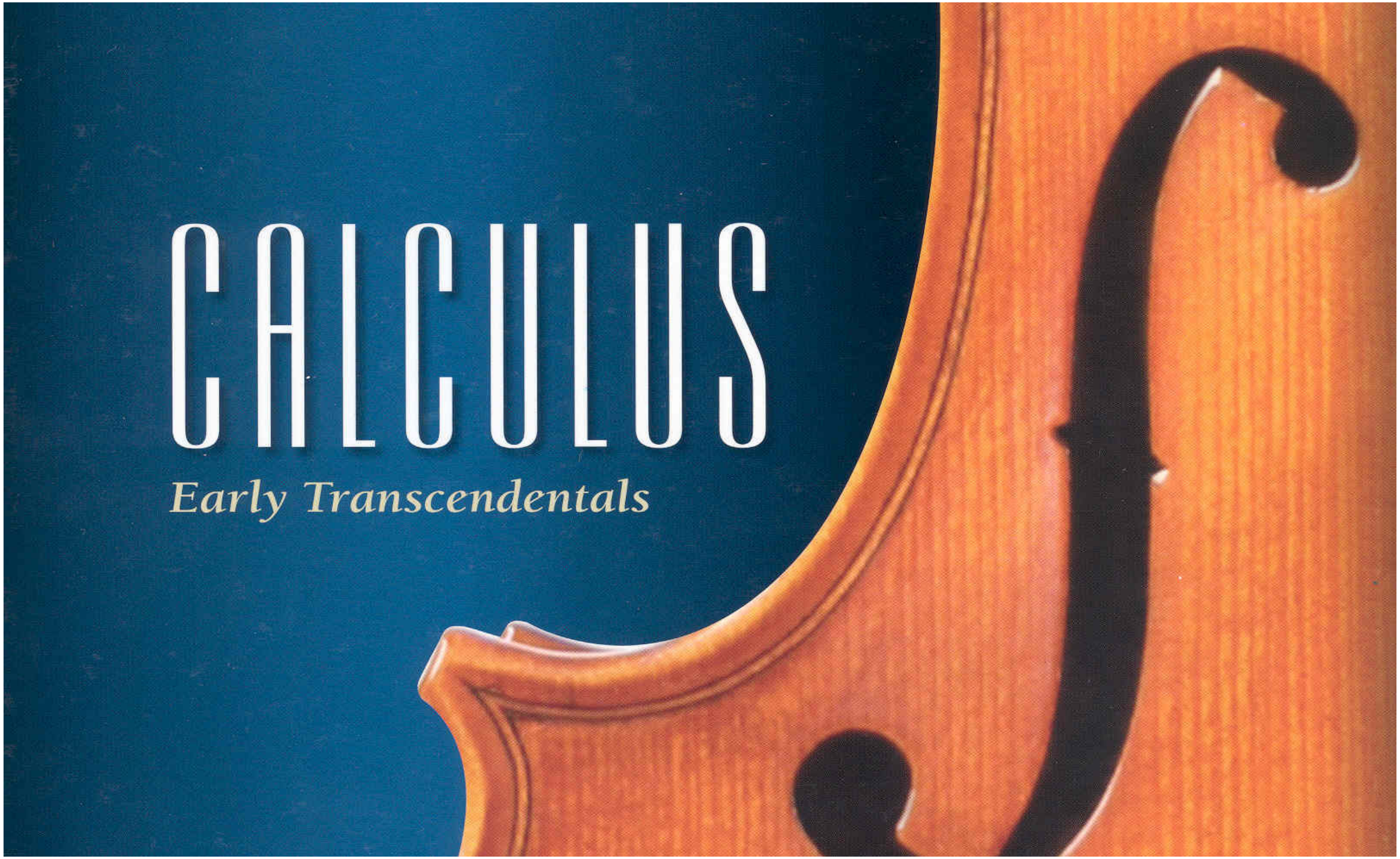


Tutorial Lab 4

29–44 Find the critical numbers of the function.

36. $h(p) = \dfrac{p-1}{p^2+4}$

(sol) $h'(p) = \dfrac{(p^2+4)(1)-(p-1)(2p)}{(p^2+4)^2} = \dfrac{p^2+4-2p^2+2p}{(p^2+4)^2} = \dfrac{-p^2+2p+4}{(p^2+4)^2}.$

$\Rightarrow \quad h'(p) = 0 \quad \Rightarrow \quad p = \dfrac{-2 \pm \sqrt{4+16}}{-2} = 1 \pm \sqrt{5}.$

The critical numbers are $1 \pm \sqrt{5}$. [$h'(p)$ exists for all real numbers.]

40. $g(x) = x^{1/3} - x^{-2/3}$

(sol) $g'(x) = \frac{1}{3}x^{-2/3} + \frac{2}{3}x^{-5/3} = \frac{1}{3}x^{-5/3}(x+2) = \frac{x+2}{3x^{5/3}}$.

$g'(-2)=0$ and $g'(0)$ does not exist, but 0 is not in the domain of g ,

so the only critical number is –2.

44. $f(x) = x^{-2} \ln x$

(sol) $f'(x) = x^{-2}(1/x) + (\ln x)(-2x^{-3})$

$$= x^{-3} - 2x^{-3} \ln x = x^{-3}(1 - 2\ln x) = \frac{1 - 2\ln x}{x^3}.$$

$f'(x) = 0 \quad \Rightarrow \quad 1 - 2\ln x = 0 \quad \Rightarrow \quad \ln x = \frac{1}{2}$

$\ln x = \frac{1}{2} \quad \Rightarrow \quad x = e^{1/2} \approx 1.65.$ $f'(0)$ does not exist,

but 0 is not in the domain of f, so the only critical number is $\sqrt{e}$.

47–62 Find the absolute maximum and absolute minimum values of f on the given interval.

54. $f(x) = \dfrac{x^2 - 4}{x^2 + 4}$, $[-4, 4]$

(sol) $$f'(x) = \frac{(x^2+4)(2x) - (x^2-4)(2x)}{(x^2+4)^2} = \frac{16x}{(x^2+4)^2} = 0 \Leftrightarrow x = 0.$$

$$f(\pm 4) = \frac{12}{20} = \frac{3}{5} \text{ and } f(0) = -1.$$

So $f(\pm 4) = \dfrac{3}{5}$ is the absolute maximum value and

$f(0) = -1$ is the absolute minimum value.

60. $f(x) = x - \ln x, \quad \left[\frac{1}{2}, 2\right]$

(sol) $f(x) = x - \ln x$, $[\frac{1}{2}, 2]$. $f'(x) = 1 - \frac{1}{x} = \frac{x-1}{x}$.

$f'(x) = 0 \quad \Rightarrow \quad x = 1$. [Note that 0 is not in the domain of f.]

$f(\frac{1}{2}) = \frac{1}{2} - \ln\frac{1}{2} \approx 1.19$, $f(1) = 1$, and $f(2) = 2 - \ln 2 \approx 1.31$.

So $f(2) = 2 - \ln 2$ is the absolute maximum value and

$f(1) = 1$ is the absolute minimum value.

62. $f(x) = e^{-x} - e^{-2x}$, $[0, 1]$

(sol) $f'(x) = e^{-x}(-1) - e^{-2x}(-2) = \frac{2}{e^{2x}} - \frac{1}{e^{x}} = \frac{2-e^{x}}{e^{2x}} = 0 \Leftrightarrow e^{x} = 2 \Leftrightarrow x = \ln 2 \approx 0.69.$

$f(0) = 0$,

$f(\ln 2) = e^{-\ln 2} - e^{-2\ln 2} = \left(e^{\ln 2}\right)^{-1} - \left(e^{\ln 2}\right)^{-2} = 2^{-1} - 2^{-2} = \frac{1}{2} - \frac{1}{4} = \frac{1}{4}$,

$f(1) = e^{-1} - e^{-2} \approx 0.233$.

So $f(\ln 2) = \frac{1}{4}$ is the absolute maximum value and

$f(0) = 0$ is the absolute minimum value.

75. Prove that the function

$$f(x) = x^{101} + x^{51} + x + 1$$

has neither a local maximum nor a local minimum.

(pf) $f(x)=x^{101}+x^{51}+x+1 \Rightarrow f'(x)=101x^{100}+51x^{50}+1 \geq 1$ for all x ,

so $f'(x)=0$ has no solution.

Thus, $f(x)$ has no critical number,

so $f(x)$ can have no local maximum or minimum.

11–14 Verify that the function satisfies the hypotheses of the Mean Value Theorem on the given interval. Then find all numbers c that satisfy the conclusion of the Mean Value Theorem.

13. $f(x) = e^{-2x}$, $[0, 3]$

(sol) f is continuous and differentiable on R ,

so it is continuous on $[0,3]$ and differentiable on $(0,3)$.

$$f'(c)=\frac{f(b)-f(a)}{b-a} \Leftrightarrow -2e^{-2c}=\frac{e^{-6}-e^{0}}{3-0} \Leftrightarrow e^{-2c}=\frac{1-e^{-6}}{6} \Leftrightarrow -2c=\ln\left(\frac{1-e^{-6}}{6}\right)$$

$$\Leftrightarrow c=-\frac{1}{2}\ln\left(\frac{1-e^{-6}}{6}\right)\approx 0.897 \text{ , which is in } (0,3).$$

15. Let $f(x) = (x-3)^{-2}$. Show that there is no value of c in $(1, 4)$ such that $f(4) - f(1) = f'(c)(4-1)$. Why does this not contradict the Mean Value Theorem?

(pf) $f(x) = (x-3)^{-2} \quad \Rightarrow \quad f'(x) = -2(x-3)^{-3}$.

$$f(4) - f(1) = f'(c)(4-1) \quad \Rightarrow \quad \frac{1}{1^2} - \frac{1}{(-2)^2} = \frac{-2}{(c-3)^3} \cdot 3$$

$$\Rightarrow \quad \frac{3}{4} = \frac{-6}{(c-3)^3} \quad \Rightarrow \quad (c-3)^3 = -8 \quad \Rightarrow \quad c-3 = -2$$

$\Rightarrow \quad c = 1$, which is not in the open interval $(1, 4)$.

This does not contradict the Mean Value Theorem

since f is not continuous at $x = 3$.

18. Show that the equation $2x - 1 - \sin x = 0$ has exactly one real root.

(pf) Let $f(x)=2x-1-\sin x$. Then $f(0)=-1<0$ and $f(\pi/2)=\pi-2>0$.

f is the sum of the polynomial $2x-1$

and the scalar multiple $(-1)\cdot\sin x$ of the trigonometric function $\sin x$,

so f is continuous (and differentiable) for all x.

By the Intermediate Value Theorem,

there is a number c in $(0,\pi/2)$ such that $f(c)=0$.

Thus, the given equation has at least one real root.

If the equation has distinct real roots a and b with $a<b$, then $f(a)=f(b)=0$.

Since f is continuous on $[a,b]$ and differentiable on (a,b) ,

Rolle's Theorem implies that there is a number r in (a,b) such that $f'(r)=0$.

But $f'(r)=2-\cos r > 0$ since $\cos r \leq 1$.

This contradiction shows that the given equation can't have two distinct real roots,

so it has exactly one real root.

22. (a) Suppose that f is differentiable on $\mathbb{R}$ and has two roots. Show that f' has at least one root.

(pf) Suppose that $f(a)=f(b)=0$ where $a<b$.

By Rolle's Theorem applied to f on [a,b]

there is a number c such that $a<c<b$ and $f'(c)=0$.

(b) Suppose f is twice differentiable on $\mathbb{R}$ and has three roots. Show that f'' has at least one real root.

(pf) Suppose that $f(a)=f(b)=f(c)=0$ where $a<b<c$.

By Rolle's Theorem applied to $f(x)$ on [a,b] and [b,c]

there are numbers $a<d<b$ and $b<e<c$ with $f'(d)=0$ and $f'(e)=0$.

By Rolle's Theorem applied to $f'(x)$ on [d,e]

there is a number g with $d<g<e$ such that $f''(g)=0$.

(c) Can you generalize parts (a) and (b)?

(sol) Suppose that f is n times differentiable on R and has $n+1$ distinct real roots.

Then $f^{(n)}$ has at least one real root.

24. Suppose that $3 \leqslant f'(x) \leqslant 5$ for all values of x. Show that $18 \leqslant f(8) - f(2) \leqslant 30$.

(pf) If $3 \le f'(x) \le 5$ for all x, then by the Mean Value Theorem,

$$f(8) - f(2) = f'(c) \cdot (8-2) \text{ for some } c \text{ in } [2,8].$$

(f is differentiable for all x, so, in particular,

f is differentiable on $(2,8)$ and continuous on $[2,8]$.

Thus, the hypotheses of the Mean Value Theorem are satisfied.)

Since $f(8) - f(2) = 6f'(c)$ and $3 \le f'(c) \le 5$,

it follows that $6 \cdot 3 \le 6f'(c) \le 6 \cdot 5 \Rightarrow 18 \le f(8) - f(2) \le 30$.

[숙제아님]

26. Suppose that f and g are continuous on $[a, b]$ and differentiable on (a, b). Suppose also that $f(a) = g(a)$ and $f'(x) < g'(x)$ for $a < x < b$. Prove that $f(b) < g(b)$. [*Hint*: Apply the Mean Value Theorem to the function $h = f - g$.]

27. Show that $\sqrt{1+x} < 1 + \frac{1}{2}x$ if $x > 0$.

(pf) We use Exercise 26 with $f(x)=\sqrt{1+x}$, $g(x)=1+\frac{1}{2}x$, and $a=0$.

Notice that $f(0)=1=g(0)$ and $f'(x)=\frac{1}{2\sqrt{1+x}} < \frac{1}{2} = g'(x)$ for $x>0$.

So by Exercise 26, $f(b)<g(b) \Rightarrow \sqrt{1+b} < 1+\frac{1}{2}b$ for $b>0$.

Another method: Apply the Mean Value Theorem directly to

either $f(x)=1+\frac{1}{2}x-\sqrt{1+x}$ or $g(x)=\sqrt{1+x}$ on $[0,b]$.

29. Use the Mean Value Theorem to prove the inequality

$$|\sin a - \sin b| \leqslant |a - b| \qquad \text{for all } a \text{ and } b$$

(pf) Let $f(x)=\sin x$ and let $b<a$.

Then $f(x)$ is continuous on [b,a] and differentiable on (b,a) .

By the $\sin a-\sin b=f(a)-f(b)=f'(c)(a-b)=(\cos c)(a-b)$.

Thus, $|\sin a-\sin b|\leq|\cos c|\,|b-a|\leq|a-b|$.

If $a<b$, then $|\sin a-\sin b|=|\sin b-\sin a|\leq|b-a|=|a-b|$.

If $a=b$, both sides of the inequality are 0.

36. A number a is called a **fixed point** of a function f if $f(a) = a$. Prove that if $f'(x) \neq 1$ for all real numbers x, then f has at most one fixed point.

(pf) Assume that f is differentiable (and hence continuous) on R and that $f'(x) \neq 1$ for all x.

Suppose f has more than one fixed point.

Then there are numbers a and b such that $a<b$, $f(a)=a$, and $f(b)=b$.

Applying the Mean Value Theorem to the function f on $[a,b]$,

we find that there is a number c in (a,b) such that $f'(c)=\dfrac{f(b)-f(a)}{b-a}$.

But then $f'(c)=\dfrac{b-a}{b-a}=1$,

contradicting our assumption that $f'(x)\neq 1$ for every real number x .

This shows that our supposition was wrong,

that is, that f cannot have more than one fixed point.

9–18

(a) Find the intervals on which f is increasing or decreasing.
(b) Find the local maximum and minimum values of f.
(c) Find the intervals of concavity and the inflection points.

12. $f(x) = \dfrac{x^2}{x^2 + 3}$

(sol) **(a)** $f'(x) = \dfrac{(x^2+3)(2x) - x^2(2x)}{(x^2+3)^2} = \dfrac{6x}{(x^2+3)^2}$.

The denominator is positive so the sign of $f'(x)$ is determined by the sign of x.

Thus, $f'(x) > 0 \Leftrightarrow x > 0$ and $f'(x) < 0 \Leftrightarrow x < 0$.

So f is increasing on $(0, \infty)$ and f is decreasing on $(-\infty, 0)$.

(b) f changes from decreasing to increasing at $x=0$.

Thus, $f(0)=0$ is a local minimum value.

(c) $$f''(x) = \frac{\{(x^2+3)^2(6)-6x\cdot 2(x^2+3)(2x)}{[(x^2+3)^2]^2} = \frac{6(x^2+3)[x^2+3-4x^2]}{(x^2+3)^4}$$

$$= \frac{6(3-3x^2)}{(x^2+3)^3} = \frac{-18(x+1)(x-1)}{(x^2+3)^3}.$$

$f''(x)>0 \Leftrightarrow -1<x<1$ and $f''(x)<0 \Leftrightarrow x<-1$ or $x>1$.

Thus, f is concave upward on $(-1,1)$ and concave downward on $(-\infty,-1)$ and $(1,\infty)$.

There are inflection points at $\left(\pm 1, \frac{1}{4}\right)$.

16. $f(x) = x^2 \ln x$

(sol) (a) $f(x) = x^2 \ln x \quad \Rightarrow \quad f'(x) = x^2(1/x) + (\ln x)(2x)$

$$= x + 2x \ln x = x(1 + 2 \ln x).$$

The domain of f is $(0, \infty)$, so the sign of f' is determined solely by the factor $1 + 2 \ln x$.

$f'(x) > 0 \quad \Leftrightarrow \quad \ln x > -\frac{1}{2} \quad \Leftrightarrow \quad x > e^{-1/2}$ $[\approx 0.61]$ and $f'(x) < 0$

$$\Leftrightarrow \quad 0 < x < e^{-1/2}.$$

So f is increasing on $(e^{-1/2}, \infty)$ and f is decreasing on $(0, e^{-1/2})$.

(b) f changes from decreasing to increasing at $x = e^{-1/2}$.

Thus, $f(e^{-1/2}) = (e^{-1/2})^2 \ln(e^{-1/2}) = e^{-1}(-1/2) = -1/(2e)$

$[\approx -0.18]$ is a local minimum value.

(c) $f'(x) = x(1 + 2\ln x) \quad \Rightarrow$

$f''(x) = x(2/x) + (1 + 2\ln x) \cdot 1 = 2 + 1 + 2\ln x = 3 + 2\ln x.$

$f''(x) > 0 \quad \Leftrightarrow \quad 3 + 2\ln x > 0 \quad \Leftrightarrow \quad \ln x > -3/2 \Leftrightarrow \quad x > e^{-3/2}$ $[\approx 0.22]$.

Thus, f is concave upward on $(e^{-3/2}, \infty)$

and f is concave downward on $(0, e^{-3/2})$.

$f(e^{-3/2}) = (e^{-3/2})^2 \ln e^{-3/2} = e^{-3}(-3/2) = -3/(2e^3)$ $[\approx -0.07]$.

There is a point of inflection at $\left(e^{-3/2}, f(e^{-3/2})\right) = \left(e^{-3/2}, -3/(2e^3)\right)$.

24–29 Sketch the graph of a function that satisfies all of the given conditions.

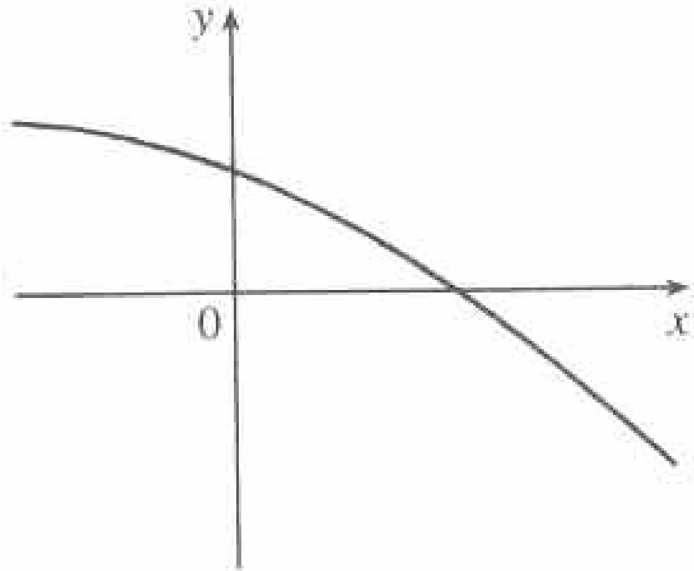
29. $f'(x) < 0$ and $f''(x) < 0$ for all x

(sol) The function must be always decreasing

(since the first derivative is always negative)

and concave downward

(since the second derivative is always negative).



45–52

(a) Find the vertical and horizontal asymptotes.
(b) Find the intervals of increase or decrease.
(c) Find the local maximum and minimum values.
(d) Find the intervals of concavity and the inflection points.
(e) Use the information from parts (a)–(d) to sketch the graph of f.

45. $f(x) = \dfrac{1 + x^2}{1 - x^2}$

(sol) (a) $\lim_{x \to \pm\infty} \dfrac{1+x^2}{1-x^2} = \lim_{x \to \pm\infty} \dfrac{(1/x^2)+1}{(1/x^2)-1} = -1$, so $y = -1$ is a HA.

$$\lim_{x \to 1^-} \frac{1+x^2}{1-x^2} = \infty, \qquad \lim_{x \to 1^+} \frac{1+x^2}{1-x^2} = -\infty,$$

$$\lim_{x \to -1^-} \frac{1+x^2}{1-x^2} = -\infty, \qquad \lim_{x \to -1^+} \frac{1+x^2}{1-x^2} = \infty.$$

So $x = 1$ and $x = -1$ are VA.

(b) $f(x) = \frac{1+x^2}{1-x^2} = -1 + \frac{2}{1-x^2} \Rightarrow \quad f'(x) = \frac{4x}{(1-x^2)^2} > 0$

$\Leftrightarrow \quad x > 0 \; [x \neq 1]$, so f increases on $(0,1)$, $(1,\infty)$

and decreases on $(-\infty,-1)$, $(-1,0)$.

(c) $f(0) = 1$ is a local minimum.

(d) $f''(x) = \dfrac{4(1-x^2)^2 - 4x \cdot 2(1-x^2)(-2x)}{(1-x^2)^4} = \dfrac{4(1+3x^2)}{(1-x^2)^3}$ Since the

numerator is always positive, the sign of $f''(x)$ is the same as the sign of

$1-x^2$. Thus, $f''(x) > 0 \iff 1-x^2 > 0 \iff x^2 < 1 \iff -1 < x < 1$,

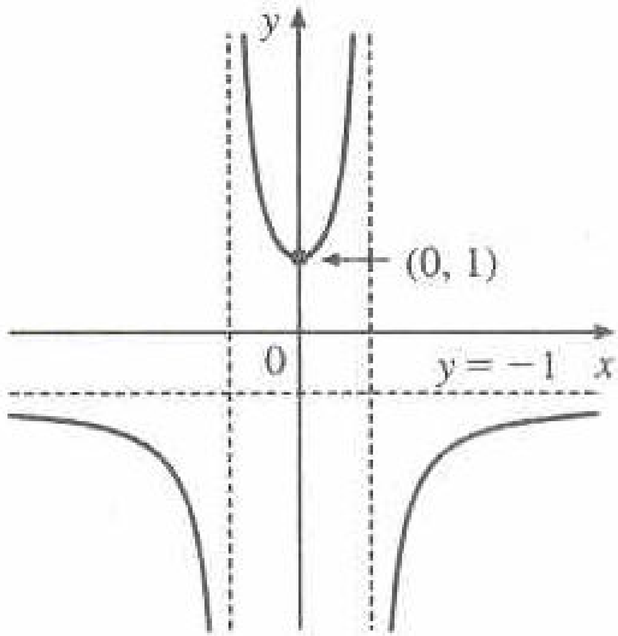
so f is CU on $(-1, 1)$ and

on $(-\infty, -1)$ and $(1, \infty)$.

There is no IP since $x = \pm 1$

are not in the domain of f.

(e)



49. $f(x) = \ln(1 - \ln x)$

(sol) $f(x)=\ln(1-\ln x)$ is defined when $x>0$ (so that $\ln x$ is defined)

and $1-\ln x>0$ [so that $\ln(1-\ln x)$ is defined].

The second condition is equivalent to $1>\ln x \Leftrightarrow x<e$, so f has domain $(0,e)$.

(a) As $x\to 0^+, \ln x\to -\infty$, so $1-\ln x\to\infty$ and $f(x)\to\infty$.

As $x\to e^-, \ln x\to 1^-$, so $1-\ln x\to 0^+$ and $f(x)\to -\infty$.

Thus, $x=0$ and $x=e$ are vertical asymptotes. There is no horizontal asymptote.

(b) $f'(x)=\dfrac{1}{1-\ln x}\left(-\dfrac{1}{x}\right)=-\dfrac{1}{x(1-\ln x)}<0$ on $(0,e)$.

Thus, f is decreasing on its domain, $(0,e)$.

(c) $f'(x) \neq 0$ on $(0, e)$, so f has no local maximum or minimum value.

(d) $$f''(x) = -\frac{-[x(1-\ln x)]'}{[x(1-\ln x)]^2} = \frac{x(-1/x)+(1-\ln x)}{x^2(1-\ln x)^2}$$

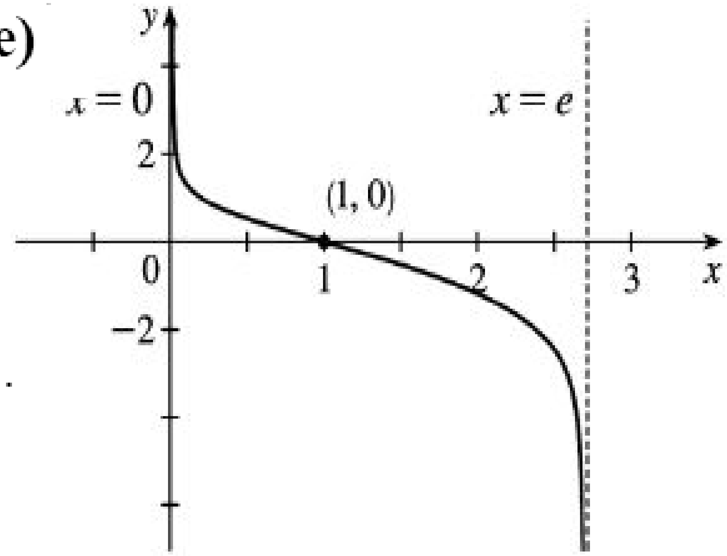
$$= -\frac{\ln x}{x^2(1-\ln x)^2}$$

so $f''(x) > 0 \Leftrightarrow \ln x < 0 \Leftrightarrow 0 < x < 1$.

Thus, f is CU on $(0,1)$ and CD on $(1,e)$.

There is an inflection point at $(1,0)$.

(e)



68. For what values of the numbers a and b does the function

$$f(x) = axe^{bx^2}$$

have the maximum value $f(2) = 1$?

(sol) $f'(x) = a\left[xe^{bx^2} \cdot 2bx + e^{bx^2} \cdot 1\right] = ae^{bx^2}\left(2bx^2+1\right)$.

For $f(2)=1$ to be a maximum value, we must have $f'(2)=0$.

$f(2)=1 \Rightarrow 1=2ae^{4b}$ and $f'(2)=0 \Rightarrow 0=(8b+1)ae^{4b}$.

So $8b+1=0[a\neq 0] \Rightarrow b=-\dfrac{1}{8}$ and now $1=2ae^{-1/2} \Rightarrow a=\sqrt{e}/2$.

81. Show that the function $g(x) = x|x|$ has an inflection point at $(0, 0)$ but $g''(0)$ does not exist.

(pf) Using the fact that $|x| = \sqrt{x^2}$,

we have that $g(x) = x\sqrt{x^2} \Rightarrow g'(x) = \sqrt{x^2} + \sqrt{x^2} = 2\sqrt{x^2} = 2|x|$

$\Rightarrow \quad g''(x) = 2x\left(x^2\right)^{-1/2} = \dfrac{2x}{|x|} < 0$ for $x < 0$ and $g''(x) > 0$ for $x > 0$,

so $(0,0)$ is an inflection point.

But $g''(0)$ does not exist.

5–64 Find the limit. Use l'Hospital's Rule where appropriate. If there is a more elementary method, consider using it. If l'Hospital's Rule doesn't apply, explain why.

6. $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2}$

(sol) $\lim_{x \to 2} \frac{x^2 + x - 6}{x - 2} = \lim_{x \to 2} \frac{(x+3)(x-2)}{x-2} = \lim_{x \to 2} (x+3) = 2 + 3 = 5$

10. $\lim_{x \to 0} \frac{\sin 4x}{\tan 5x}$

(sol) $\lim_{x \to 0} \frac{\sin 4x}{\tan 5x} \overset{H}{=} \lim_{x \to 0} \frac{4 \cos 4x}{5 \sec^2(5x)} = \frac{4(1)}{5(1)^2} = \frac{4}{5}$

14. $\lim_{x \to \pi} \frac{\tan x}{x}$

(sol) $$\lim_{x \to \pi} \frac{\tan x}{x} = \frac{\tan \pi}{\pi} = \frac{0}{\pi} = 0.$$

L'Hospital's Rule does not apply because the denominator doesn't approach 0.

20. $\lim_{x \to 1} \frac{\ln x}{\sin \pi x}$

(sol) $$\lim_{x \to 1} \frac{\ln x}{\sin \pi x} \overset{H}{=} \lim_{x \to 1} \frac{1/x}{\pi \cos \pi x} = \frac{1}{\pi(-1)} = -\frac{1}{\pi}$$

22. $\lim_{x\to 0} \dfrac{e^x - 1 - x - \frac{1}{2}x^2}{x^3}$

(sol) $\lim_{x\to 0} \dfrac{e^x - 1 - x - x^2/2}{x^3} \overset{H}{=} \lim_{x\to 0} \dfrac{e^x - 1 - x}{3x^2} \overset{H}{=} \lim_{x\to 0} \dfrac{e^x - 1}{6x} \overset{H}{=} \lim_{x\to 0} \dfrac{e^x}{6} = \dfrac{1}{6}$

24. $\lim_{x\to 0} \dfrac{x - \sin x}{x - \tan x}$

(sol) $\lim_{x\to 0} \dfrac{x - \sin x}{x - \tan x} \overset{H}{=} \lim_{x\to 0} \dfrac{1 - \cos x}{1 - \sec^2 x} \overset{H}{=} \lim_{x\to 0} \dfrac{-(-\sin x)}{-2\sec x\,(\sec x\ \tan x)}$

$= -\dfrac{1}{2} \lim_{x\to 0} \dfrac{\sin x \left(\dfrac{\cos x}{\sin x}\right)}{\sec^2 x} = -\frac{1}{2} \lim_{x\to 0} \cos^3 x = -\frac{1}{2}(1)^3 = -\frac{1}{2}$

44. $\lim_{x \to \pi/4} (1 - \tan x) \sec x$

(sol) $\lim_{x \to \pi/4} (1 - \tan x) \sec x = (1-1)\sqrt{2} = 0$. L'Hospital's Rule does not apply.

48. $\lim_{x \to 0} (\csc x - \cot x)$

(sol) $\lim_{x \to 0} (\csc x - \cot x) = \lim_{x \to 0} \left(\frac{1}{\sin x} - \frac{\cos x}{\sin x} \right) = \lim_{x \to 0} \frac{1 - \cos x}{\sin x} \overset{H}{=} \lim_{x \to 0} \frac{\sin x}{\cos x} = 0$

54. $\lim_{x\to 0^+} (\tan 2x)^x$

(sol) $y=(\tan 2x)^x \Rightarrow \ln y = x\cdot \ln \tan 2x$,

so $\lim_{x\to 0^+} \ln y = \lim_{x\to 0^+} x\cdot \ln \tan 2x = \lim_{x\to 0^+} \dfrac{\ln \tan 2x}{1/x}$

$$\overset{H}{=} \lim_{x\to 0^+} \frac{(1/\tan 2x)(2\sec^2 2x)}{-1/x^2} = \lim_{x\to 0^+} \frac{-2x^2 \cos 2x}{\sin 2x \cos^2 2x}$$

$$= \lim_{x\to 0^+} \frac{2x}{\sin 2x} \cdot \lim_{x\to 0^+} \frac{-x}{\cos 2x} = 1\cdot 0 = 0$$

$$\Rightarrow \quad \lim_{x\to 0^+} (\tan 2x)^x = \lim_{x\to 0^+} e^{\ln y} = e^0 = 1 \ .$$

56. $\lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}$

(sol) . $y=\left(1+\frac{a}{x}\right)^{bx} \Rightarrow \ln y = bx\ln\left(1+\frac{a}{x}\right)$, so

$$\lim_{x\to\infty}\ln y=\lim_{x\to\infty}\frac{b\ln(1+a/x)}{1/x}\overset{H}{=}\lim_{x\to\infty}\frac{b\left(\frac{1}{1+a/x}\right)\left(-\frac{a}{x^2}\right)}{-1/x^2}=\lim_{x\to\infty}\frac{ab}{1+a/x}=ab$$

$$\Rightarrow \lim_{x\to\infty}\left(1+\frac{a}{x}\right)^{bx}=\lim_{x\to\infty}e^{\ln y}=e^{ab}.$$

62. $\lim_{x\to 1}(2-x)^{\tan(\pi x/2)}$

(sol) $y=(2-x)^{\tan(\pi x/2)} \quad\Rightarrow\quad \ln y=\tan\left(\frac{\pi x}{2}\right)\ln(2-x)$

$$\Rightarrow \lim_{x\to 1}\ln y=\lim_{x\to 1}\left[\tan\left(\frac{\pi x}{2}\right)\ln(2-x)\right]$$

$$=\lim_{x\to 1}\frac{\ln(2-x)}{\cot\left(\frac{\pi x}{2}\right)} \overset{H}{=} \lim_{x\to 1}\frac{\frac{1}{2-x}(-1)}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}}=\frac{2}{\pi}\lim_{x\to 1}\frac{\sin^2\left(\frac{\pi x}{2}\right)}{2-x}$$

$$=\frac{2}{\pi}\cdot\frac{1^2}{1}=\frac{2}{\pi}$$

$$\Rightarrow \quad \lim_{x\to 1}(2-x)^{\tan(\pi x/2)}=\lim_{x\to 1}e^{\ln y}=e^{(2/\pi)}$$

76. A metal cable has radius r and is covered by insulation, so that the distance from the center of the cable to the exterior of the insulation is R. The velocity v of an electrical impulse in the cable is

$$v = -c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)$$

where c is a positive constant. Find the following limits and interpret your answers.

(a) $\lim_{R \to r^+} v$ (b) $\lim_{r \to 0^+} v$

(sol) (a) $\lim_{R \to r+} v = \lim_{R \to r+} \left[-c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -cr^2 \lim_{R \to r+} \left[\left(\frac{1}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right]$

$$= -cr^2 \cdot \frac{1}{r^2} \cdot \ln 1 = -c \cdot 0 = 0$$

As the insulation of a metal cable becomes thinner,

the velocity of an electrical impulse in the cable approaches zero.

(b) $\lim_{r\to 0^+} v = \lim_{r\to 0^+}\left[-c\left(\frac{r}{R}\right)^2 \ln\left(\frac{r}{R}\right)\right] = -\frac{c}{R^2}\lim_{r\to 0^+}\left[r^2 \ln\left(\frac{r}{R}\right)\right]$ [form is $0\cdot\infty$]

$$= -\frac{c}{R^2}\lim_{r\to 0^+}\frac{\ln\left(\frac{r}{R}\right)}{\frac{1}{r^2}} \quad \text{[form is } \infty/\infty\text{]}$$

$$\overset{\text{H}}{=} -\frac{c}{R^2}\lim_{r\to 0^+}\frac{\frac{R}{r}\cdot\frac{1}{R}}{\frac{-2}{r^3}} = -\frac{c}{R^2}\lim_{r\to 0^+}\left(-\frac{r^2}{2}\right) = 0$$

As the radius of the metal cable approaches zero,

the velocity of an electrical impulse in the cable approaches zero.

77. The first appearance in print of l'Hospital's Rule was in the book *Analyse des Infiniment Petits* published by the Marquis de l'Hospital in 1696. This was the first calculus *textbook* ever published and the example that the Marquis used in that book to illustrate his rule was to find the limit of the function

$$y = \frac{\sqrt{2a^3x - x^4} - a\sqrt[3]{aax}}{a - \sqrt[4]{ax^3}}$$

as x approaches a, where $a > 0$. (At that time it was common to write aa instead of a^2.) Solve this problem.

(sol) We see that both numerator and denominator approach 0 ,

so we can use l'Hospital's Rule:

$$\lim_{x\to a}\frac{\sqrt{2a^3x-x^4}-a\sqrt[3]{aax}}{a-\sqrt[4]{ax^3}} \overset{H}{=} \lim_{x\to a}\frac{\frac{1}{2}\left(2a^3x-x^4\right)^{-1/2}\left(2a^3-4x^3\right)-a\left(\frac{1}{3}\right)(aax)^{-2/3}a^2}{-\frac{1}{4}\left(ax^3\right)^{-3/4}\left(3ax^2\right)}$$

$$= \frac{\frac{1}{2}\left(2a^3a-a^4\right)^{-1/2}\left(2a^3-4a^3\right)-\frac{1}{3}a^3\left(a^2a\right)^{-2/3}}{-\frac{1}{4}\left(aa^3\right)^{-3/4}\left(3aa^2\right)}$$

$$= \frac{\left(a^4\right)^{-1/2}\left(-a^3\right)-\frac{1}{3}a^3\left(a^3\right)^{-2/3}}{-\frac{3}{4}a^3\left(a^4\right)^{-3/4}} = \frac{-a-\frac{1}{3}a}{-\frac{3}{4}}$$

$$=\frac{4}{3}\left(\frac{4}{3}a\right)=\frac{16}{9}a$$