Lecture Notes 1

Partial Fraction Expansion

Ex 1

$$\frac{2s^2 - 3s}{(s-2)(s-1)^2} = \frac{A}{s-2} + \frac{B}{s-1} + \frac{C}{(s-1)^2}$$

Multiply by $s-2$ and substitute $s=2$, we obtain

$A = 2.$

Multiply by $(s-1)^2$ and substitute $s=1$, we obtain

$C = 1$

$$\text{RHS} = \frac{2}{s-2} + \frac{B}{s-1} + \frac{1}{(s-1)^2}$$

$$= \frac{2(s-1)^2 + B(s-1)(s-2) + (s-2)}{(s-2)(s-1)^2}$$

Comparing co-efficient of s^2 we have

$B = 0$

$$\therefore \text{RHS} = \frac{2}{s-2} + \frac{1}{(s-1)^2}$$

Ex 2

$$\frac{s^2}{[(s+1)^2+1]^2} = \frac{As+B}{(s+1)^2+1} + \frac{Cs+D}{[(s+1)^2+1]^2}$$

$$RHS = \frac{(As+B)\left((s+1)^2+1\right)+(Cs+D)}{[(s+1)^2+1]^2}$$

comparing co-eff of s^3 we get $A=0$

" " " s^2 " " $B=1$

$$RHS = \frac{(s^2+2s+2)+(Cs+D)}{[(s+1)^2+1]^2}$$

Comparing co-effs of s & 1 we get

$$\boxed{C=-2,\ D=-2}$$

$$RHS = \frac{1}{(s+1)^2+1} - \frac{2s+2}{[(s+1)^2+1]^2}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2}\right) = t$$

$$\mathcal{L}^{-1}\left(\frac{2}{s-2}\right) = 2e^{2t}$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s-1)^2}\right) = te^{t}$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{2s^2-3s}{(s-2)(s-1)^2}\right) = 2e^{2t} + te^{t}}$$

$$\mathcal{L}^{-1}\left(\frac{1}{s^2+1}\right) = \sin t$$

$$\mathcal{L}^{-1}\left(\frac{1}{(s+1)^2+1}\right) = e^{-t}\sin t$$

$$\mathcal{L}^{-1}\left(\frac{s}{(s^2+1)^2}\right) = \frac{t}{2}\sin t$$

$$\mathcal{L}^{-1}\left(\frac{2s+2}{[(s+1)^2+1]^2}\right) = 2\,\frac{t}{2}\,e^{-t}\sin t$$

$$= te^{-t}\sin t$$

$$\boxed{\mathcal{L}^{-1}\left(\frac{s^2}{[(s+1)^2+1]^2}\right) = e^{-t}\sin t\,(1-t)}$$

$$\frac{d}{ds}\left(\frac{As+B}{s^2+\sigma^2}\right) = \frac{(s^2+\sigma^2)A - (As+B)2s}{(s^2+\sigma^2)^2}$$

$$= \frac{As^2+\sigma^2 A - 2As^2 - 2Bs}{(s^2+\sigma^2)^2}.$$

$$= \frac{-As^2 - 2Bs + \sigma^2 A}{(s^2+\sigma^2)^2}$$

Define

$$F(s) = \frac{s}{s^2+\sigma^2} \qquad f(t) = \cos\sigma t.$$

$$\mathcal{L}(t\cos\sigma t) = -F'(s)$$

$$= \frac{s^2-\sigma^2}{(s^2+\sigma^2)^2}.$$

$$\mathcal{L}^{-1}\left(\frac{\sigma^2}{(s^2+\sigma^2)^2}\right) = \frac{1}{2\sigma}(\sin\sigma t - \sigma t\cos\sigma t).$$

$$= \frac{1}{2\sigma}\sin\sigma t - \frac{t}{2}\cos\sigma t.$$

$$\therefore \mathcal{L}^{-1}\left(\frac{s^2}{(s^2+\sigma^2)^2}\right) = \frac{1}{2\sigma}\sin\sigma t + \frac{t}{2}\cos\sigma t.$$

$$= \frac{1}{2\sigma}(\sin\sigma t + \sigma t\cos\sigma t).$$

Ex 3

Calculate

$$\mathcal{L}^{-1} \frac{s^3}{\left[(s+1)^2+1\right]^2}$$

writing

$$\frac{s^3}{\left[(s+1)^2+1\right]^2} = \frac{As+B}{(s+1)^2+1} + \frac{Cs+D}{\left[(s+1)^2+1\right]^2}$$

$$RHS = \frac{(As+B)\left[(s+1)^2+1\right]+(Cs+D)}{\left[(s+1)^2+1\right]^2}$$

comparing co-eff of s^3 we get $A=1$

" " " s^2 " " $B+2A=0$

$\Rightarrow B=-2$

" " " s we get $A+2B+C=0$

$\Rightarrow C=3$

" " " 1 we get $2B+D=0$

$\Rightarrow D=4$

RHS =

$$\frac{s-2}{(s+1)^2+1} + \frac{3s+4}{\left[(s+1)^2+1\right]^2}$$

$$= \frac{s+1}{(s+1)^2+1} - \frac{3}{(s+1)^2+1}$$

$$+ \frac{3(s+1)}{\left[(s+1)^2+1\right]^2} + \frac{1}{\left[(s+1)^2+1\right]^2}$$

$$\mathcal{L}^{-1} \frac{s^3}{\left[(s+1)^2+1\right]^2} =$$

$$e^{-t}\cos t - 3e^{-t}\sin t$$

$$+3e^{-t}\frac{t}{2}\sin t + e^{-t}\frac{1}{2}\left[\sin t - t\cos t\right]$$

$$= \left[e^{-t}\cos t - \frac{5}{2}e^{-t}\sin t + \frac{3}{2}te^{-t}\sin t - \frac{1}{2}te^{-t}\cos t\right]$$

Fourier Series Expansion

A function $f(x)$ is called periodic if

$\forall x, \exists p > 0$:

$$f(x+p) = f(x).$$

This number p is called the period of $f(x)$.

$\boxed{\text{Assume } p = 2\pi}$

Let $f(x)$ be a periodic $f^{\underline{n}}$ with period 2π, a fourier series representation of $f(x)$ over the interval $-\pi \leq x \leq \pi$ is an expression of the form.

$$f(x) = a_0 + a_1 \cos x + b_1 \sin x$$
$$+ a_2 \cos 2x + b_2 \sin 2x.$$
$$+ \cdots$$
$$+ \cdots$$

Where

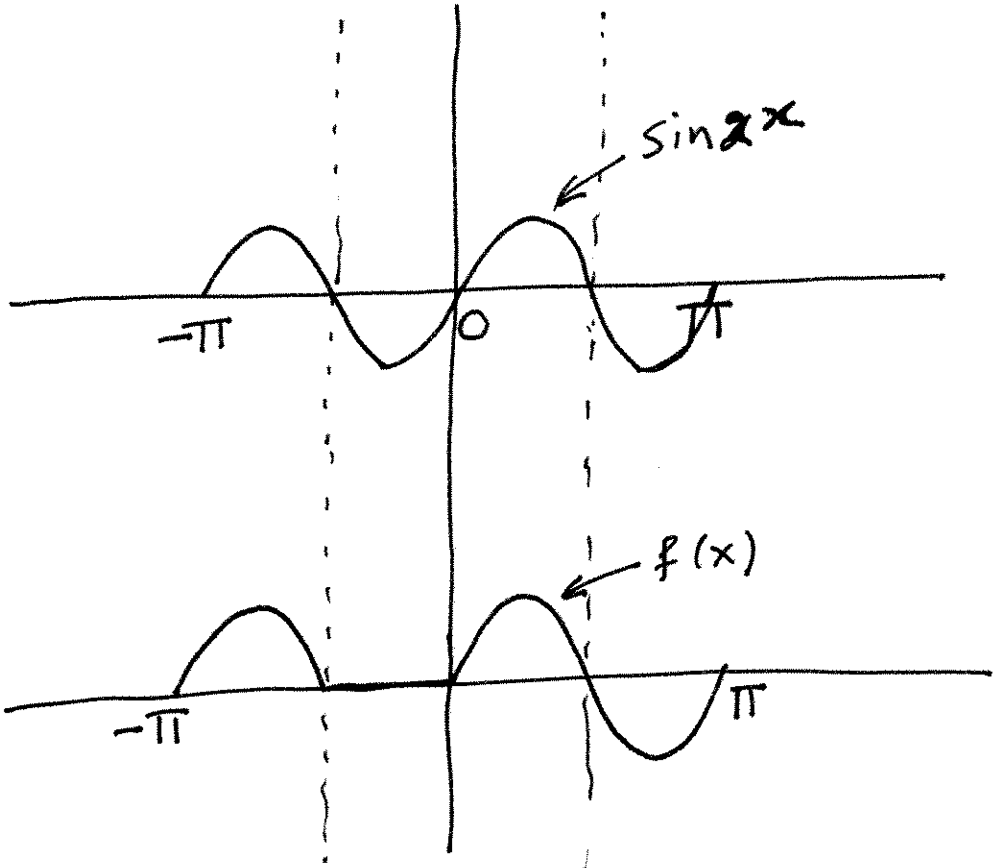
$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)\, dx .$$

$$a_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos nx \, dx .$$

$$b_j = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin nx \, dx .$$

$\boxed{\text{Eulers Formulas}}$

Illustrative example



$$f(x) = \begin{cases} \sin 2x & -\pi < x < -\pi/2 \\ & 0 < x < \pi \\ 0 & -\pi/2 < x < 0 \end{cases}$$

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\,dx = \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} \sin 2x\,dx$$

$$+ \frac{1}{2\pi}\int_{0}^{\pi} \sin 2x\,dx$$

$$\frac{1}{2\pi}\left(-\frac{\cos 2x}{2}\right)\Bigg|_{-\pi}^{-\pi/2} + \frac{1}{2\pi}\left(-\frac{\cos 2x}{2}\right)\Bigg|_{0}^{\pi}$$

$$= \frac{1}{2\pi} + 0 = \frac{1}{2\pi} \qquad \boxed{a_0 = \frac{1}{2\pi}}$$

— x —

Let us calculate a_2 as follows.

$$a_2 = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos 2x\,dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \cos 2x\,dx$$

$$+ \frac{1}{\pi}\int_{0}^{\pi} \sin 2x \cos 2x\,dx.$$

$$= \frac{1}{2\pi}\int_{-\pi}^{-\pi/2} \sin 4x\,dx + \frac{1}{2\pi}\int_{0}^{\pi} \sin 4x\,dx = 0$$

Finally we calculate a_n for $n \neq 2$.

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos nx\, dx.$$

$$= \frac{1}{\pi}\int_{-\pi}^{-\pi/2} \sin 2x \cos nx\, dx + \frac{1}{\pi}\int_{0}^{\pi} \sin 2x \cos nx\, dx$$

$$= -\frac{2}{\pi}\left[\frac{\cos n\pi + \cos\frac{n\pi}{2}}{n^2-4}\right]_{-\pi}^{-\pi/2} + \frac{2}{\pi}\left[\frac{\cos n\pi - 1}{n^2-4}\right]_0^{\pi}$$

$$= -\frac{2}{\pi}\left[\frac{1+\cos\frac{n\pi}{2}}{n^2-4}\right] \quad n \neq 2.$$

Likewise one can show that

$$b_2 = \frac{3}{4} \quad \& \quad b_n = \frac{2}{\pi}\frac{\sin\frac{n\pi}{2}}{n^2-4} \qquad n \neq 2.$$

$$a_0 = \frac{1}{2\pi},\ a_1 = \frac{2}{3\pi},\ a_2 = 0,\ a_3 = -\frac{2}{5\pi},\ a_4 = -\frac{1}{3\pi}, \dots$$

$$b_1 = -\frac{2}{3\pi},\ b_2 = \frac{3}{4},\ b_3 = -\frac{2}{5\pi},\ b_4 = 0, \dots$$

Fourier series looks like

$$f(x) = \frac{1}{2\pi} + \frac{1}{\pi}\left[\frac{2}{3}\cos x - \frac{2}{5}\cos 3x - \frac{1}{3}\cos 4x - \frac{2}{21}\cos 5x \cdots\right]$$

$$+\frac{1}{\pi}\left[-\frac{2}{3}\sin x + \frac{3\pi}{4}\sin 2x - \frac{2}{5}\sin 3x + \frac{2}{21}\sin 5x \cdots\right]$$

—— x ——

Assume $p = 2L$

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

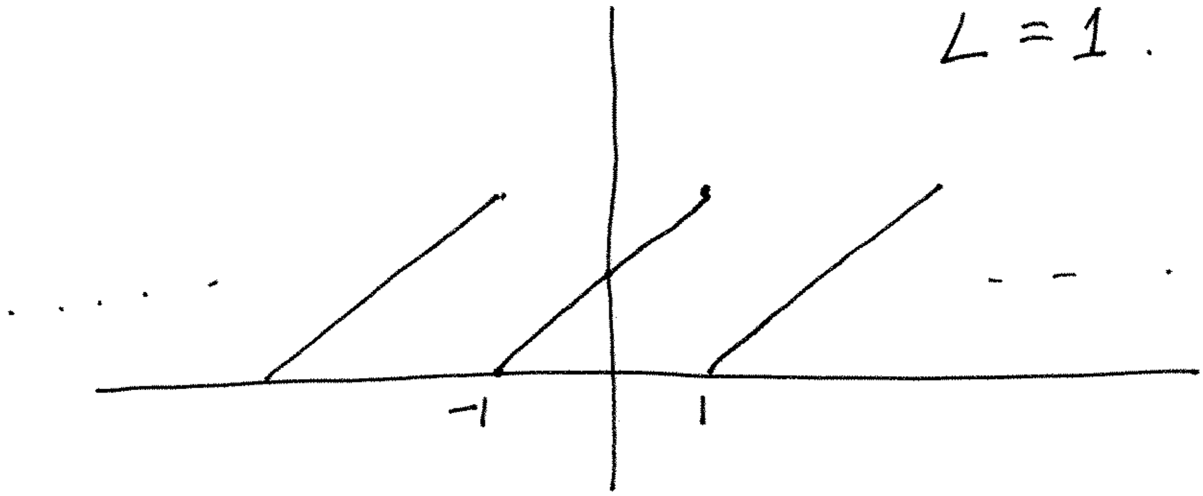
$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx.$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Ex:

$$f(x) = x + 1 \qquad -1 \leq x \leq 1$$

$L = 1.$



$$a_0 = \frac{1}{2}\int_{-1}^{1} (x+1)\, dx = 1$$

$$a_n = \int_{-1}^{1} (x+1) \cos(n\pi x)\, dx = 0$$

$$b_n = \int_{-1}^{1} (x+1) \sin(n\pi x)\, dx = \frac{2}{n\pi}(-1)^{n+1}.$$

$$n = 1, 2, \cdots$$

$$f(x) = 1 + \frac{2}{\pi} \sum_{n} \frac{(-1)^{n+1}}{n} \sin n\pi x.$$

$$-1 \leq x \leq 1.$$