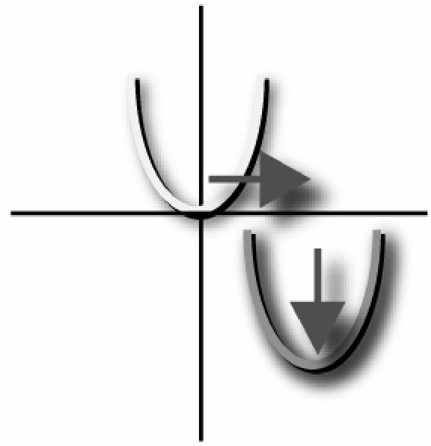
## Pure Math 30:

# TRANSFORMATIONS



# LESSON 4

## Other Transformations

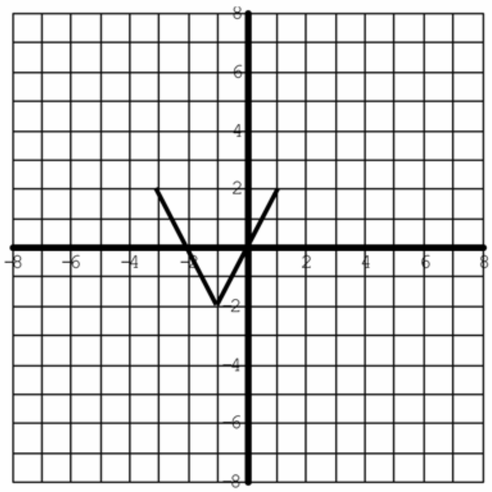
Pure Math 30: EXPLAINED!

By Barry Mabillard

# Transformations Lesson 4
## Part 1: Stretches About Other Lines

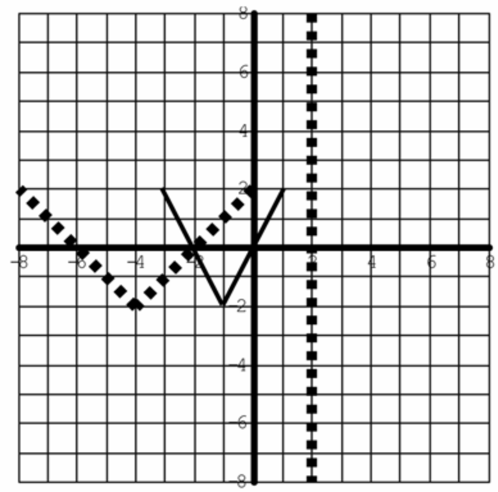
***Stretches about other lines:*** **Stretches about lines other than the x & y axis are frequently required.**

**Example 1: Stretch the graph horizontally by a factor of 2 about the line x = 2**



First draw in the vertical line x = 2.

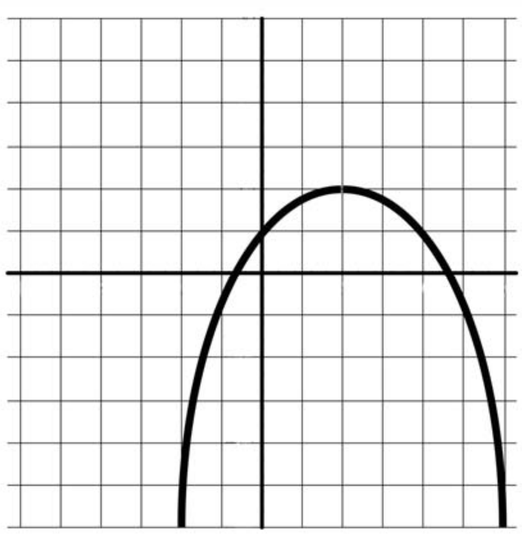
Count out how far each point is away from the line, then multiply by the stretch factor to find the new distance from the line.



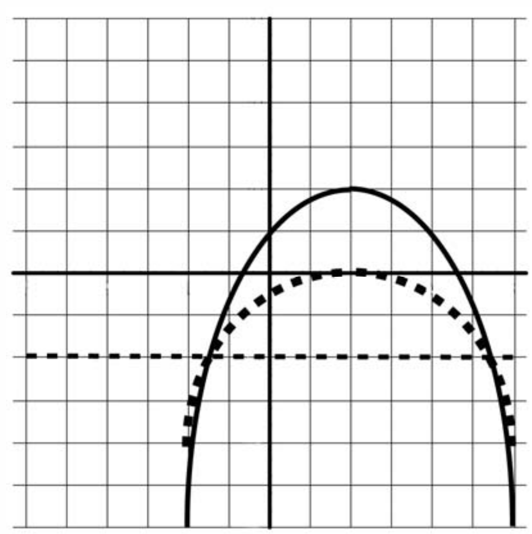
Notice how the top right point was 1 unit from the line, now it's 2 units away.

The top left point was 5 units from the line, now it's 10 units away.

**Example 2: Stretch the graph vertically by a factor of $\frac{1}{2}$ about the line y = -2**

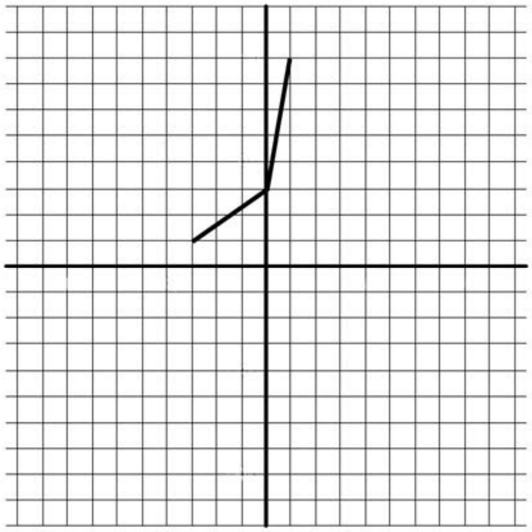


First draw in the horizontal line y = -2. Count out how far each point is away from the line, then multiply by the stretch factor to find the new distance from the line.



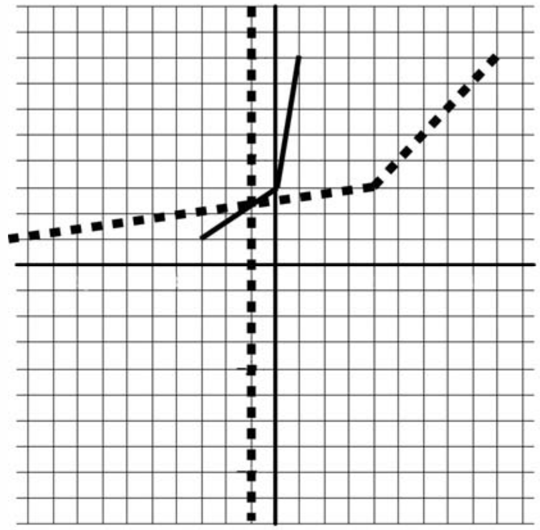
Notice how the top of the parabola was 4 units from the line, now it's 2 units away.

**Example 3:** Stretch the graph horizontally by a factor of 5 about the line x = -1



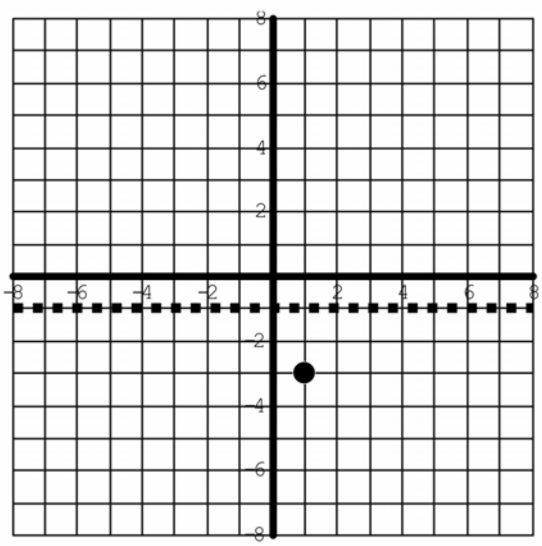
First draw in the vertical line x = -1.

Then apply the horizontal stretch by counting the spaces, multiplying by 5, and drawing in the new points.

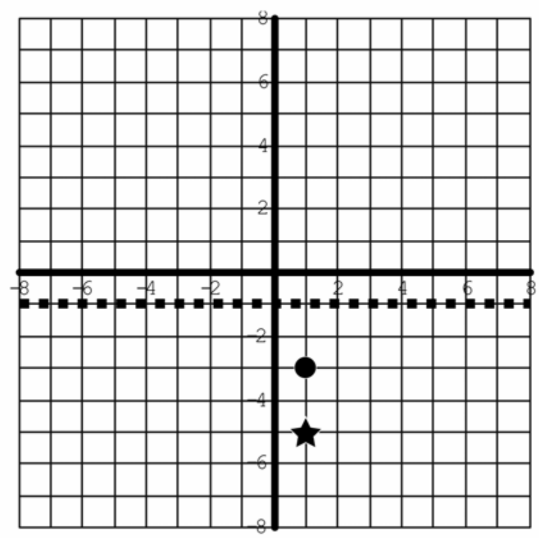


**Example 4:** Find the new coordinates of the point (1, -3) after a vertical stretch by a factor of 2 about the line y = -1

First draw in the horizontal line y = -1 and the point (1, -3)




The point is 2 units away from the line initially, so after multiplying by 2 it will be 4 units away from the line. New Point = (1, -5)

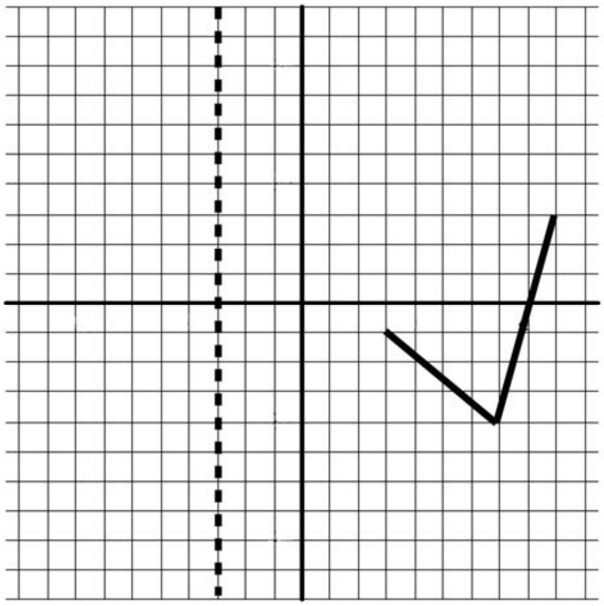



In these types of stretches, the invariant points are on the line you are stretching about.

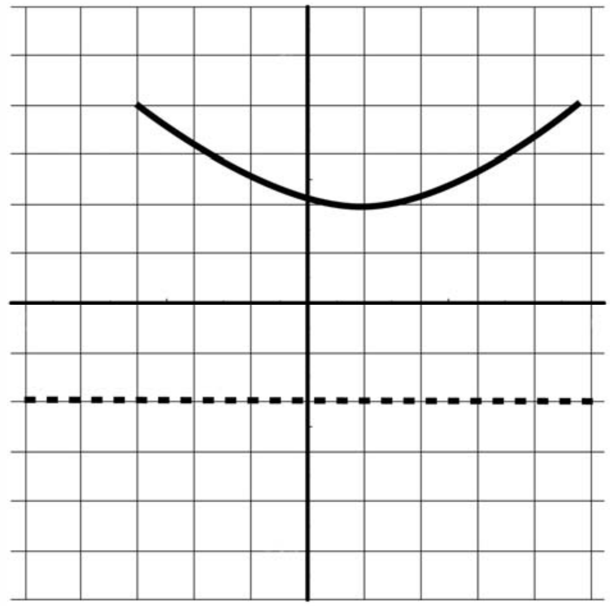
## Part 1: Stretches About Other Lines

**Questions:** For each of the following, draw in the transformed graph:

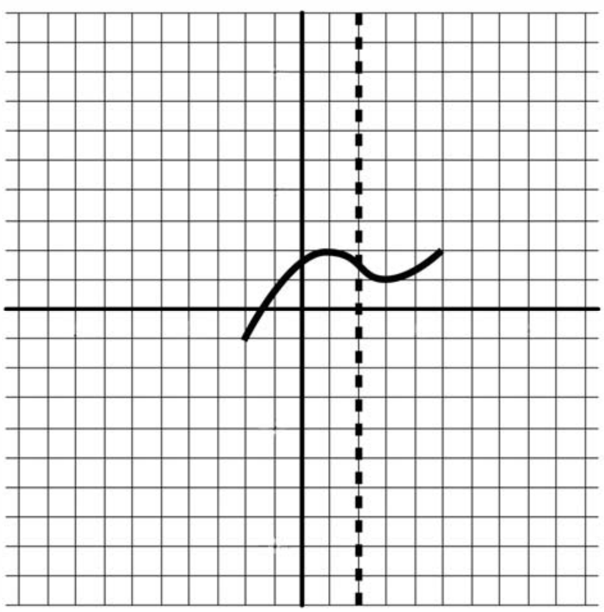
1) Horizontal stretch by a factor of 1/2 about the line x = -3



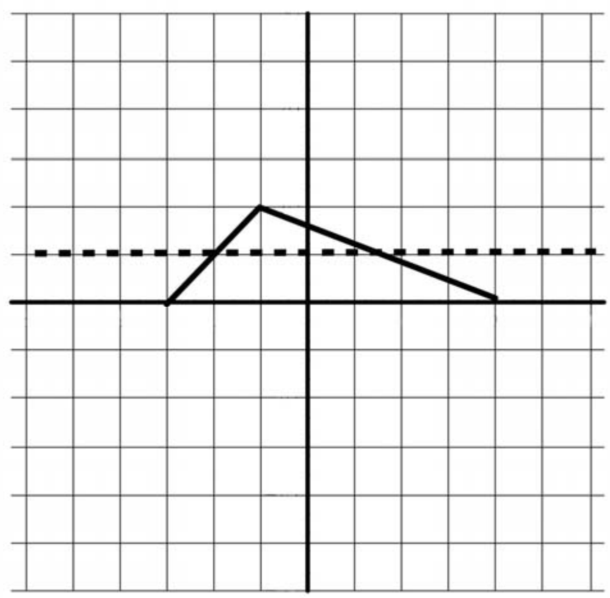
2) Vertical stretch by a factor of 1/2 about the line y = -2



3) Horizontal stretch by a factor of 2 about the line x = 2



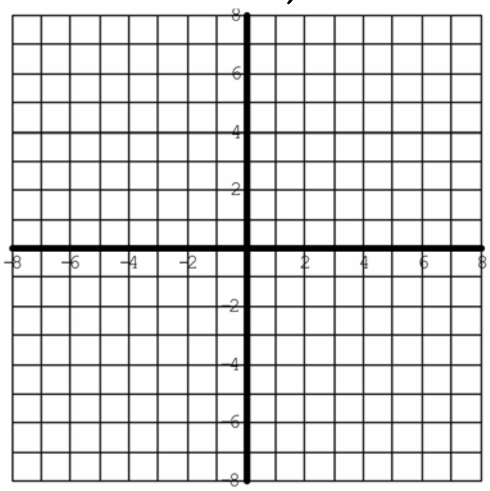
4) Vertical stretch by a factor of 5 about the line y = 1



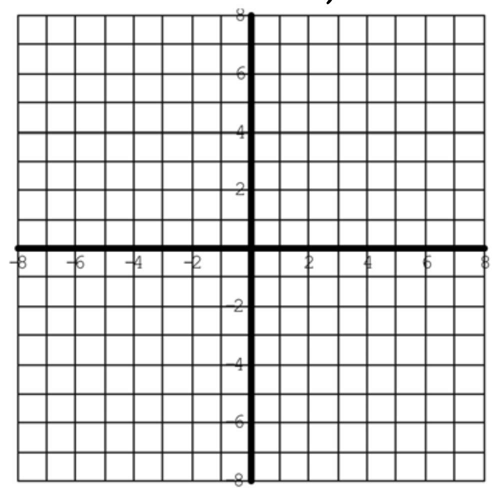
## Questions: Continued

**5) Find the new coordinates of the point (3, -2) after the following transformation:**

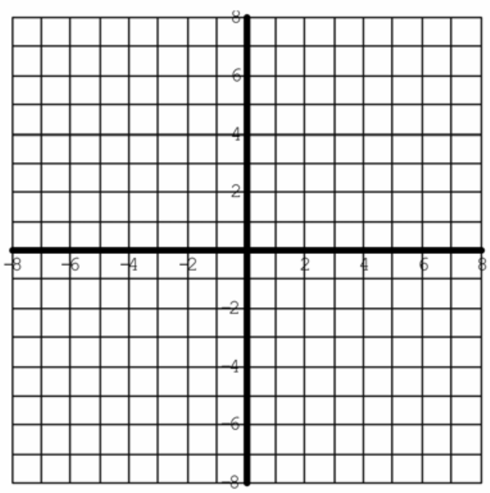
a) Vertical stretch by a factor of 3 about the line y = -1



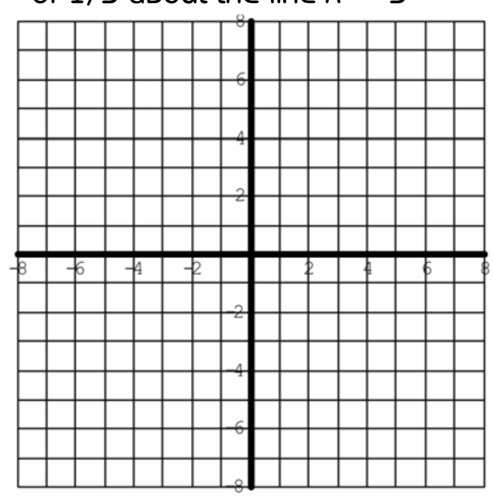
b) Vertical stretch by a factor of 1/2 about the line y = 2



c) Horizontal stretch by a factor of 2 about the line x = 1

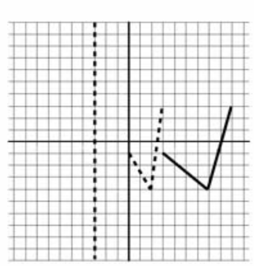


d) Horizontal stretch by a factor of 1/3 about the line x = -3

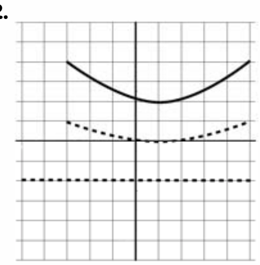


## Answers:

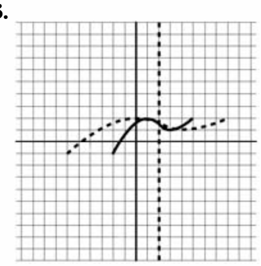
**1.**



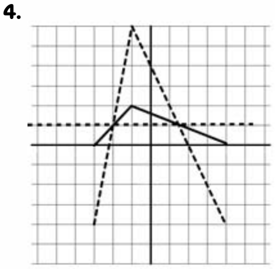
**2.**



**3.**



**4.**

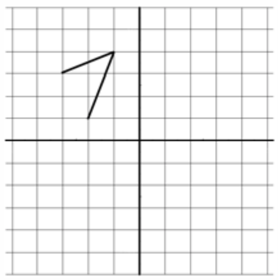


**5.**
a) (3, -4)
b) (3, 0)
c) (5, -2)
d) (-1, -2)

# Transformations Lesson 4
## Part II: Inverses

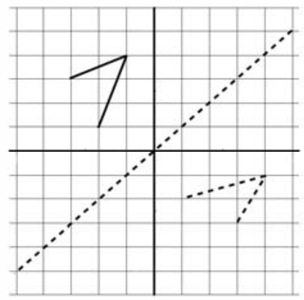
***Inverses:*** **An inverse is defined as $x = f(y)$ or $y = f^{-1}(x)$, and may be obtained by interchanging the x & y values. As such, inverses are reflected over the line $y = x$.**

**Example 1:** **Draw in the inverse of the given graph.**



First draw in the line y=x. Now graph the inverse by interchanging the x & y values.

*e.g.* The point (-2, 1) will become (1, -2)



**Example 2:** **Determine the equation of the inverse to $f(x) = 3x - 4$**

First rewrite f(x) as y : $y = 3x - 4$

Now interchange x & y : $x = 3y - 4$

Solve for the new y : $x + 4 = 3y$

$$y = \frac{x+4}{3}$$

Inverses are written with the notation $f^{-1}(x)$, so express the final answer as : $f^{-1}(x) = \frac{x+4}{3}$

**Example 3:** **Find the equation of the inverse to $f(x) = (x - 3)^2 - 4$**

$y = (x - 3)^2 - 4$

$x = (y - 3)^2 - 4$

$x + 4 = (y - 3)^2$

$\pm\sqrt{x+4} = y - 3$ Remember that you need the ± when taking a square root

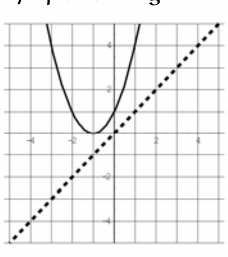
$f^{-1}(x) = \pm\sqrt{x+4} + 3$

**For inverses, the invariant points are on y = x**

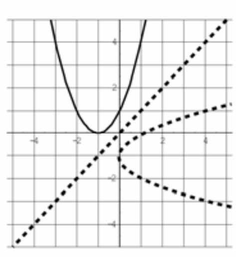
**Example 4:** **How would you restrict the domain of $y = (x + 1)^2$ such that the inverse is a function?**

First find the equation of the inverse : $f^{-1}(x) = \pm\sqrt{x} - 1$

Graph the original



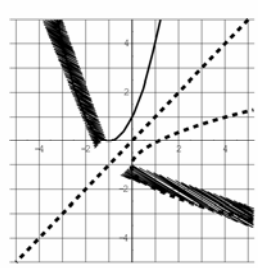
Now draw the inverse



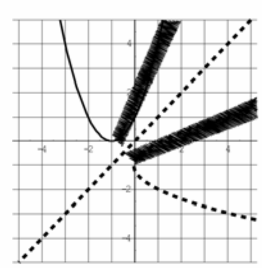
Notice that the inverse is NOT a function, since it doesn't pass the vertical line test.

To make the inverse a function, we need to restrict the domain of the original so the inverse *will* pass the vertical line test.

The domain of the original could be restricted to $x \geq -1$ This will force the inverse to be a function.

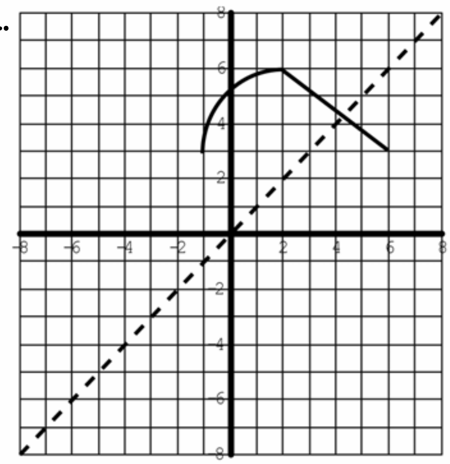


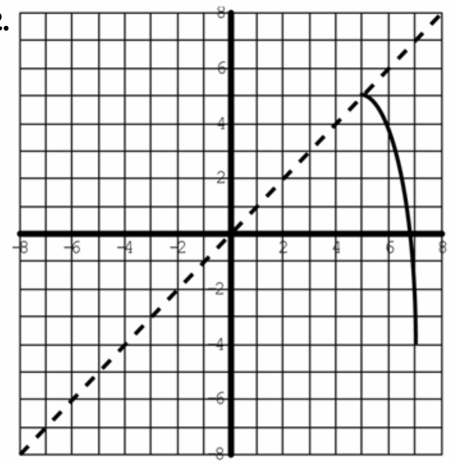
Or, the domain of the original could be restricted to $x \leq -1$ This will also force the inverse to be a function.

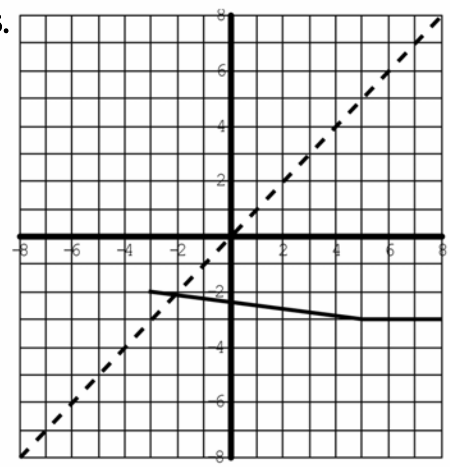


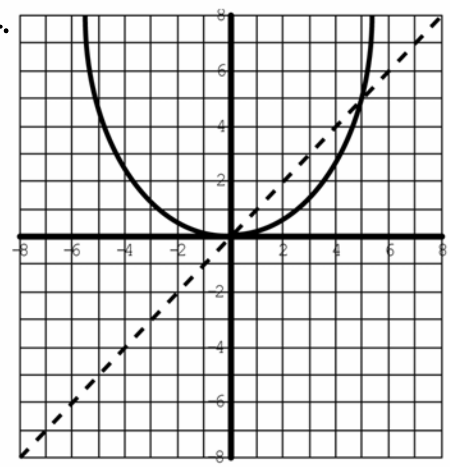
**Questions:**

Draw in the inverse for each of the following graphs.

1.


2.


3.


4.


**5) Find the equation of the inverse for each of the following:**

a) $f(x) = 4x - 5$

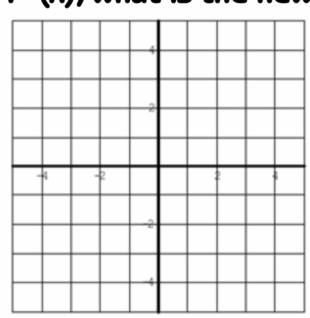
b) $f(x) = x^2 - 4$

c) $f(x) = (x + 2)^2$

d) $f(x) = \sqrt{x - 1}$

**6) If the point (-3, 4) undergoes the transformation $y = f^{-1}(x)$, what is the new point?**

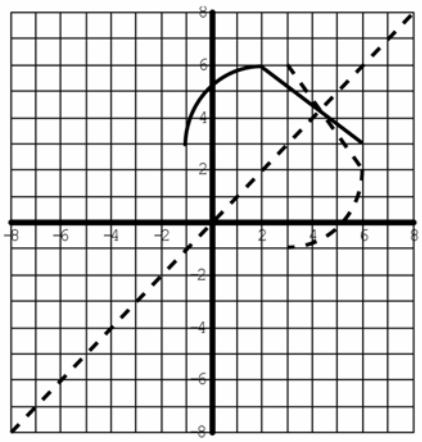
**7) How would you restrict the domain of $y = (x - 2)^2$ such that the inverse $f^{-1}(x) = \pm\sqrt{x} + 2$ is a function?**



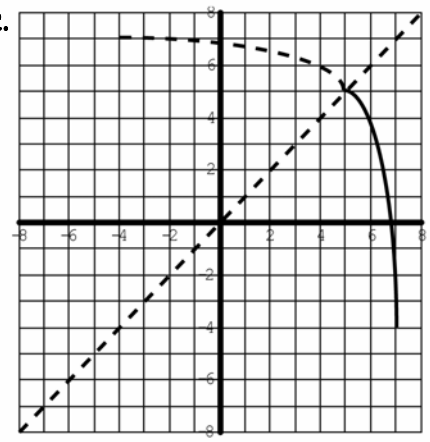
# Transformations Lesson 4
## Part II: Inverses

**Answers:**

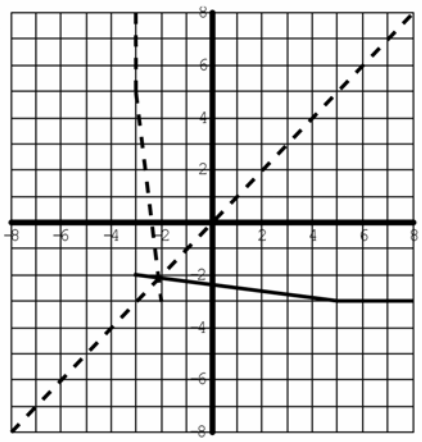
**1.**



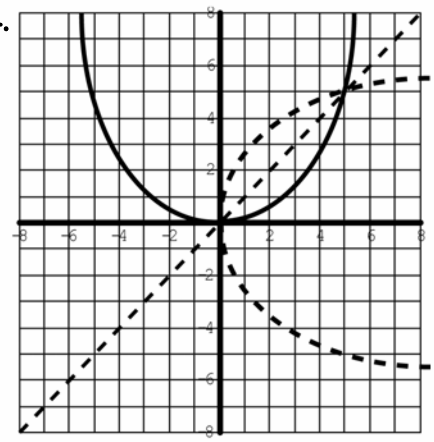
**2.**



**3.**



**4.**



**5.**

a) $f^{-1}(x) = \dfrac{x+5}{4}$

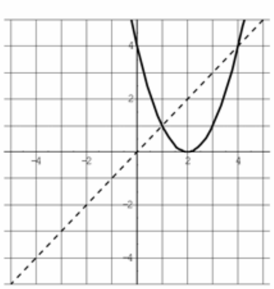
b) $f^{-1}(x) = \pm\sqrt{x+4}$

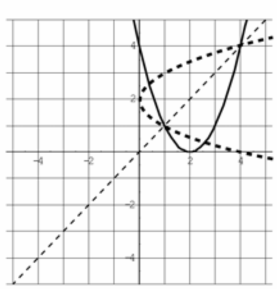
c) $f^{-1}(x) = \pm\sqrt{x} - 2$

d) $f^{-1}(x) = x^2 + 1$

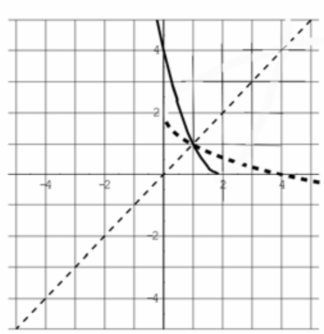
**6.** The new point is found by interchanging x & y, (4, -3)

**7.** First graph the function & the inverse.

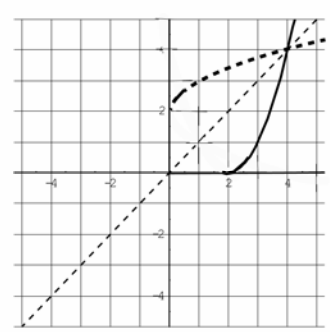




If the domain of the original is restricted to $x \leq 2$, this will force the inverse to be a function.



Or, the domain of the original could be restricted to $x \geq 2$. This will also force the inverse to be a function.



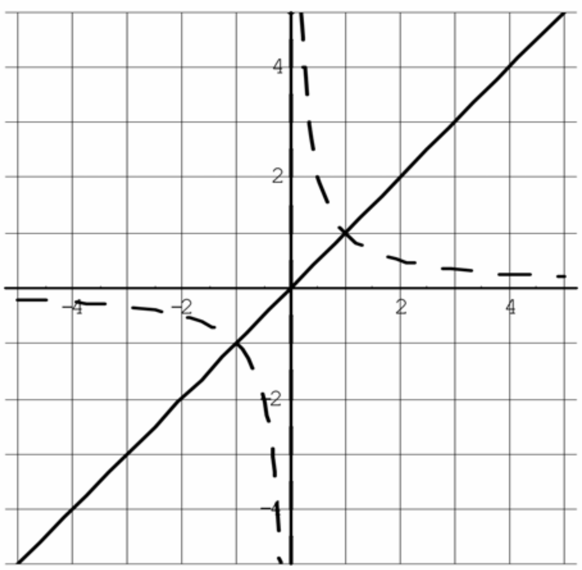
**Reciprocal Functions:** A reciprocal function is represented by $y = \dfrac{1}{f(x)}$

Note that $\dfrac{1}{f(x)}$ and $f^{-1}(x)$ do NOT mean the same thing! The first is a reciprocal graph, the second is an inverse graph. They are very different.

**Example 1:** Draw $y = x$ and the reciprocal graph, $y = \dfrac{1}{x}$

Black: $y = x$

Dashed: $y = \dfrac{1}{x}$




| $f(x)$ | $\dfrac{1}{f(x)}$ |
|---|---|
| Obtain the values in this column by using the y-values of the original graph. | Reciprocal the numbers from the first column. |
| -4 | -0.25 |
| -2 | -0.5 |
| -1 | -1 |
| -0.5 | -2 |
| -0.1 | -10 |
| 0 | UNDEFINED |
| 0.1 | 10 |
| 0.5 | 2 |
| 1 | 1 |
| 2 | 0.5 |
| 4 | 0.25 |

*Remember what happens in basic division of numbers:*

As the denominator becomes larger, the resulting number becomes smaller.

$\dfrac{1}{2} = 0.5$ $\quad \dfrac{1}{4} = 0.25$ $\quad \dfrac{1}{100} = 0.01$

As the denominator becomes smaller, the resulting number becomes larger.

$\dfrac{1}{0.5} = 2$ $\quad \dfrac{1}{0.25} = 4$ $\quad \dfrac{1}{0.01} = 100$

Remember that dividing by zero is undefined, so at the x-intercepts of the original, the reciprocal graph has no corresponding point.

Indicate this on the graph by drawing a ***vertical asymptote***, since the graph will approach this line from both sides but never actually reach it.

**Note the following characteristics of reciprocal graphs:**

**1) The reciprocal graph will always be on the same side of the x-axis as the original.**
*If the original is above the x-axis, the reciprocal is also above.*
*If the original is below, the reciprocal is also below.*

**2) The vertical asymptotes are drawn at the x-intercepts of the original.**
*The equation of a vertical line is of the form: x = constant.*
*In the graph above, the equation of the vertical asymptote is x = 0*

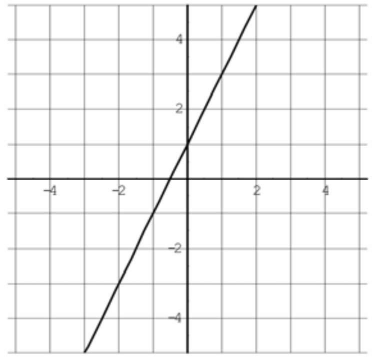
**3) Horizontal asymptotes are may also by present in the reciprocal graph.**
*The equation of a horizontal line is of the form: y = constant.*
*In the graph above, the equation of the horizontal asymptote is y = 0*

**4) Invariant points are points that don't change position when a transformation is applied.**
*In reciprocal graphs, the invariant points are located at y = -1 and y = 1.*

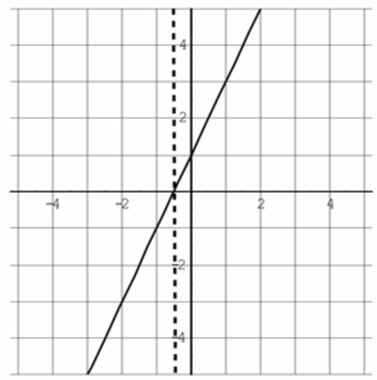
# Transformations Lesson 4
## Part III: Linear Reciprocals

**Example 2:** **Draw the graph of $y = 2x + 1$ and it's reciprocal, $y = \dfrac{1}{2x+1}$**

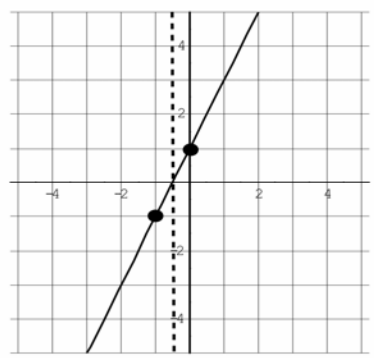
*Step 1:* Draw $y = 2x + 1$



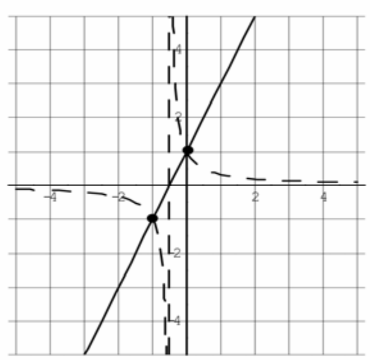

*Step 2:* Draw in the vertical asymptote at the x-intercept.



*Step 3:* Place dots at the invariant points (wherever y = ±1 on the original)



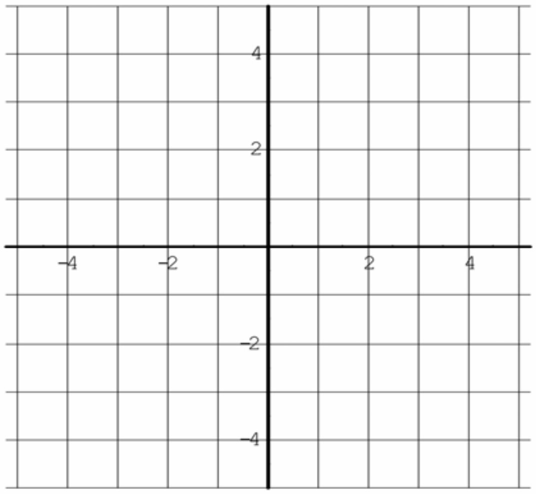
*Step 4:* Draw in the reciprocal graph.



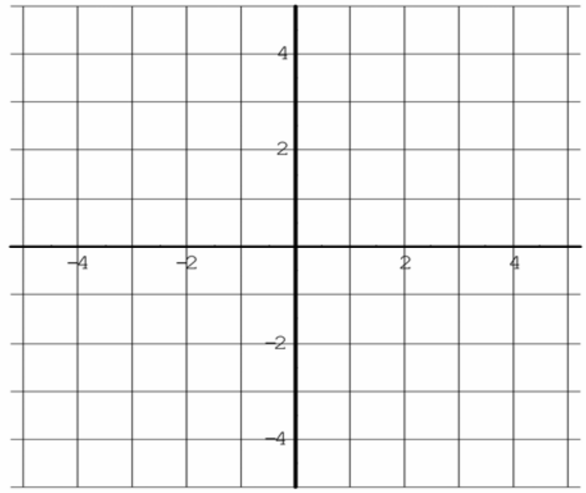
Domain: $x \neq -0.5$
Range: $y \neq 0$

***Questions:*** **Draw the original & reciprocal graphs for each of the following functions. State the domain and range for the reciprocal.**

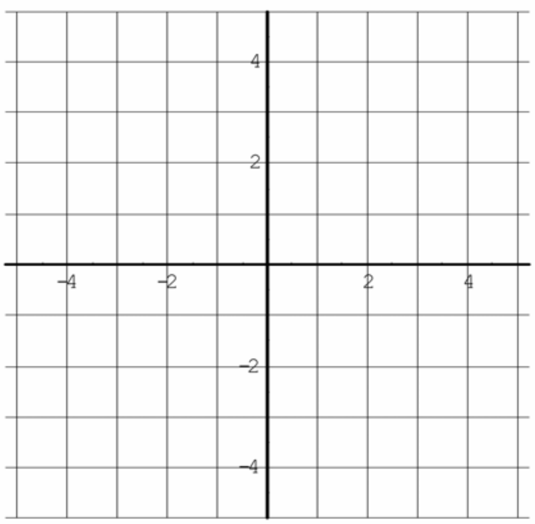
**1)** $y = -x$



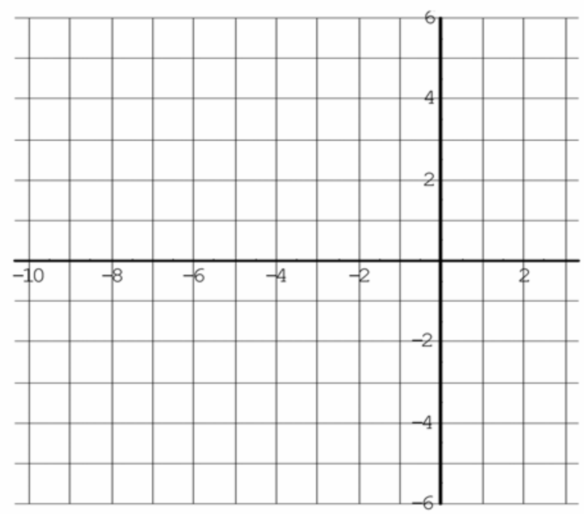

**2)** $y = x - 3$




**3)** $y = 2x + 2$

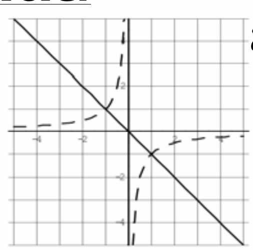



**4)** $y = -\frac{1}{2}x - 2$



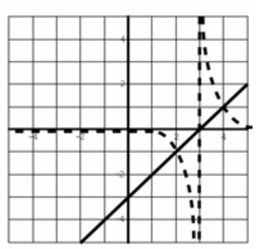

***Answers:***

**1)**



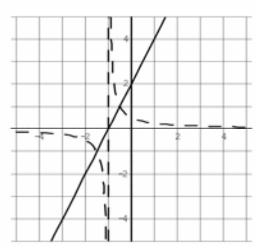
Domain: $x \neq 0$
Range: $y \neq 0$

**2)**



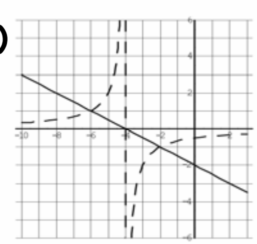
Domain: $x \neq 3$
Range: $y \neq 0$

**3)**



Domain: $x \neq -1$
Range: $y \neq 0$

**4)**

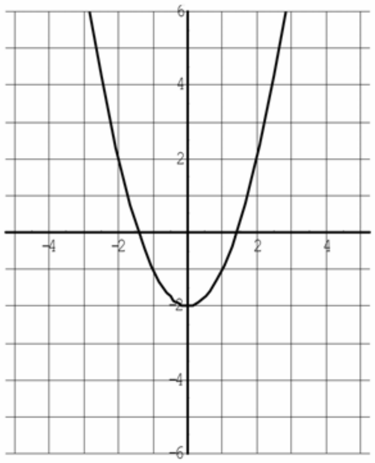


Domain: $x \neq -4$
Range: $y \neq 0$

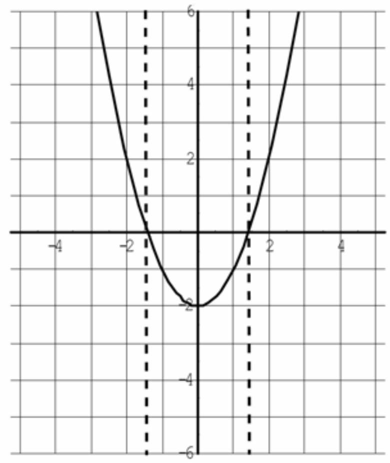
**Reciprocals of Non-Linear Graphs:** In these graphs, you have to be careful about where a reciprocal passes through, touches, or completely misses the original graph.

**Example 1:** Draw the reciprocal of $y = x^2 - 2$.

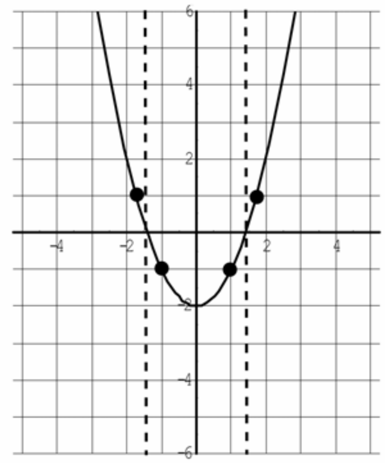
***Step 1:*** Draw $y = x^2 - 2$



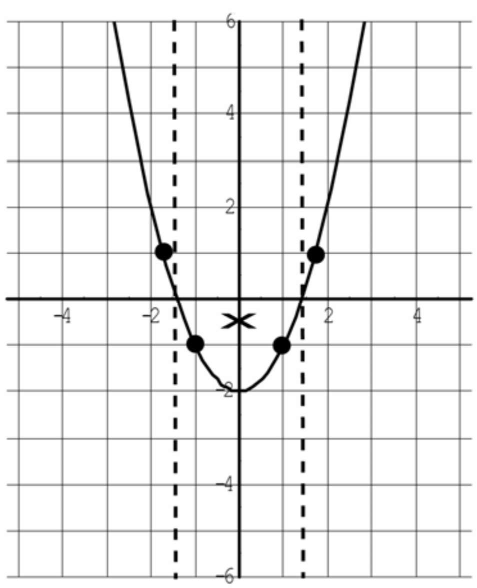
***Step 2:*** Draw in vertical asymptotes at the x-intercepts



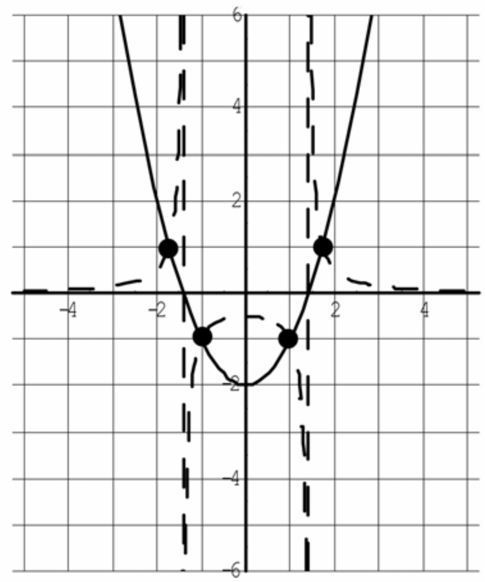
***Step 3***: Draw in dots at the invariant points.



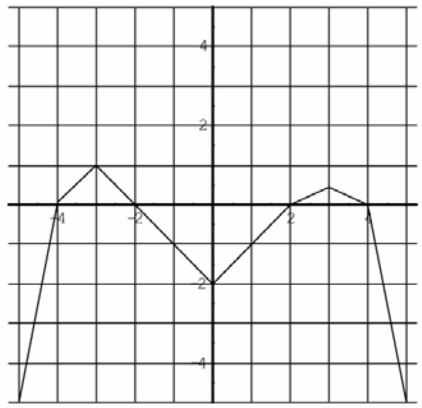
***Step 4:*** Look at the bottom tip of the parabola. The y-value there is -2, so the reciprocal value is -0.5. Put a tick there since the reciprocal graph must pass through that point.



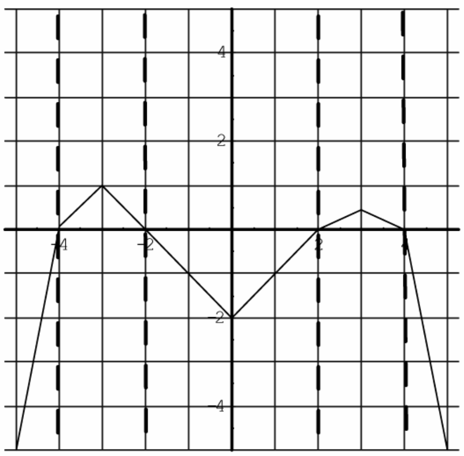
***Step 5:*** Draw in the graph



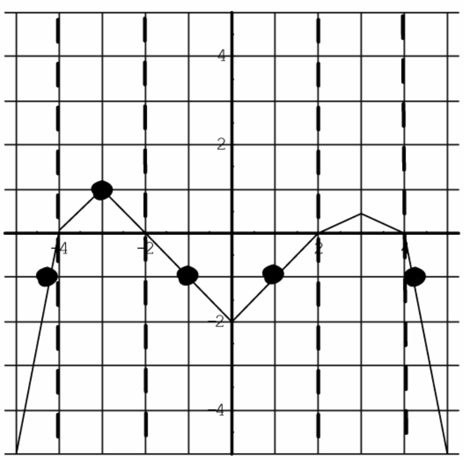
**Example 2: Draw in the reciprocal of the following graph:**



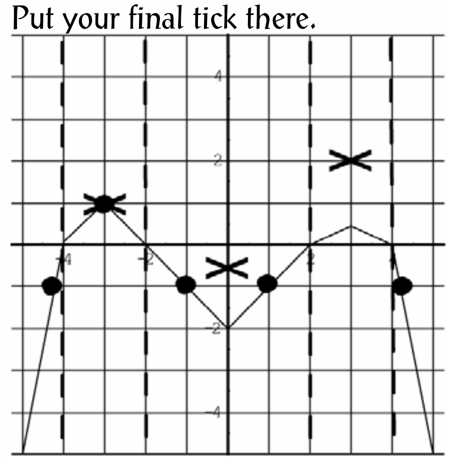
***Step 1:*** Draw in the asymptotes



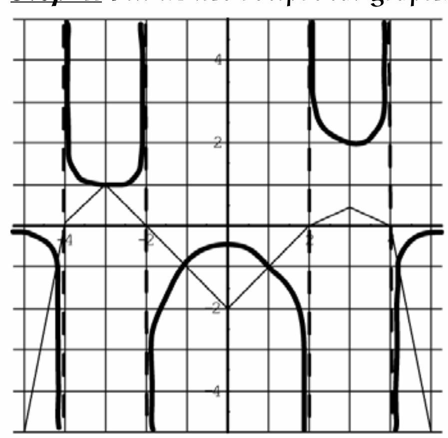
***Step 2:*** Draw in dots at all invariant points.



***Step 3:***
The first tip is at y = 1, and the reciprocal of this is also 1. Place a tick there.
The next tip is at -2, with a reciprocal of -0.5. Place another tick there.
The last tip is at 0.5, and the reciprocal is 2. Put your final tick there.

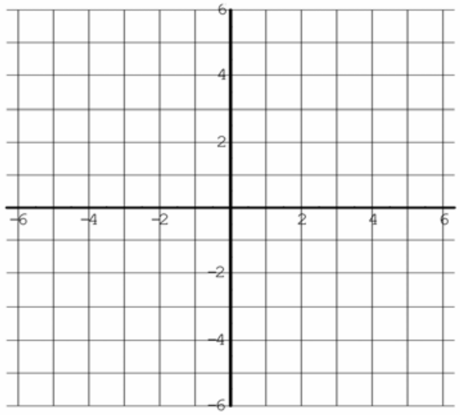


***Step 4:*** Fill in the reciprocal graph.

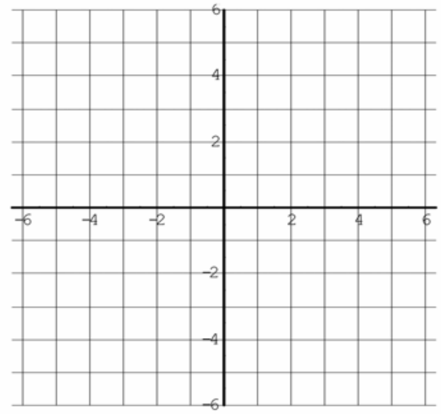


**Questions:** Draw the original & reciprocal graphs.

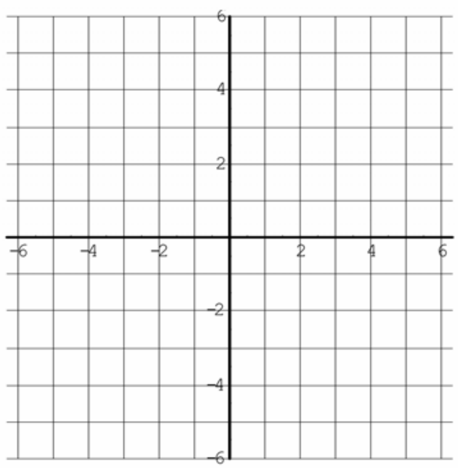
**1.** $y = \frac{1}{2}x^2 - \frac{1}{2}$



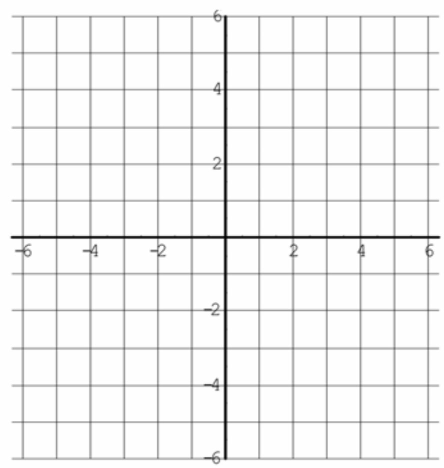
**2.** $y = -(x-2)^2 + \frac{1}{2}$



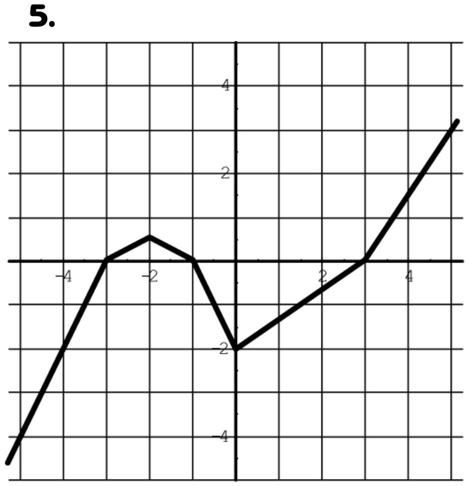
**3.** $y = -(x+2)^2 + 1$



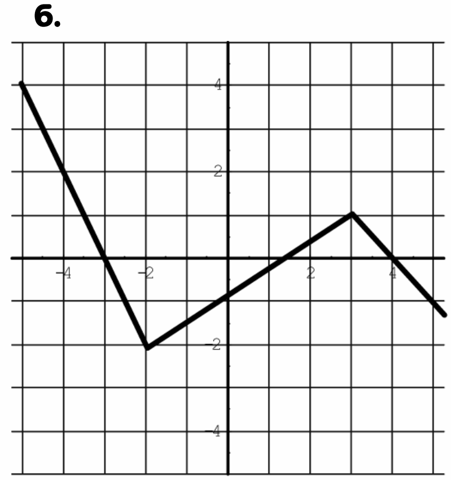
**4.** $y = (x-3)^2 - 2$



**5.**

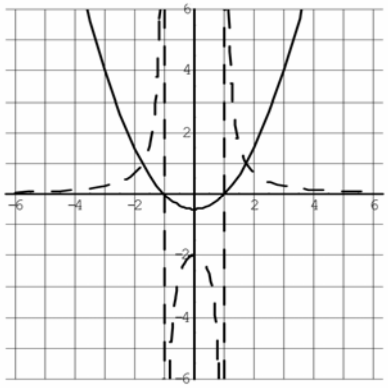


**6.**

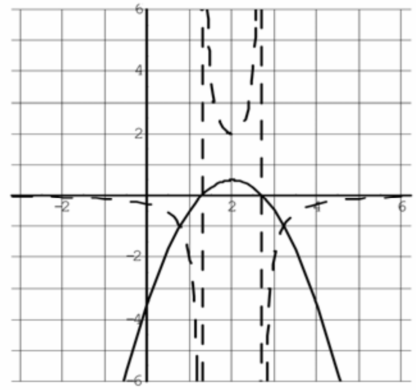


***Answers:***

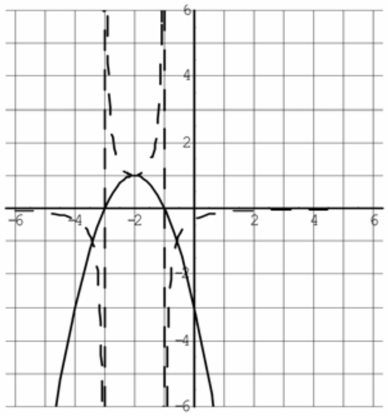
**1.**



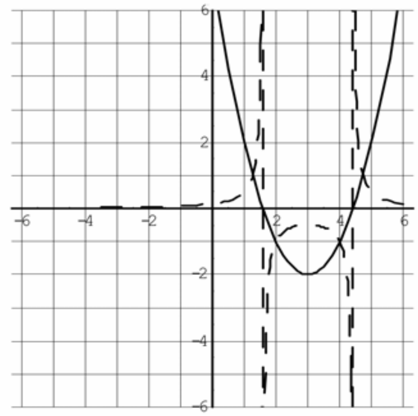
**2.**



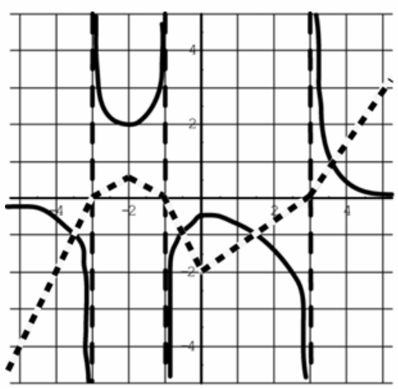
**3.**



**4.**



**5.**



**6.**