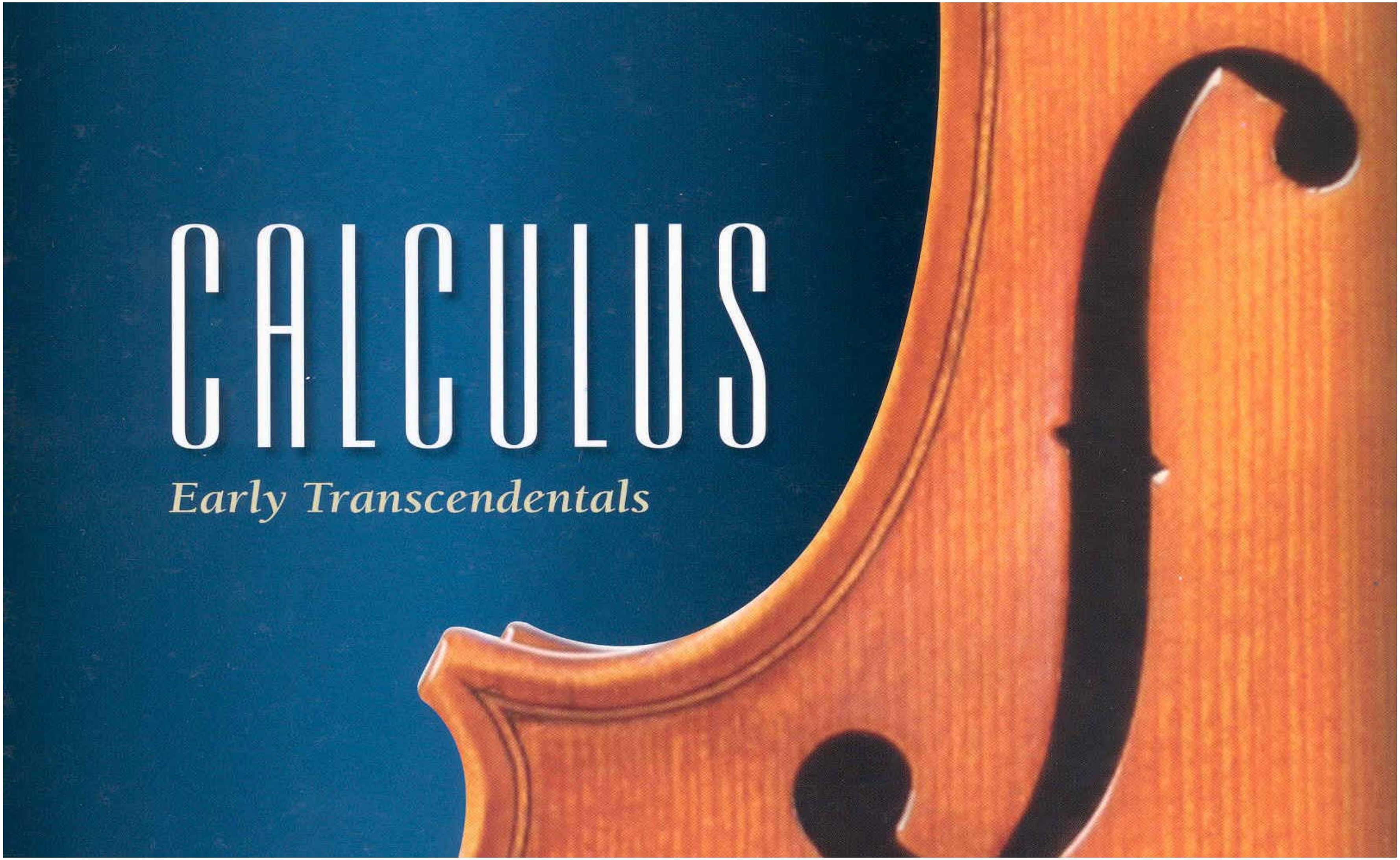


Tutorial Lab 6

1–6 Evaluate the integral by making the given substitution.

4. $\displaystyle\int \frac{dt}{(1-6t)^4}$, $\quad u = 1 - 6t$

(sol) Let $u = 1 - 6t$.

Then $du = -6\,dt$ and $dt = -\frac{1}{6}\,du$, so

$$\int \frac{dt}{(1-6t)^4} = \int \frac{-\frac{1}{6}\,du}{u^4} = -\frac{1}{6}\int u^{-4}\,du$$

$$= -\frac{1}{6}\frac{u^{-3}}{-3} + C = \frac{1}{18u^3} + C = \frac{1}{18(1-6t)^3} + C.$$

6. $\int e^{\sin\theta} \cos\theta \, d\theta, \quad u = \sin\theta$

(sol) Let $u = \sin\theta$.

Then $du = \cos\theta \, d\theta$, so

$$\int e^{\sin\theta} \cos\theta \, d\theta = \int e^u \, du = e^u + C = e^{\sin\theta} + C.$$

7–46 Evaluate the indefinite integral.

22. $\int \sqrt{x}\,\sin(1+x^{3/2})\,dx$

(sol) Let $u = 1 + x^{3/2}$.

Then $du = \frac{3}{2}x^{1/2}\,dx$ and $\sqrt{x}\,dx = \frac{2}{3}\,du$, so

$$\int \sqrt{x}\,\sin(1+x^{3/2})\,dx = \int \sin u\left(\tfrac{2}{3}\,du\right)$$

$$= \tfrac{2}{3}\cdot(-\cos u) + C = -\tfrac{2}{3}\cos(1+x^{3/2}) + C.$$

26. $\int e^{\cos t} \sin t \, dt$

(sol) Let $u = \cos t$.

Then $du = -\sin t \, dt$ and $\sin t \, dt = -du$, so

$$\int e^{\cos t} \sin t \, dt = \int e^u (-du) = -e^u + C = -e^{\cos t} + C.$$

30. $\int \frac{\sin(\ln x)}{x} dx$

(sol) Let $u = \ln x$.

Then $du = (1/x)\, dx$, so

$$\int \frac{\sin(\ln x)}{x} dx = \int \sin u \, du = -\cos u + C = -\cos(\ln x) + C.$$

38. $\displaystyle\int \frac{dt}{\cos^2 t\sqrt{1 + \tan t}}$

(sol) Let $u = 1 + \tan t$.

Then $du = \sec^2 t\, dt$, so

$$\int \frac{dt}{\cos^2 t\sqrt{1+\tan t}} = \int \frac{\sec^2 t\, dt}{\sqrt{1+\tan t}} = \int \frac{du}{\sqrt{u}}$$

$$= \int u^{-1/2}\, du = \frac{u^{1/2}}{1/2} + C = 2\sqrt{1+\tan t} + C.$$

51–70 Evaluate the definite integral.

52. $\int_0^7 \sqrt{4 + 3x}\, dx$

(sol) Let $u = 4 + 3x$, so $du = 3\,dx$.

When $x = 0$, $u = 4$; when $x = 7$, $u = 25$. Thus,

$$\int_0^7 \sqrt{4+3x}\,dx = \int_4^{25} \sqrt{u}\left(\tfrac{1}{3}\,du\right)$$

$$= \frac{1}{3}\left[\frac{u^{3/2}}{3/2}\right]_4^{25}$$

$$= \tfrac{2}{9}(25^{3/2} - 4^{3/2}) = \tfrac{2}{9}(125 - 8)$$

$$= \tfrac{234}{9} = 26.$$

60. $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6}\, dx$

(sol) $\int_{-\pi/2}^{\pi/2} \frac{x^2 \sin x}{1 + x^6}\, dx = 0$ by Theorem 7(b) since $f(x) = \frac{x^2 \sin x}{1 + x^6}$ is an odd function.

65. $\int_1^2 x\sqrt{x - 1}\, dx$

(sol) Let $u = x - 1$, so $u + 1 = x$ and $du = dx$.

When $x = 1$, $u = 0$; when $x = 2$, $u = 1$.

Thus,

$$\int_1^2 x\sqrt{x-1}\, dx = \int_0^1 (u+1)\sqrt{u}\, du = \int_0^1 (u^{3/2} + u^{1/2})\, du$$

$$= \left[\tfrac{2}{5}u^{5/2} + \tfrac{2}{3}u^{3/2}\right]_0^1 = \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{16}{15}.$$

81. If f is continuous and $\int_0^4 f(x)\,dx = 10$, find $\int_0^2 f(2x)\,dx$.

(sol) Let $u = 2x$.

Then $du = 2\,dx$, so

$$\int_0^2 f(2x)\,dx = \int_0^4 f(u)\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^4 f(u)\,du = \tfrac{1}{2}(10) = 5.$$

82. If f is continuous and $\int_0^9 f(x)\,dx = 4$, find $\int_0^3 xf(x^2)\,dx$.

(sol) Let $u = x^2$.

Then $du = 2x\,dx$, so

$$\int_0^3 xf(x^2)\,dx = \int_0^9 f(u)\left(\tfrac{1}{2}\,du\right) = \tfrac{1}{2}\int_0^9 f(u)\,du = \tfrac{1}{2}(4) = 2.$$

85. If a and b are positive numbers, show that

$$\int_0^1 x^a(1-x)^b\,dx = \int_0^1 x^b(1-x)^a\,dx$$

(sol) Let $u = 1 - x$.

Then $x = 1 - u$ and $dx = -du$, so

$$\int_0^1 x^a(1-x)^b\,dx = \int_1^0 (1-u)^a\,u^b(-du)$$

$$= \int_0^1 u^b(1-u)^a\,du = \int_0^1 x^b(1-x)^a\,dx.$$

86. If f is continuous on $[0, \pi]$, use the substitution $u = \pi - x$ to show that

$$\int_0^{\pi} x f(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx$$

(pf) Let $u = \pi - x$.

Then $du = -dx$.

When $x = \pi$, $u = 0$ and when $x = 0$, $u = \pi$.

So

$$\int_0^{\pi} x f(\sin x)\, dx = -\int_{\pi}^{0} (\pi - u)\, f(\sin(\pi - u))\, du = \int_0^{\pi} (\pi - u)\, f(\sin u)\, du$$

$$= \pi \int_0^{\pi} f(\sin u)\, du - \int_0^{\pi} u\, f(\sin u)\, du$$

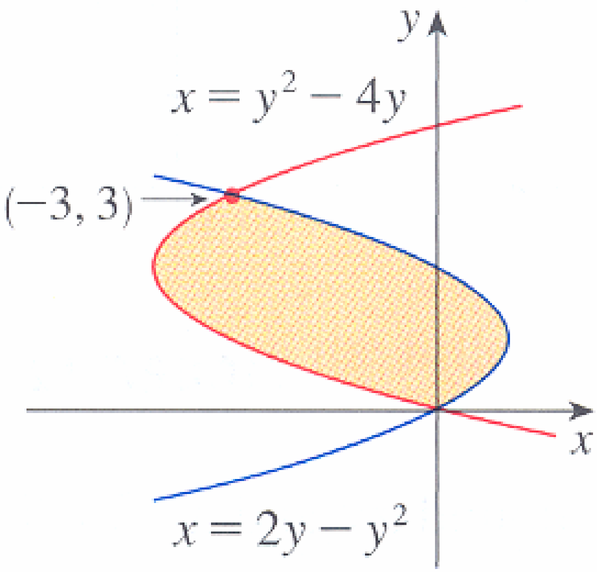
$$= \pi \int_0^{\pi} f(\sin x)\, dx - \int_0^{\pi} x\, f(\sin x)\, dx$$

$$\Rightarrow \quad 2 \int_0^{\pi} x\, f(\sin x)\, dx = \pi \int_0^{\pi} f(\sin x)\, dx$$

$$\Rightarrow \quad \int_0^{\pi} x\, f(\sin x)\, dx = \frac{\pi}{2} \int_0^{\pi} f(\sin x)\, dx.$$

1–4 Find the area of the shaded region.

4.



(sol) $A = \int_0^3 [(2y - y^2) - (y^2 - 4y)]\, dy$

$= \int_0^3 (-2y^2 + 6y)\, dy$

$= \left[-\frac{2}{3}y^3 + 3y^2\right]_0^3$

$= (-18 + 27) - 0 = 9$

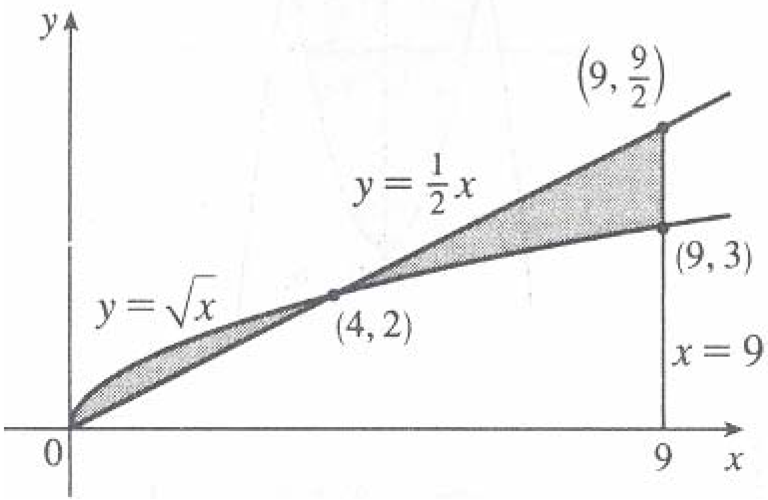
5–28 Sketch the region enclosed by the given curves. Decide whether to integrate with respect to x or y. Draw a typical approximating rectangle and label its height and width. Then find the area of the region.

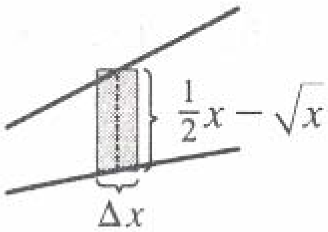
17. $y = \sqrt{x}, \quad y = \frac{1}{2}x, \quad x = 9$

(sol) $\frac{1}{2}x = \sqrt{x} \;\Rightarrow\; \frac{1}{4}x^2 = x \;\Rightarrow\; x^2 - 4x = 0 \Rightarrow\; x(x-4) = 0 \;\Rightarrow\; x = 0$ or 4, so

$$A = \int_0^4 \left(\sqrt{x} - \tfrac{1}{2}x\right) dx + \int_4^9 \left(\tfrac{1}{2}x - \sqrt{x}\right) dx = \left[\tfrac{2}{3}x^{3/2} - \tfrac{1}{4}x^2\right]_0^4 + \left[\tfrac{1}{4}x^2 - \tfrac{2}{3}x^{3/2}\right]_4^9$$

$$= \left[\left(\tfrac{16}{3} - 4\right) - 0\right] + \left[\left(\tfrac{81}{4} - 18\right) - \left(4 - \tfrac{16}{3}\right)\right] = \tfrac{81}{4} + \tfrac{32}{3} - 26 = \tfrac{59}{12}$$





For $4 < x < 9$

1–18 Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region, the solid, and a typical disk or washer.

10. $y = x^{2/3}$, $x = 1$, $y = 0$; about the y-axis

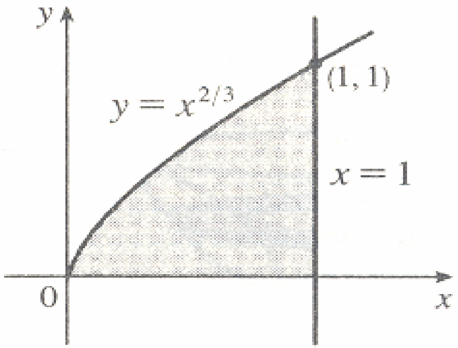
(sol) $y = x^{2/3} \Leftrightarrow x = y^{3/2}$,

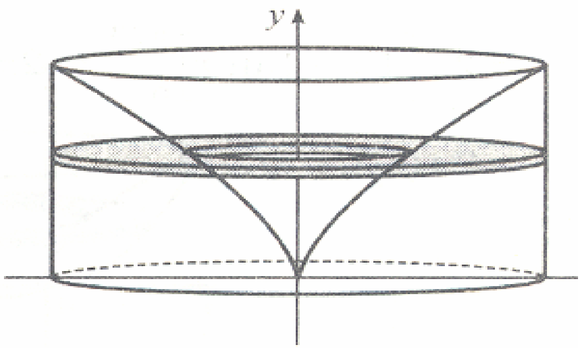
so a cross-section is a washer with inner radius $y^{3/2}$ and outer radius 1,

and its area is $A(y) = \pi(1)^2 - \pi(y^{3/2})^2 = \pi(1 - y^3)$.

$$V = \int_0^1 A(y)\,dy = \pi \int_0^1 (1 - y^3)\,dy$$

$$= \pi\left[y - \tfrac{1}{4}y^4\right]_0^1 = \tfrac{3}{4}\pi$$





18. $y = x,\ y = 0,\ x = 2,\ x = 4;$ about $x = 1$

(sol)

For $0 \le y < 2$, a cross-section is an annulus with inner radius $2 - 1$ and outer radius $4 - 1$,

the area of which is $A_1(y) = \pi(4-1)^2 - \pi(2-1)^2$.

For $2 \le y \le 4$, a cross-section is an annulus with inner radius $y - 1$ and outer radius $4 - 1$,

the area of which is $A_2(y) = \pi(4-1)^2 - \pi(y-1)^2$.

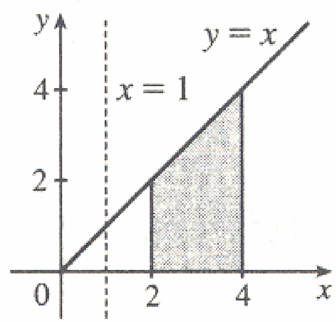
$$V = \int_0^4 A(y)\,dy = \pi \int_0^2 \left[(4-1)^2 - (2-1)^2\right] dy + \pi \int_2^4 \left[(4-1)^2 - (y-1)^2\right] dy$$

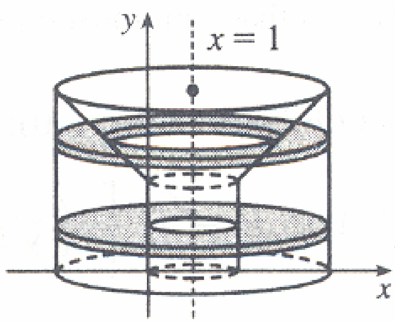
$$= \pi\left[8y\right]_0^2 + \pi \int_2^4 (8 + 2y - y^2)\,dy$$

$$= 16\pi + \pi\left[8y + y^2 - \tfrac{1}{3}y^3\right]_2^4$$

$$= 16\pi + \pi\left[\left(32 + 16 - \tfrac{64}{3}\right) - \left(16 + 4 - \tfrac{8}{3}\right)\right]$$

$$= \frac{76}{3}\pi$$





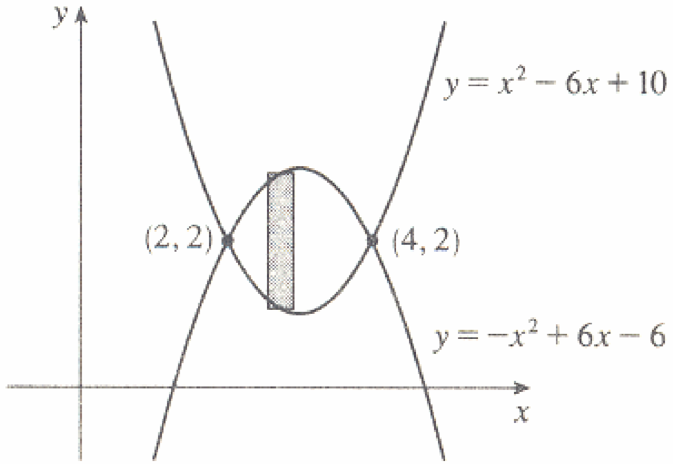
3–7 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the y-axis. Sketch the region and a typical shell.

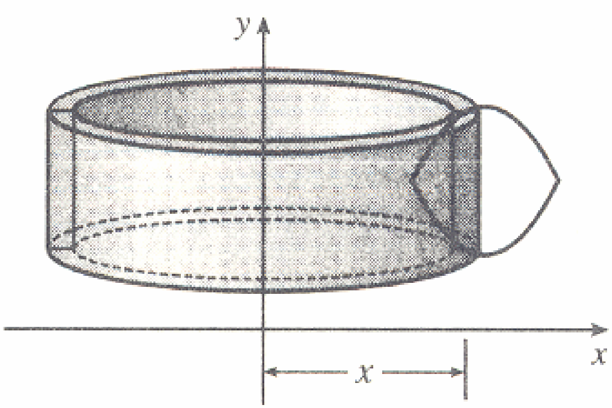
6. $y = x^2 - 6x + 10, \quad y = -x^2 + 6x - 6$

(sol) $V = \int_2^4 2\pi x[(-x^2 + 6x - 6) - (x^2 - 6x + 10)]\,dx = 2\pi \int_2^4 x(-2x^2 + 12x - 16)\,dx$

$= 4\pi \int_2^4 (-x^3 + 6x^2 - 8x)\,dx = 4\pi \left[-\frac{1}{4}x^4 + 2x^3 - 4x^2\right]_2^4$

$= 4\pi[(-64 + 128 - 64) - (-4 + 16 - 16)] = 16\pi$



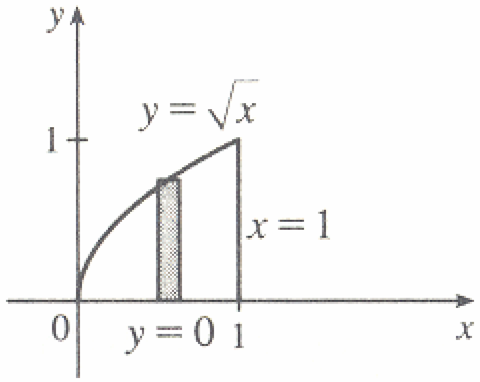


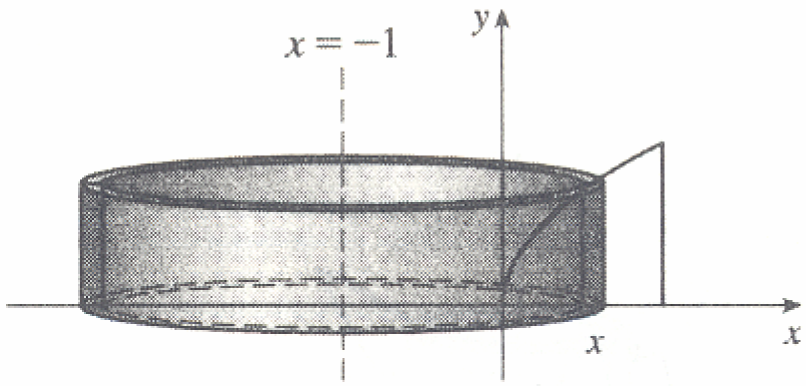
15–20 Use the method of cylindrical shells to find the volume generated by rotating the region bounded by the given curves about the specified axis. Sketch the region and a typical shell.

16. $y = \sqrt{x}$, $y = 0$, $x = 1$; about $x = -1$

(sol) The shell has radius $x - (-1) = x + 1$, circumference $2\pi(x+1)$, and height $\sqrt{x}$.

$$V = \int_0^1 2\pi(x+1)\sqrt{x}\,dx$$
$$= 2\pi \int_0^1 (x^{3/2} + x^{1/2})\,dx$$
$$= 2\pi\left[\tfrac{2}{5}x^{5/2} + \tfrac{2}{3}x^{3/2}\right]_0^1$$
$$= 2\pi\left[\left(\tfrac{2}{5} + \tfrac{2}{3}\right) - 0\right] = 2\pi\left(\tfrac{16}{15}\right) = \tfrac{32}{15}\pi$$





19. $y = x^3$, $y = 0$, $x = 1$; about $y = 1$

(sol) The shell has radius $1 - y$, circumference $2\pi(1-y)$,

and height $1 - \sqrt[3]{y}$ $\left[y = x^3 \;\Leftrightarrow\; x = \sqrt[3]{y}\right]$.

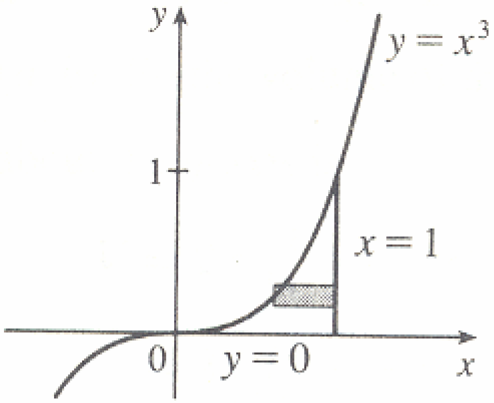
$$V = \int_0^1 2\pi(1-y)(1-y^{1/3})\,dy$$

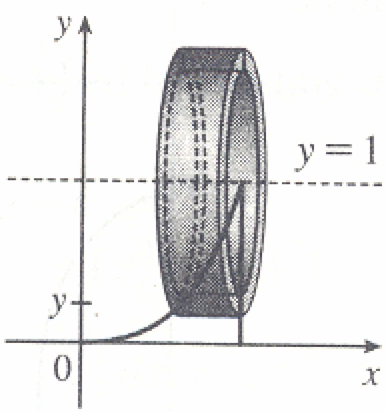
$$= 2\pi \int_0^1 (1 - y - y^{1/3} + y^{4/3})\,dy$$

$$= 2\pi \left[y - \tfrac{1}{2}y^2 - \tfrac{3}{4}y^{4/3} + \tfrac{3}{7}y^{7/3} \right]_0^1$$

$$= 2\pi \left[\left(1 - \tfrac{1}{2} - \tfrac{3}{4} + \tfrac{3}{7}\right) - 0 \right]$$

$$= 2\pi\left(\tfrac{5}{28}\right) = \tfrac{5}{14}\pi$$

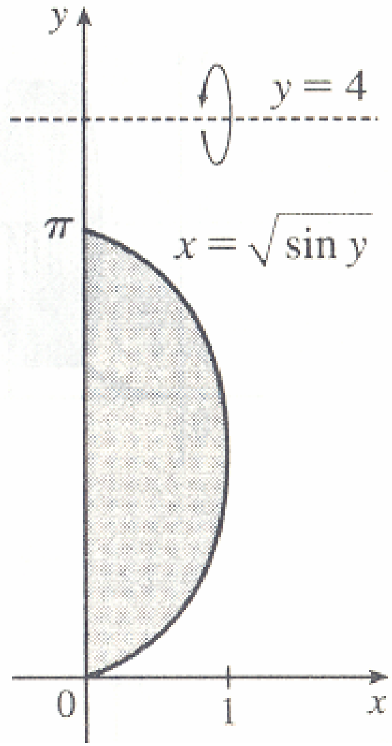




21–26 Set up, but do not evaluate, an integral for the volume of the solid obtained by rotating the region bounded by the given curves about the specified axis.

25. $x = \sqrt{\sin y}$, $0 \leqslant y \leqslant \pi$, $x = 0$; about $y = 4$

(sol) $V = \int_0^{\pi} 2\pi(4 - y)\sqrt{\sin y}\, dy$



2. Find the work done if a constant force of 100 lb is used to pull a cart a distance of 200 ft.

(sol) $W = Fd = (100)(200) = 20{,}000$ ft-lb