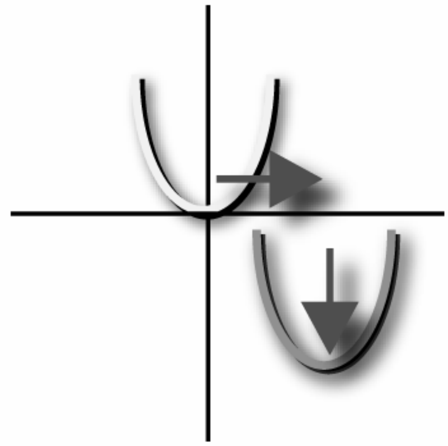
Pure Math 30:

TRANSFORMATIONS



LESSON 3

Algebraic Transformations

Pure Math 30: EXPLAINED!

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Algebraic Transformations of Functions: **If you know the equation of a particular function, you can "insert" a transformation to derive the new function.**

Example 1: Given the function 2x - 1, find the equation of:

a) $y = 2f(x)$

The 2 in front of f(x) tells you to multiply the entire function by 2.

$y = 2f(x)$
$y = 2(2x-1)$
$y = 4x - 2$

b) $y = f(3x)$

The 3x inside the function brackets tells you that wherever there is an x in the original function, you must replace it with 3x.

$y = f(3x)$
$y = 2(3x)-1$
$y = 6x -1$

c) $y = -f(x)$

The - in front of f(x) tells you to multiply the entire function by -1.

$y = -f(x)$
$y = -1(2x-1)$
$y = -2x + 1$

d) $f(-x)$

The -x inside the function brackets tells you that wherever there is an x in the original function, you must replace it with -x.

$y = f(-x)$
$y = 2(-x) -1$
$y = -2x - 1$

e) $y = f(x) + 3$

The +3 tells you to add 3 units to the original function.

$y = f(x) + 3$
$y = 2x - 1 + 3$
$y = 2x + 2$

f) $y = f(x - 4)$

The x-4 inside the brackets tells you that wherever there is an x in the original function, you must replace it with x - 4.

$y = f(x - 4)$
$y = 2(x - 4) - 1$
$y = 2x - 8 - 1$
$y = 2x - 9$

Example 2: Given the function
$f(x) = 3x^2 - 3x + 4$,
find the equation of:

a) $2f(x)$
$y = 2(3x^2 - 3x + 4)$
$y = 6x^2 - 6x + 8$

b) $f(3x)$
$y = 3(3x)^2 - 3(3x) + 4$
$y = 3(9x^2) - 3(3x) + 4$
$y = 27x^2 - 9x + 4$

c) $-f(x)$
$y = -(3x^2 - 3x + 4)$
$y = -3x^2 + 3x - 4$

d) $f(-x)$
$y = 3(-x)^2 - 3(-x) + 4$
$y = 3x^2 + 3x + 4$

e) $y = f(x) + 3$
$y = 3x^2 - 3x + 4 + 3$
$y = 3x^2 - 3x + 7$

f) $y = f(x-4)$
$y = 3(x - 4)^2 - 3(x - 4) + 4$
$y = 3(x^2 - 8x + 16) - 3x + 12 + 4$
$y = 3x^2 - 24x + 48 - 3x + 12 + 4$
$y = 3x^2 - 27x + 64$

Example 3: Given the function
$f(x) = \sqrt{3x+6}$, find the equation of:

a) $y = 4f(x - 2)$
$y = 4\sqrt{3(x-2)+6}$
$y = 4\sqrt{3x-6+6}$
$y = 4\sqrt{3x}$

b) $y = -f(-3x)$
$y = -\sqrt{3(-3x)+6}$
$y = -\sqrt{-9x+6}$

c) $y = -f(x + 2) + 4$
$y = -\sqrt{3(x+2)+6} + 4$
$y = -\sqrt{3x+6+6} + 4$
$y = -\sqrt{3x+12} + 4$

d) $y = \frac{1}{2}f(-x - 2)$
$y = \frac{1}{2}\sqrt{3(-x-2)+6}$
$y = \frac{1}{2}\sqrt{-3x-6+6}$
$y = \frac{1}{2}\sqrt{-3x}$

e) $y = -2f(x) + 3$
$y = -2\sqrt{3x+6} + 3$

f) $y = -4f(-2x) - 5$
$y = -4\sqrt{3(-2x)+6} - 5$
$y = -4\sqrt{-6x+6} - 5$

TRANSFORMATIONS LESSON 3

PART I: ALGEBRAIC TRANSFORMATIONS

Questions:

1) $f(x) = 3x - 7$

a) $y = 3f(x)$

b) $y = f(6x)$

c) $y = -f(x)$

d) $y = f(-x)$

e) $y = f(x) + 5$

f) $y = f(x - 6)$

2) $f(x) = (2x - 3)^2$

a) $y = \frac{1}{2}f(x)$

b) $y = f(\frac{1}{2}x)$

c) $y = -3f(x)$

d) $y = f(-2x)$

e) $y = f(x) - 4$

f) $y = f(x + 5)$

3) $f(x) = -\sqrt{-2x - 3}$

a) $y = 2f(x - 3)$

b) $y = f(3x + 4)$

c) $y = -f(-x)$

d) $y = 2f(-x) + 4$

e) $y = -\frac{1}{2}f(x) + 3$

f) $y = f(-x - 4) + 9$

4) $f(x) = |3x - 4|$

a) $y = 2f(x) - 8$

b) $y = -f(7x - 1)$

c) $y = -\frac{1}{3}f(x) - 3$

d) $y = \frac{1}{2}f(-x)$

e) $y = 2f(x - 5) + 3$

f) $y = -f(-x - 4)$

Answers:

1. a) $y = 9x - 21$
b) $y = 18x - 7$
c) $y = -3x + 7$
d) $y = -3x - 7$
e) $y = 3x - 2$
f) $y = 3x - 25$

2. a) $y = \frac{1}{2}(2x - 3)^2$
b) $y = (x - 3)^2$
c) $y = -3(2x - 3)^2$
d) $y = (-4x - 3)^2$
e) $y = (2x - 3)^2 - 4$
f) $y = (2x + 7)^2$

3. a) $y = -2\sqrt{-2x + 3}$
b) $y = -\sqrt{-6x - 11}$
c) $y = \sqrt{2x - 3}$
d) $y = -2\sqrt{2x - 3} + 4$
e) $y = \frac{1}{2}\sqrt{-2x - 3} + 3$
f) $y = -\sqrt{2x + 5} + 9$

4. a) $y = 2|3x - 4| - 8$
b) $y = -|21x - 7|$
c) $y = -\frac{1}{3}|3x - 4| - 3$
d) $y = \frac{1}{2}|-3x - 4|$
e) $y = 2|3x - 19| + 3$
f) $y = -|-3x - 16|$

Describing Transformations: **This section deals with verbal descriptions of transformations.**

Example 1: How does the graph of $y = -3f(x - 4) + 2$ compare to $y = f(x)$?
Vertical Stretch by a factor of 3
Reflection in the x-axis
Translation of 4 Right and 2 Up.

Example 2: How does the graph of $y = 2(x + 4)^2 - 1$ compare to the graph of $y = x^2$
Vertical Stretch by a factor of 2
Translation of 4 Left and 1 Down.

Example 3: How does the graph of $y = (3x + 12)^2$ compare to the graph of $y = x^2$
First factor out the 3 from the x: $y = [3(x+4)]^2$
Horizontal stretch by a factor of 1/3.
Translation of 4 units left

Example 4: Given the graph of $y = \sqrt{x}$, write the new equation after a vertical stretch by a factor of 1/2, a horizontal stretch by a factor of 1/3, and a vertical translation of 3 units up.

First apply the stretches: $y = \frac{1}{2}\sqrt{3x}$

Now apply the translations: $y = \frac{1}{2}\sqrt{3x} + 3$

Example 5: The graph of $y = (x + 2)^2 + 1$ is shifted 6 units left and 4 units down. Determine the equation of the transformed function.
The best way to do this type of question is to find a point on the graph, then apply the transformation to that point.

We know the point (-2, 1) is on the graph. (It's the vertex of the parabola)
After moving 6 left & 4 down, it will become (-8, -3)
Rewrite the equation using these values: $y = (x + 8)^2 - 3$

PART II: DESCRIBING TRANSFORMATIONS

Questions:

1) Describe how each transformed function compares to the original:

a) Original is $y = f(x)$

Transformed is $y = -3f(\frac{1}{4}x)$

b) Original is $y = f(x)$

Transformed is $y = -\frac{1}{2}f(-x) - 4$

c) Original is $y = f(x)$
Transformed is $y = f(-x + 3)$

d) Original is $y = f(x - 2)$
Transformed is $y = f(x + 5)$

e) Original is $y = f(x + 7) + 2$
Transformed is $y = f(x + 4) - 6$

f) Original is $y = f(x)$
Transformed is $4y = f(x)$

g) Original is $y = |x|$

Transformed is $y = -3|x|$

h) Original is $y = x^2$
Transformed is $y = 2(-3x - 12)^2 + 3$

i) Original is $y = \sqrt{x}$

Transformed is $y = -\sqrt{2x + 4}$

j) Original is $y = (x - 3)^2$
Transformed is $y = (x - 4)^2$

k) Original is $y = (x + 8)^2 - 4$
Transformed is $y = (x + 6)^2 - 3$

l) Original is $y = x^2$
Transformed is $y + 3 = x^2$

Questions: *Continued*

2) Write out the transformation given the following information:

a) The graph of $y = x^2$ is transformed by a vertical stretch of a factor of $\frac{2}{3}$, a horizontal stretch by a factor of 3, and a horizontal translation of 3 units right.

b) The graph of $y = \sqrt{x}$ is reflected in the x-axis, and shifted down by 4 units.

c) The graph of $y = f(x)$ is vertically stretched by a factor of 4, reflected in the y-axis, and moved 7 units left.

d) The graph of $y = x^3$ is horizontally stretched by a factor of 3, reflected in the x-axis, and then moved 5 units down.

e) The graph of $y = |x|$ is vertically stretched by a factor of 4/3, horizontally stretched by a factor of 6, reflected in both the x & y axis, then shifted 2 units right and 2 units down.

f) The graph of $y = (x - 4)^2$ is shifted 5 units to the right.

g) The graph of $y = (x + 4)^2 - 6$ is shifted 2 units to the left and 5 units up.

h) The graph of $y = f(x - 1) - 3$ is shifted 7 units to the right and 3 units down.

i) The graph of $y = f(x)$ has y replaced with $\frac{1}{2}y$

j) The graph of $y = f(x)$ has y replaced with $y - 2$

Answers:

1.

a) Vertical Stretch by a factor of 3
Horizontal Stretch by a factor of 4
Reflection in the x-axis

b) Vertical Stretch by a factor of ½
Reflection in both the x & y axis
Translated 4 units down

c) *Rewrite as* $y = f\left[-(x-3)\right]$
Reflection in the y-axis
Translated 3 units right

d) *Original point = (2, 0)*
Transformed point = (-5, 0)
Translated 7 units left

e) *Original Point = (-7, 2)*
Transformed Point = (-4, -6)
Translated 3 units right and 8 units down.

f) *Rewrite as* $y = \frac{1}{4}f(x)$
Vertical Stretch by a factor of 1/4

g) Vertical Stretch by a factor of 3
Reflected in the x-axis

h) *Rewrite as* $y = 2\left[-3(x+4)\right]^2 + 3$
Vertical Stretch by a factor of 2
Horizontal stretch by a factor of 1/3
Reflected in the y-axis
Translated 4 units left and 3 units up

i) *Rewrite as* $y = -\sqrt{2(x+2)}$
Horizontal stretch by a factor of 1/2
Reflection in the x-axis
Translated 2 units left

j) *Original Point = (3, 0)*
Transformed Point = (4, 0)
Translated 1 unit right

k) *Original Point = (-8, -4)*
Transformed Point = (-6, -3)
Translated 2 units right and 1 unit up.

l) *Rewrite as* $y = x^2 - 3$
Translated 3 units down

2.

a) $y = \frac{2}{3}\left(\frac{1}{3}(x-3)\right)^2$

b) $y = -\sqrt{x} - 4$

c) $y = 4f\left[-\left(x+7\right)\right]$

d) $y = -\left(\frac{1}{3}x\right)^3 - 5$

e) $y = -\frac{4}{3}\left|-\frac{1}{6}(x-2)\right| - 2$

f) *Original Point = (4, 0)*
Transformed Point = (9, 0)
$y = (x-9)^2$

g) *Original Point = (-4, -6)*
Transformed Point = (-6, -1)
$y = (x+6)^2 - 1$

h) *Original Point = (1, -3)*
Transformed Point = (8, -6)
$y = f(x-8) - 6$

i) *Replace y with* $\frac{1}{2}y$:

$\frac{1}{2}y = f(x)$

$y = 2f(x)$

j) *Replace y with y – 2:*
$y - 2 = f(x)$
$y = f(x) + 2$

TRANSFORMATIONS LESSON 3

PART III: TRANSFORMING A POINT

Transforming a point: **Always transform a point by doing stretches / reflections first, followed by translations.**

Example 1: What will the point (-3, 4) become after a transformation of $y = -2f(-x - 4)$?

First rewrite the transformation as $y = -2f\left[-(x+4)\right]$

Multiply the x-values by -1, and the y-values by -2 to get (+3, -8)
Move four units left to get (-1, -8)

Example 2: What will a y-intercept of -2 become after a transformation of $y = -f(4x - 28) + 5$?

First rewrite the transformation as $y = -f\left[4(x-7)\right] + 5$

The original point is (0, -2)
Multiply the x-values by ¼ and the y-values by -1 to get (0, 2)
Move 7 right and 5 up to get (7, 7)

Example 3: If the function $f(x) = 2x^2 + 3x - 5$ is multiplied by a constant value *m*, the graph of $g(x) = mf(x)$ passes through the point (2, -27). Determine the value of *m*.

First rewrite the equation as $y = m(2x^2 - 3x + 5)$
Then plug in the given point:

$$-27 = m\left(2(2)^2 + 3(2) - 5\right)$$

$$-27 = 9m$$

$$m = -3$$

Questions:

Given the point (-5, 12), find the new point after each of the following transformations:

1) $y = -3f(\frac{1}{4}x)$

2) $y = -\frac{1}{2}f(-x) - 4$

3) $f(-x - 4) + 9$

4) $y = -2f(-5x - 15) - 6$

5) $y = \frac{1}{2}f(-x - 2)$

6) $y = f(\frac{1}{2}x)$

7) If the function $f(x) = x^2 + 3x - 7$ is multiplied by a constant value *m*, the graph of $g(x) = mf(x)$ passes through the point (-1, -18). Determine the value of *m*.

Answers:

1) (-20, -36)
2) (5, -10)
3) (1, 21)
4) (-2, -30)
5) (3, 6)
6) (-10, 12)
7) $m = 2$