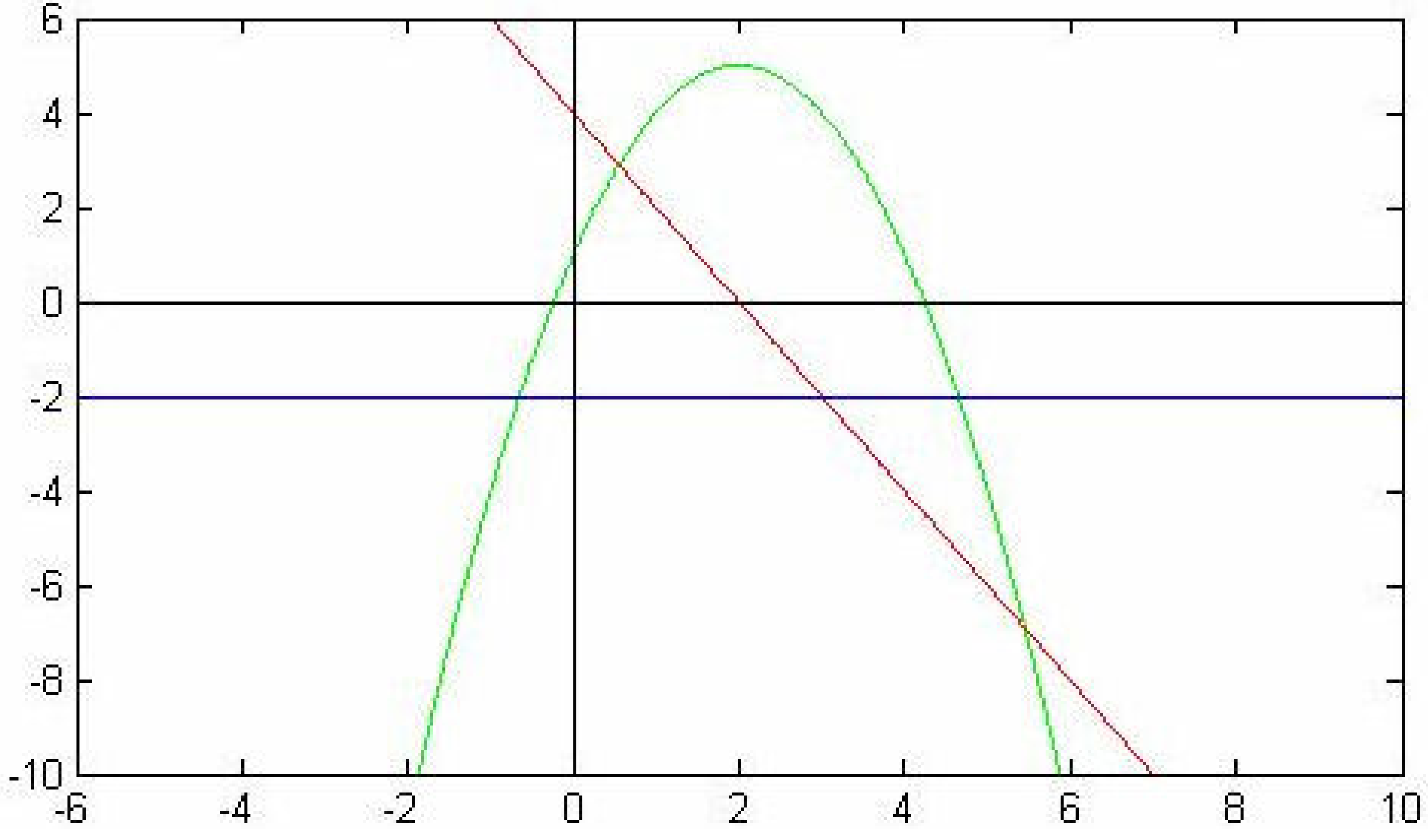
(sol) $$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1 + 4(x+h) - (x+h)^2 - (1 + 4x - x^2)}{h}$$

$$= \lim_{h \to 0} \frac{1}{h}(1 + 4x + 4h - x^2 - 2xh - h^2 - 1 - 4x + x^2)$$

$$= \lim_{h \to 0} \frac{1}{h}(4h - 2xh - h^2) = \lim_{h \to 0}(4 - 2x + h) = 4 - 2x$$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{1}{h}(4 - 2(x+h) - (4 - 2x))$$

$$= \lim_{h \to 0} \frac{1}{h}(-2h) = -2$$


6
4
2
0
-2
-4
-6
-8
-10
-6
-4
-2
0
2
4
6
8
10

(sol) $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left(\frac{1}{x+h} - \frac{1}{x}\right)$

$= \lim_{h \to 0} \frac{1}{h}\left(\frac{x - x - h}{x(x+h)}\right) = \lim_{h \to 0} \frac{-1}{x(x+h)} = -\frac{1}{x^2}$

$$f''(x) = \lim_{h \to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h \to 0} \frac{1}{h}\left(-\frac{1}{(x+h)^2} + \frac{1}{x^2}\right)$$

$$= \lim_{h \to 0} \frac{1}{h}\left(\frac{(x+h)^2 - x^2}{x^2(x+h)^2}\right) = \lim_{h \to 0} \frac{1}{h}\left(\frac{2xh + h^2}{x^2(x+h)^2}\right)$$

$$= \lim_{h \to 0}\left(\frac{2x+h}{x^2(x+h)^2}\right) = \frac{2x}{x^4} = \frac{2}{x^3}$$

3–26 Differentiate.

20. $z = w^{3/2}(w + ce^{w})$

(sol) $z = w^{3/2}(w + ce^{w}) = w^{5/2} + cw^{3/2}e^{w}$

$$\Rightarrow z' = \frac{5}{2}w^{3/2} + c\left(w^{3/2} \cdot e^{w} + e^{w} \cdot \frac{3}{2}w^{1/2}\right) = \frac{5}{2}w^{3/2} + \frac{1}{2}cw^{1/2}e^{w}(2w+3)$$

21. $f(t) = \dfrac{2t}{2 + \sqrt{t}}$

(sol) $$f'(t) = \frac{2(2+\sqrt{t}) - 2t\frac{1}{2}t^{-\frac{1}{2}}}{(2+\sqrt{t})^2} = \frac{4+2\sqrt{t} - \sqrt{t}}{(2+\sqrt{t})^2} = \frac{4+\sqrt{t}}{(2+\sqrt{t})^2}$$

33–34 Find equations of the tangent line and normal line to the given curve at the specified point.

34. $y = \dfrac{\sqrt{x}}{x+1}$, $(4, 0.4)$

(sol) $y = \dfrac{\sqrt{x}}{x+1} \Rightarrow y' = \dfrac{(x+1)\left(\dfrac{1}{2\sqrt{x}}\right) - \sqrt{x}\,(1)}{(x+1)^2} = \dfrac{(x+1)-(2x)}{2\sqrt{x}\,(x+1)^2} = \dfrac{1-x}{2\sqrt{x}\,(x+1)^2}$.

At $(4, 0.4)$, $y' = \dfrac{-3}{100} = -0.03$, and an equation of the tangent line is

$$y - 0.4 = -0.03(x-4), \text{ or } y = -0.03x + 0.52.$$

36. (a) The curve $y = x/(1 + x^2)$ is called a **serpentine**. Find an equation of the tangent line to this curve at the point (3, 0.3).

(sol) **(a)** $y=f(x)=\dfrac{x}{1+x^2} \Rightarrow f'(x)=\dfrac{(1+x^2)1-x(2x)}{(1+x^2)^2}=\dfrac{1-x^2}{(1+x^2)^2}$

So the slope of the tangent line at the point

(3,0.3) is $f'(3)=\dfrac{-8}{100}$ and its equation is $y-0.3=-0.08(x-3)$

or $y=-0.08x+0.54$.

43. Suppose that $f(5) = 1$, $f'(5) = 6$, $g(5) = -3$, and $g'(5) = 2$. Find the following values.

(a) $(fg)'(5)$ (b) $(f/g)'(5)$

(c) $(g/f)'(5)$

(sol) We are given that $f(5)=1$, $f'(5)=6$, $g(5)=-3$, and $g'(5)=2$.

(a) $(fg)'(5)=f(5)g'(5)+g(5)f'(5)=(1)(2)+(-3)(6)=2-18=-16$

(b) $\left(\frac{f}{g}\right)'(5)=\frac{g(5)f'(5)-f(5)g'(5)}{[g(5)]^2}=\frac{(-3)(6)-(1)(2)}{(-3)^2}=-\frac{20}{9}$

(c) $\left(\frac{g}{f}\right)'(5)=\frac{f(5)g'(5)-g(5)f'(5)}{[f(5)]^2}=\frac{(1)(2)-(-3)(6)}{(1)^2}=20$

1–16 Differentiate.

2. $f(x) = x \sin x$

(sol) $f'(x) = x \cdot \cos x + (\sin x) \cdot 1 = x\cos x + \sin x$

10. $y = \dfrac{1 + \sin x}{x + \cos x}$

(sol) $$y' = \frac{(x+\cos x)(\cos x)-(1+\sin x)(1-\sin x)}{(x+\cos x)^2} = \frac{x\cos x+\cos^2 x-(1-\sin^2 x)}{(x+\cos x)^2}$$

$$= \frac{x\cos x+\cos^2 x-(\cos^2 x)}{(x+\cos x)^2} = \frac{x\cos x}{(x+\cos x)^2}$$

13. $y = \dfrac{\sin x}{x^2}$

(sol) $y' = \dfrac{x^2\cos x - (\sin x)(2x)}{\left(x^2\right)^2} = \dfrac{x(x\cos x - 2\sin x)}{x^4} = \dfrac{x\cos x - 2\sin x}{x^3}$

16. $y = x^2 \sin x \tan x$

(sol) $y' = 2x\sin x\tan x + x^2\cos x\tan x + x^2\sin x\sec^2 x$

$= 2x\sin x\tan x + x^2\sin x + x^2\sin x\sec x$

$= x\sin x(2\tan x + x + x\sec^2 x)$

21–24 Find an equation of the tangent line to the curve at the given point.

23. $y = x + \cos x$, $(0, 1)$

(sol) $y = x + \cos x \Rightarrow y' = 1 - \sin x$.

At $(0,1)$, $y' = 1$, and an equation of the tangent line is

$y - 1 = 1(x - 0)$, or $y = x + 1$.

39–48 Find the limit.

40. $\lim_{x \to 0} \frac{\sin 4x}{\sin 6x}$

(sol) $\lim_{x \to 0} \frac{\sin 4x}{\sin 6x} = \lim_{x \to 0} \left(\frac{\sin 4x}{x} \cdot \frac{x}{\sin 6x} \right) = \lim_{x \to 0} \frac{4\sin 4x}{4x} \cdot \lim_{x \to 0} \frac{6x}{6\sin 6x}$

$= 4 \lim_{x \to 0} \frac{\sin 4x}{4x} \cdot \frac{1}{6} \lim_{x \to 0} \frac{6x}{\sin 6x} = 4(1) \cdot \frac{1}{6}(1) = \frac{2}{3}$

42. $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta}$

(sol) $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\sin \theta} = \lim_{\theta \to 0} \frac{\frac{\cos \theta - 1}{\theta}}{\frac{\sin \theta}{\theta}} = \frac{\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta}}{\lim_{\theta \to 0} \frac{\sin \theta}{\theta}} = \frac{0}{1} = 0$

45. $\lim_{\theta \to 0} \frac{\sin\theta}{\theta + \tan\theta}$

(sol) Divide numerator and denominator by θ. ($\sin(\theta)$ also works.)

$$\lim_{\theta \to 0} \frac{\sin\theta}{\theta+\tan\theta} = \lim_{\theta \to 0} \frac{\frac{\sin\theta}{\theta}}{1+\frac{\sin\theta}{\theta}\cdot\frac{1}{\cos\theta}}$$

$$= \frac{\lim_{\theta \to 0}\frac{\sin\theta}{\theta}}{1+\lim_{\theta \to 0}\frac{\sin\theta}{\theta}\ \lim_{\theta \to 0}\frac{1}{\cos\theta}} = \frac{1}{1+1\cdot 1} = \frac{1}{2}$$

47. $\lim_{x \to \pi/4} \dfrac{1 - \tan x}{\sin x - \cos x}$

(sol)

$$\lim_{x \to \frac{\pi}{4}} \frac{\frac{\cos x - \sin x}{\cos x}}{\sin x - \cos x} = \lim_{x \to \frac{\pi}{4}} \frac{\cos x - \sin x}{\cos x(\sin x - \cos x)} = \lim_{x \to \frac{\pi}{4}} \left(-\frac{1}{\cos x}\right) = -\frac{1}{\frac{1}{\sqrt{2}}} = -\sqrt{2}$$

7–46 Find the derivative of the function.

28. $y = \dfrac{e^{2u}}{e^{u} + e^{-u}}$

(sol) $$y' = \frac{(e^{u}+e^{-u})(e^{2u}\cdot 2) - e^{2u}(e^{u}-e^{-u})}{(e^{u}+e^{-u})^{2}}$$

$$= \frac{e^{2u}(2e^{u}+2e^{-u}-e^{u}+e^{-u})}{(e^{u}+e^{-u})^{2}} = \frac{e^{2u}(e^{u}+3e^{-u})}{(e^{u}+e^{-u})^{2}}$$

30. $G(y) = \left(\dfrac{y^2}{y+1}\right)^5$

(sol) $G'(y) = 5\cdot\left(\dfrac{y^2}{y+1}\right)^4\cdot\left(\dfrac{y^2}{y+1}\right)'$

$$= 5\cdot\left(\frac{y^2}{y+1}\right)^4\cdot\frac{2y(y+1)-y^2}{(y+1)^2}$$

$$= \frac{5y^8(y^2+2y)}{(y+1)^6} = \frac{5y^9(y+2)}{(y+1)^6}$$

36. $f(t) = \sqrt{\dfrac{t}{t^2 + 4}}$

(sol) $f'(t) = \frac{1}{2}(\frac{t}{t^2+4})^{-\frac{1}{2}} \cdot (\frac{t}{t^2+4})'$

$= \frac{1}{2}(\frac{t}{t^2+4})^{-\frac{1}{2}} \cdot \frac{(t^2+4)-2t \cdot t}{(t^2+4)^2}$

$= -\frac{1}{2}(\frac{t}{t^2+4})^{-\frac{1}{2}} \cdot \frac{t^2-4}{(t^2+4)^2}$

38. $y = e^{k \tan \sqrt{x}}$

(sol) $y = e^{k\tan\sqrt{x}} \Rightarrow y' = e^{k\tan\sqrt{x}} \cdot \frac{d}{dx}(k\tan\sqrt{x})$

$$= e^{k\tan\sqrt{x}}\left(k\sec^2\sqrt{x} \cdot \frac{1}{2}x^{-1/2}\right)$$

$$= \frac{k\sec^2\sqrt{x}}{2\sqrt{x}}\, e^{k\tan\sqrt{x}}$$

40. $y = \sin(\sin(\sin x))$

(sol) $y' = \cos(\sin(\sin x)) \dfrac{d}{dx} (\sin(\sin x))$

$= \cos(\sin(\sin x)) \cdot \cos(\sin x) \cdot \cos x$

63. A table of values for f, g, f', and g' is given.

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	3	2	4	6
2	1	8	5	7
3	7	2	7	9

(a) If $h(x) = f(g(x))$, find $h'(1)$.
(b) If $H(x) = g(f(x))$, find $H'(1)$.

(sol) **(a)** $h(x)=f(g(x)) \Rightarrow h'(x)=f'(g(x))\cdot g'(x)$,

so $h'(1)=f'(g(1))\cdot g'(1)=f'(2)\cdot 6=5\cdot 6=30$.

(b) $H(x)=g(f(x)) \Rightarrow H'(x)=g'(f(x))\cdot f'(x)$,

so $H'(1)=g'(f(1))\cdot f'(1)=g'(3)\cdot 4=9\cdot 4=36.$

64. Let f and g be the functions in Exercise 63.

(a) If $F(x) = f(f(x))$, find $F'(2)$.

(b) If $G(x) = g(g(x))$, find $G'(3)$.

(sol) **(a)** $F(x)=f(f(x)) \Rightarrow F'(x)=f'(f(x))\cdot f'(x)$,

so $F'(2)=f'(f(2))\cdot f'(2)=f'(1)\cdot 5=4\cdot 5=20$.

(b) $G(x)=g(g(x)) \Rightarrow G'(x)=g'(g(x))\cdot g'(x)$,

so $G'(3)=g'(g(3))\cdot g'(3)=g'(2)\cdot 9=7\cdot 9=63$.

67. Suppose f is differentiable on $\mathbb{R}$. Let $F(x) = f(e^x)$ and $G(x) = e^{f(x)}$. Find expressions for (a) $F'(x)$ and (b) $G'(x)$.

(sol) **(a)** $F(x)=f(e^x) \Rightarrow F'(x)=f'(e^x)\dfrac{d}{dx}(e^x)=f'(e^x)e^x$

(b) $G(x)=e^{f(x)} \Rightarrow G'(x)=e^{f(x)}\dfrac{d}{dx}f(x)=e^{f(x)}f'(x)$

77. The displacement of a particle on a vibrating string is given by the equation

$$s(t) = 10 + \tfrac{1}{4}\sin(10\pi t)$$

where s is measured in centimeters and t in seconds. Find the velocity of the particle after t seconds.

(sol) $$s(t)=10+\frac{1}{4}\sin(10\pi t)$$

$\Rightarrow$ the velocity after t seconds is

$$v(t)=s'(t)=\frac{1}{4}\cos(10\pi t)(10\pi)=\frac{5\pi}{2}\cos(10\pi t)\ \text{cm / s.}$$

80. In Example 4 in Section 1.3 we arrived at a model for the length of daylight (in hours) in Ankara, Turkey, on the tth day of the year:

$$L(t) = 12 + 2.8 \sin\left[\frac{2\pi}{365}(t - 80)\right]$$

(sol) $L'(t) = 2.8\cos\left(\frac{2\pi}{365}(t-80)\right)\left(\frac{2\pi}{365}\right)$.

On March 21, $t=80$, and $L'(80) \approx 0.0482$ hours per day.

On May 21, $t=141$, and $L'(141) \approx 0.02398$,

which is approximately one–half of $L'(80)$.