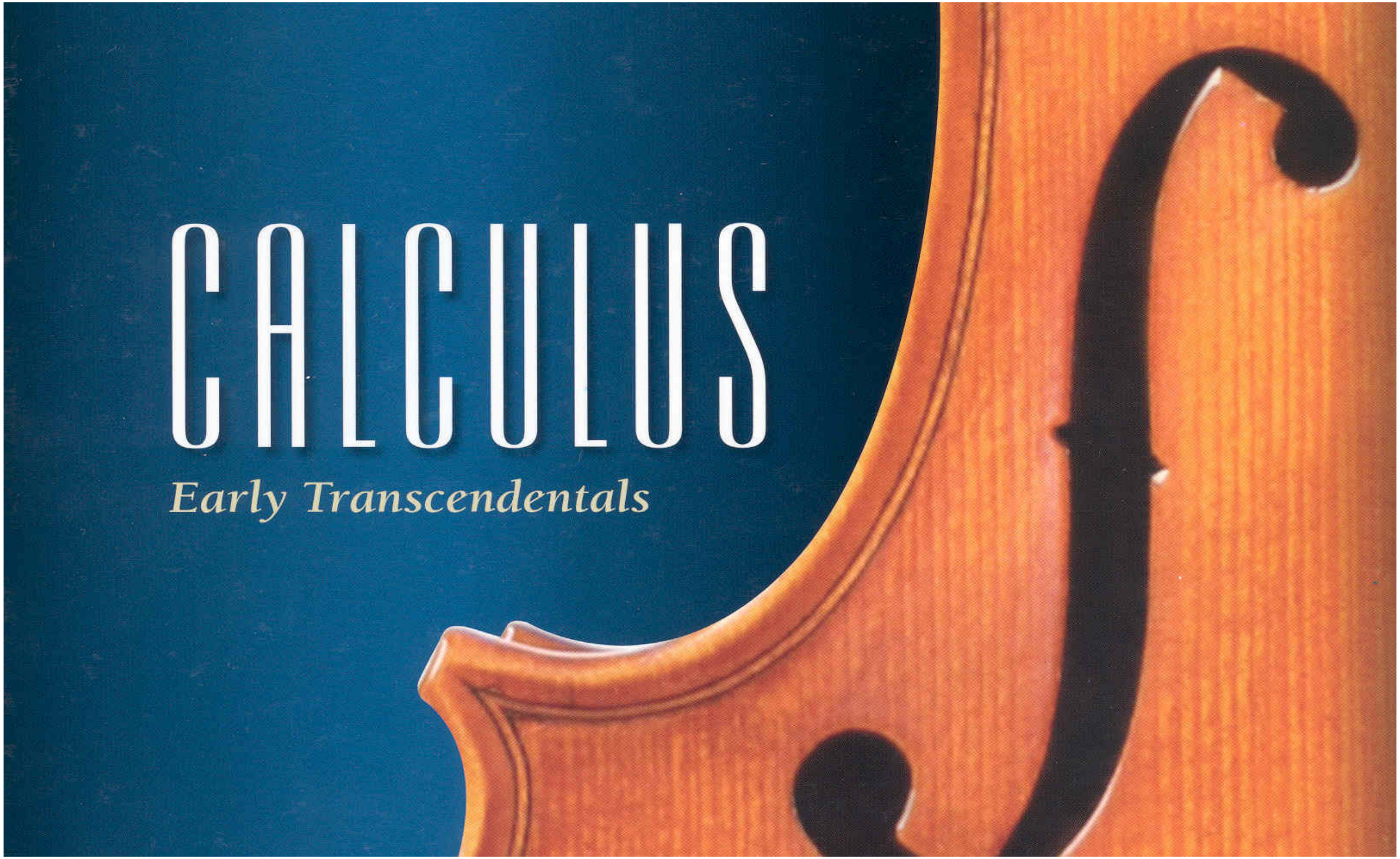


Tutorial Lab 3

5–20 Find dy/dx by implicit differentiation.

6. $2\sqrt{x} + \sqrt{y} = 3$

(sol) $2\sqrt{x} + \sqrt{y} = 3 \;\Rightarrow\; 2\frac{1}{2}x^{-\frac{1}{2}} + \frac{1}{2}y^{-\frac{1}{2}}y' = 0$

$\Rightarrow \frac{1}{\sqrt{x}} + \frac{1}{2\sqrt{y}}y' = 0 \;\Rightarrow\; \frac{1}{2\sqrt{y}}y' = -\frac{1}{\sqrt{x}} \qquad \therefore y' = -\frac{2\sqrt{y}}{\sqrt{x}}$

8. $x^2 - 2xy + y^3 = c$

(Sol) $\frac{d}{dx}(x^2 - 2xy + y^3) = \frac{d}{dx}(c) \quad \Rightarrow 2x - 2(xy' + y\cdot 1) + 3y^2y' = 0$

$\Rightarrow 2x - 2y = 2xy' - 3y^2y' \Rightarrow 2x - 2y = y'(2x - 3y^2)$

$\Rightarrow \; y' = \frac{2x-2y}{2x-3y^2}$

10. $y^5 + x^2y^3 = 1 + ye^{x^2}$

(Sol) $$\frac{d}{dx}(y^5+x^2y^3)=\frac{d}{dx}\left(1+ye^{x^2}\right)$$

$$\Rightarrow 5y^4y' + (x^2\cdot 3y^2y' + y^3\cdot 2x) = 0 + y\cdot e^{x^2}\cdot 2x + e^{x^2}\cdot y'$$

$$\Rightarrow y'\left(5y^4+3x^2y^2-e^{x^2}\right)=2xye^{x^2}-2xy^3$$

$$\Rightarrow y' = \frac{2xy\left(e^{x^2}-y^2\right)}{5y^4+3x^2y^2-e^{x^2}}$$

18. $\tan(x - y) = \dfrac{y}{1 + x^2}$

(Sol) $\tan(x-y) = \dfrac{y}{1+x^2} \Rightarrow (1+x^2)\tan(x-y) = y$

$$\Rightarrow (1+x^2)\sec^2(x-y)\cdot(1-y') + \tan(x-y)\cdot 2x = y'$$

$$\Rightarrow (1+x^2)\sec^2(x-y) - (1+x^2)\sec^2(x-y)\cdot y' + 2x\tan(x-y) = y'$$

$$\Rightarrow (1+x^2)\sec^2(x-y) + 2x\tan(x-y) = [1+(1+x^2)\sec^2(x-y)]\cdot y'$$

$$\Rightarrow y' = \frac{(1+x^2)\sec^2(x-y) + 2x\tan(x-y)}{1+(1+x^2)\sec^2(x-y)}$$

20. $\sin x + \cos y = \sin x \cos y$

(Sol) $\sin x + \cos y = \sin x \cos y$

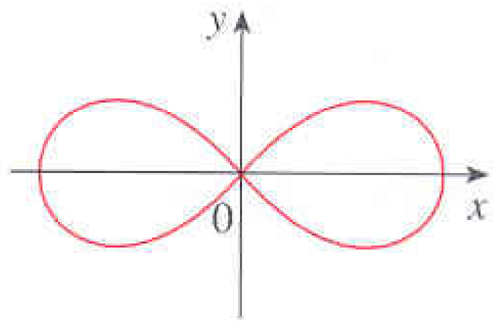
$$\Rightarrow \cos x - \sin y \cdot y' = \sin x(-\sin y \cdot y') + \cos y \cos x$$

$$\Rightarrow (\sin x \sin y - \sin y) y' = \cos x \cos y - \cos x$$

$$\Rightarrow y' = \frac{\cos x(\cos y - 1)}{\sin y(\sin x - 1)}$$

25–30 Use implicit differentiation to find an equation of the tangent line to the curve at the given point.

29. $2(x^2 + y^2)^2 = 25(x^2 - y^2)$
$(3, 1)$
(lemniscate)



(Sol) $2(x^2+y^2)^2=25(x^2-y^2) \Rightarrow 4(x^2+y^2)(2x+2yy')=25(2x-2yy')$

$\Rightarrow 4(x+yy')(x^2+y^2)=25(x-yy')$

$\Rightarrow 4yy'(x^2+y^2)+25yy'=25x-4x(x^2+y^2)$

$\Rightarrow y'=\dfrac{25x-4x(x^2+y^2)}{25y+4y(x^2+y^2)}$. When $x=3$ and $y=1$, $y'=\dfrac{75-120}{25+40}=-\dfrac{45}{65}=-\dfrac{9}{13}$

so an equation of the tangent line is $y-1=-\dfrac{9}{13}(x-3)$ or $y=-\dfrac{9}{13}x+\dfrac{40}{13}$.

44. The Power Rule can be proved using implicit differentiation for the case where n is a rational number, $n = p/q$, and $y = f(x) = x^n$ is assumed beforehand to be a differentiable function. If $y = x^{p/q}$, then $y^q = x^p$. Use implicit differentiation to show that

$$y' = \frac{p}{q} x^{(p/q)-1}$$

(Sol) $$y^q = x^p \Rightarrow q y^{q-1} y' = p x^{p-1}$$

$$\Rightarrow y' = \frac{p x^{p-1}}{q y^{q-1}} = \frac{p x^{p-1} y}{q y^q} = \frac{p x^{p-1} x^{p/q}}{q x^p} = \frac{p}{q} x^{(p/q)-1}$$

45–54 Find the derivative of the function. Simplify where possible.

46. $y = \sqrt{\tan^{-1} x}$

(Sol) $y = \sqrt{\tan^{-1} x} = (\tan^{-1} x)^{1/2}$

$$\Rightarrow \quad y' = \frac{1}{2}(\tan^{-1} x)^{-1/2} \cdot \frac{d}{dx}(\tan^{-1} x)$$

$$= \frac{1}{2\sqrt{\tan^{-1} x}} \cdot \frac{1}{1+x^2} = \frac{1}{2\sqrt{\tan^{-1} x}\,(1+x^2)}$$

68. (a) Show that $f(x) = 2x + \cos x$ is one-to-one.
(b) What is the value of $f^{-1}(1)$?
(c) Use the formula from Exercise 67(a) to find $(f^{-1})'(1)$.

(Sol) **(a)** $f(x)=2x+\cos x \Rightarrow f'(x)=2-\sin x>0$ for all x.

Thus, f is increasing for all x therefore one–to–one.

(b) Since f is one–to–one, $f^{-1}(1)=k \Leftrightarrow f(k)=1$.

By inspection, we see that $f(0)=2(0)+\cos 0=1$,

so $k=f^{-1}(1)=0$.

(c) $(f^{-1})'(1)=1/f'(f^{-1}(1))=1/f'(0)=1/(2-\sin 0)=\frac{1}{2}$

2–22 Differentiate the function.

7. $f(x) = \sqrt[5]{\ln x}$

(Sol) $f(x)=\sqrt[5]{\ln x}=(\ln x)^{1/5} \Rightarrow f'(x)=\frac{1}{5}(\ln x)^{-4/5}\frac{d}{dx}(\ln x)$

$$=\frac{1}{5(\ln x)^{4/5}}\cdot\frac{1}{x}=\frac{1}{5x\sqrt[5]{(\ln x)^4}}$$

16. $y = \ln(x^4 \sin^2 x)$

(Sol) $y=\ln(x^4\sin^2 x)=\ln x^4+\ln(\sin x)^2=4\ln x+2\ln \sin x$

$$\Rightarrow y'=4\cdot\frac{1}{x}+2\cdot\frac{1}{\sin x}\cdot\cos x=\frac{4}{x}+2\cot x$$

18. $H(z) = \ln\sqrt{\dfrac{a^2 - z^2}{a^2 + z^2}}$

(sol) $$H'(z) = \frac{\left[\left(\dfrac{(a^2 - z^2)}{(a^2 + z^2)}\right)^{\frac{1}{2}}\right]'}{\sqrt{\dfrac{(a^2 - z^2)}{(a^2 + z^2)}}}$$

we calculate numerator first,

$$\Rightarrow \frac{1}{2}\left(\frac{a^2 - z^2}{a^2 + z^2}\right)^{-\frac{1}{2}}\left(\frac{a^2 - z^2}{a^2 + z^2}\right)' = \frac{1}{2\sqrt{\dfrac{(a^2 - z^2)}{(a^2 + z^2)}}}\ \frac{-2z(a^2 + z^2) - (a^2 - z^2)2z}{(a^2 + z^2)^2}$$

$$H'(z) \Rightarrow \frac{\dfrac{-4a^2 z}{(a^2 + z^2)^2}}{2\left(\dfrac{a^2 - z^2}{a^2 + z^2}\right)} = \frac{-2a^2 z}{(a^2 - z^2)(a^2 + z^2)} = \frac{2a^2 z}{z^4 - a^4}$$

23–26 Find y' and y''.

24. $y = \dfrac{\ln x}{x^2}$

(Sol) $y = \dfrac{\ln x}{x^2} \Rightarrow y' = \dfrac{x^2(1/x)-(\ln x)(2x)}{\left(x^2\right)^2} = \dfrac{x(1-2\ln x)}{x^4} = \dfrac{1-2\ln x}{x^3}$

$\Rightarrow \quad y'' = \dfrac{x^3(-2/x)-(1-2\ln x)(3x^2)}{(x^3)^2} = \dfrac{x^2(-2-3+6\ln x)}{x^6} = \dfrac{6\ln x-5}{x^4}$

27–30 Differentiate f and find the domain of f.

28. $f(x) = \dfrac{1}{1 + \ln x}$

(Sol) $f(x) = \dfrac{1}{1+\ln x} \Rightarrow f'(x) = -\dfrac{1/x}{(1+\ln x)^2}$ [Reciprocal rule] $= -\dfrac{1}{x(1+\ln x)^2}$.

$\mathrm{Dom}(f) = \{x \mid x>0 \text{ and } \ln x \neq -1\} = \{x \mid x>0 \text{ and } x \neq 1/e\} = (0, 1/e) \cup (1/e, \infty)$.

37–48 Use logarithmic differentiation to find the derivative of the function.

47. $y = (\tan x)^{1/x}$

(sol) Using logarithmic differentiation, we have $\ln y = \frac{1}{x}\ln(\tan x)$

$$\Rightarrow \frac{y'}{y} = \frac{1}{x}\frac{\sec^2 x}{\tan x} - \frac{1}{x^2}\ln(\tan x)$$

$$\therefore\ y' = (\tan x)^{\frac{1}{x}}\left(\frac{\sec^2 x}{x\tan x} - \frac{\ln(\tan x)}{x^2}\right)$$

51. Find a formula for $f^{(n)}(x)$ if $f(x) = \ln(x - 1)$.

(Sol) $f(x)=\ln(x-1) \Rightarrow f'(x)=1/(x-1)=(x-1)^{-1} \Rightarrow f''(x)=-(x-1)^{-2}$

$\Rightarrow f'''(x)=2(x-1)^{-3} \Rightarrow f^{(4)}(x)=-2\cdot 3(x-1)^{-4} \Rightarrow \cdots$

$\Rightarrow f^{(n)}(x)=(-1)^{n-1}\cdot 2\cdot 3\cdot 4\cdot \ldots \cdot (n-1)(x-1)^{-n}=(-1)^{n-1}\dfrac{(n-1)!}{(x-1)^{n}}$

52. Find $\frac{d^9}{dx^9}(x^8 \ln x)$.

(Sol) $y = x^8 \ln x$, so $D^9 y = D^8 y' = D^8(8x^7 \ln x + x^7)$.

But the eighth derivative of x^7 is 0, so we now have

$$D^8(8x^7 \ln x) = D^7(8 \cdot 7x^6 \ln x + 8x^6) = D^7(8 \cdot 7x^6 \ln x)$$
$$= D^6(8 \cdot 7 \cdot 6x^5 \ln x) = \ldots = D(8!\, x^0 \ln x) = 8!/x.$$

20. Newton's Law of Gravitation says that the magnitude F of the force exerted by a body of mass m on a body of mass M is

$$F = \frac{GmM}{r^2}$$

where G is the gravitational constant and r is the distance between the bodies.

(a) Find dF/dr and explain its meaning. What does the minus sign indicate?

(Sol) **(a)** $F = \frac{GmM}{r^2} = (GmM)r^{-2} \Rightarrow \frac{dF}{dr} = -2(GmM)r^{-3} = -\frac{2GmM}{r^3}$

, which is the rate of change of the force with respect to the distance between the bodies.

The minus sign indicates that as the distance r between the bodies increases, the magnitude of the force F exerted by the body of mass m on the body of mass M is decreasing.

(b) Suppose it is known that the earth attracts an object with a force that decreases at the rate of 2 N/km when $r = 20{,}000$ km. How fast does this force change when $r = 10{,}000$ km?

(Sol) Given $F'(20,000)=-2$, find $F'(10,000)$.

$$-2=-\frac{2GmM}{20{,}000^3} \Rightarrow GmM=20,000^3.$$

$$F'(10{,}000)=-\frac{2\left(20{,}000^3\right)}{10{,}000^3}=-2\cdot 2^3=-16 \text{ N / km}$$

28. The frequency of vibrations of a vibrating violin string is given by

$$f = \frac{1}{2L}\sqrt{\frac{T}{\rho}}$$

where L is the length of the string, T is its tension, and ρ is its linear density. [See Chapter 11 in D. E. Hall, *Musical Acoustics*, 3d ed. (Pacific Grove, CA: Brooks/Cole, 2002).]

(a) Find the rate of change of the frequency with respect to

(i) the length (when T and ρ are constant),

(ii) the tension (when L and ρ are constant), and

(iii) the linear density (when L and T are constant).

(Sol) **(a)** (i) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-1} \Rightarrow \frac{df}{dL}=-\left(\frac{1}{2}\sqrt{\frac{T}{\rho}}\right)L^{-2}=-\frac{1}{2L^2}\sqrt{\frac{T}{\rho}}$

(ii) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{1}{2L\sqrt{\rho}}\right)T^{1/2} \Rightarrow \frac{df}{dT}=\frac{1}{2}\left(\frac{1}{2L\sqrt{\rho}}\right)T^{-1/2}=\frac{1}{4L\sqrt{T\rho}}$

(iii) $f=\frac{1}{2L}\sqrt{\frac{T}{\rho}}=\left(\frac{\sqrt{T}}{2L}\right)\rho^{-1/2} \Rightarrow \frac{df}{d\rho}=-\frac{1}{2}\left(\frac{\sqrt{T}}{2L}\right)\rho^{-3/2}=-\frac{\sqrt{T}}{4L\rho^{3/2}}$

(b) The pitch of a note (how high or low the note sounds) is determined by the frequency f. (The higher the frequency, the higher the pitch.) Use the signs of the derivatives in part (a) to determine what happens to the pitch of a note

(i) when the effective length of a string is decreased by placing a finger on the string so a shorter portion of the string vibrates,

(ii) when the tension is increased by turning a tuning peg,

(iii) when the linear density is increased by switching to another string.

(Sol)

(b) *Note:* Illustrating tangent lines on the generic figures may help to explain the results.

(i) $\frac{df}{dL} < 0$ and L is decreasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

(ii) $\frac{df}{dT} > 0$ and T is increasing $\Rightarrow f$ is increasing $\Rightarrow$ higher note

(iii) $\frac{df}{d\rho} < 0$ and ρ is increasing $\Rightarrow f$ is decreasing $\Rightarrow$ lower note

12. If a snowball melts so that its surface area decreases at a rate of 1 cm^2/min, find the rate at which the diameter decreases when the diameter is 10 cm.

(Sol) **(a)** Given: the rate of decrease of the surface area is 1 cm^2/min.

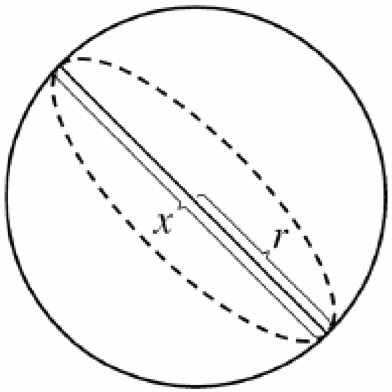
If we let t be time (in minutes) and S be the surface area (in cm^2),

then we are given that $dS/dt = -1\ cm^2/s$.

(b) Unknown: the rate of decrease of the diameter when the diameter is 10 cm.

If we let x be the diameter, then we want to find dx/dt when x=10 cm.

(c)



(d) If the radius is r and the diameter $x=2r$,

then $r=\frac{1}{2}x$ and $S=4\pi r^2=4\pi\left(\frac{1}{2}x\right)^2=\pi x^2 \Rightarrow \frac{dS}{dt}=\frac{dS}{dx}\frac{dx}{dt}=2\pi x\frac{dx}{dt}$.

(e) $-1=\frac{dS}{dt}=2\pi x\frac{dx}{dt} \Rightarrow \frac{dx}{dt}=-\frac{1}{2\pi x}$.

When $x=10$, $\frac{dx}{dt}=-\frac{1}{20\pi}$, So the rate of decrease is $\frac{1}{20\pi}$ cm/min.

22. A particle is moving along the curve $y = \sqrt{x}$. As the particle passes through the point (4, 2), its x-coordinate increases at a rate of 3 cm/s. How fast is the distance from the particle to the origin changing at this instant?

(Sol) Let D denote the distance from the origin (0,0) to the point on the curve $y=\sqrt{x}$.

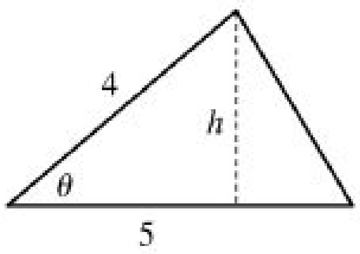
$$D=\sqrt{(x-0)^2+(y-0)^2}=\sqrt{x^2+(\sqrt{x})^2}=\sqrt{x^2+x}$$

$$\Rightarrow \frac{dD}{dt}=\frac{1}{2}(x^2+x)^{-1/2}(2x+1)\frac{dx}{dt}=\frac{2x+1}{2\sqrt{x^2+x}}\frac{dx}{dt}.$$

With $\frac{dx}{dt}=3$ when $x=4$, $\frac{dD}{dt}=\frac{9}{2\sqrt{20}}(3)=\frac{27}{4\sqrt{5}}\approx 3.02\text{ cm / s}.$

29. Two sides of a triangle are 4 m and 5 m in length and the angle between them is increasing at a rate of 0.06 rad/s. Find the rate at which the area of the triangle is increasing when the angle between the sides of fixed length is $\pi/3$.

(Sol)



$A=\frac{1}{2}bh$, but $b=5$ m and $\sin\theta=\frac{h}{4} \Rightarrow h=4\sin\theta$, so $A=\frac{1}{2}(5)(4\sin\theta)=10\sin\theta$.

We are given $\frac{d\theta}{dt}=0.06$ rad/s, so $\frac{dA}{dt}=\frac{dA}{d\theta}\frac{d\theta}{dt}=(10\cos\theta)(0.06)=0.6\cos\theta$.

When $\theta=\frac{\pi}{3}$, $\frac{dA}{dt}=0.6\left(\cos\frac{\pi}{3}\right)=(0.6)\left(\frac{1}{2}\right)=0.3\ m^2/s$.

30–47 Find the derivative. Simplify where possible.

31. $f(x) = \tanh 3x$

(Sol) $f'(x) = \text{sech}^2 3x\,(3x)' = 3\,\text{sech}^2 3x$

33. $h(x) = \cosh(x^4)$

(sol) $h'(x) = \sinh(x^4)(x^4)' = 4x^3 \sinh(x^4)$

35. $y = e^{\cosh 3x}$

(Sol) $y = e^{\cosh 3x} \Rightarrow y' = e^{\cosh 3x} \cdot \sinh 3x \cdot 3 = 3e^{\cosh 3x} \sinh 3x$

37. $f(t) = \operatorname{sech}^2(e^t)$

(sol) $f'(t) = 2\operatorname{sech}(e^t)(\operatorname{sech}(e^t))' = 2\operatorname{sech}(e^t)(-\operatorname{sech}(e^t)\tanh(e^t))e^t$

$= -2e^t \operatorname{sech}^2(e^t)\tanh(e^t)$

39. $y = \arctan(\tanh x)$

(sol) $y = \tan^{-1}(\tanh x)$

$$\Rightarrow y' = \frac{1}{1+(\tanh x)^2}(\tanh x)' = \frac{1}{1+\tanh^2 x}\,\text{sech}^2 x = \frac{\text{sech}^2 x}{1+\tanh^2 x}$$

42. $y = x^2 \sinh^{-1}(2x)$

(Sol) $y = x^2 \sinh^{-1}(2x) \Rightarrow y' = x^2 \cdot \dfrac{1}{\sqrt{1+(2x)^2}} \cdot$

$$2 + \sinh^{-1}(2x) \cdot 2x = 2x\left[\frac{x}{\sqrt{1+4x^2}} + \sinh^{-1}(2x)\right]$$

44. $y = x\tanh^{-1}x + \ln\sqrt{1-x^2}$

(Sol) $y = x\tanh^{-1}x + \ln\sqrt{1-x^2} = x\tanh^{-1}x + \frac{1}{2}\ln(1-x^2)$

$$\Rightarrow y' = \tanh^{-1}x + \frac{x}{1-x^2} + \frac{1}{2}\left(\frac{1}{1-x^2}\right)(-2x) = \tanh^{-1}x$$

46. $y = \operatorname{sech}^{-1}\sqrt{1-x^2}, \quad x > 0$

(Sol) $y = \operatorname{sech}^{-1}\sqrt{1-x^2} \Rightarrow y' = -\frac{1}{\sqrt{1-x^2}\sqrt{1-(1-x^2)}}\,\frac{-2x}{2\sqrt{1-x^2}} = \frac{x}{(1-x^2)|x|}$

47. $y = \coth^{-1}\sqrt{x^2 + 1}$

(Sol) $$y = \coth^{-1}\sqrt{x^2+1} \Rightarrow y' = \frac{1}{1-(x^2+1)} \frac{2x}{2\sqrt{x^2+1}} = -\frac{1}{x\sqrt{x^2+1}}$$