A note on a theorem of Erdös & Gallai [*]

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A sequence $d_1, d_2, \ldots, d_p$ of nonegative integers is called the *Degree Sequence* of a graph G if the vertices of G can be labeled $v_1, v_2, \ldots, v_p$ such that $deg\, v_k = d_k$ for each k, $1 \le k \le p$. A sequence $a_1, a_2, \ldots, a_p$ of nonnegative integers is called *graphical* if it is the degree sequence of some graph. Any graphical sequence clearly satisfies the two conditions $a_k \le p-1$ for each k and $\sum_{k=1}^{p} a_k$ is *even*. However, these two conditions together do not ensure that a sequence will be graphical. Necessary and sufficient conditions for a sequence of nonnegative integers to be graphical are well known. Two characterizations of graphical sequences are due to *Havel, Hakimi and Erdös & Gallai*:

Theorem H-H (Havel [3] and Hakimi [2]). A sequence of nonnegative integers $a_1, a_2, \ldots, a_p$ with $a_1 \ge a_2 \ge \cdots \ge a_p$, $a_1 \ge 1$ and $p \ge 3$ is graphical if and only if the sequence $a_2 - 1, a_3 - 1, \ldots, a_{a_1+1} - 1, a_{a_1+2}, \ldots, a_p$ is graphical.

Theorem EG (Erdös and Gallai [1]). A sequence of positive integers $a_1, a_2, \ldots, a_p$ with $a_1 \ge a_2 \ge \cdots \ge a_p$ is graphical if and only if $\sum_{k=1}^{p} a_k$ is even and for each integer n, $1 \le n \le p-1$,

$$\sum_{k=1}^{n} a_k \le n(n-1) + \sum_{k=n+1}^{p} \min(n, a_k).$$

The theorem of Erdös and Gallai (Theorem EG) requires the verification of an inequality for each n, $1 \le n \le p-1$. It is interesting to observe that the inequality need not be checked for $n > s$, where s is the *largest* integer such that $a_s \ge s-1$. For $n > s$, the inequality in Theorem EG reduces to

$$\sum_{k=1}^{n} a_k \le n(n-1) + \sum_{k=n+1}^{p} a_k.$$

Consider the difference between the right- and left-hand sides of the inequality as a function of n. Replacing n by $n+1$, this difference increases by $2(n - a_{n+1}) > 0$ since $n > s$. Thus, assuming the inequality holds for all $n \le s$, it will also hold for all $n \le p-1$. We record this observation as a

Lemma 1. *Let $\{a_1, a_2, \ldots, a_p\}$ be a sequence of positive integers with $a_1 \ge a_2 \ge \cdots \ge a_p$. Let s be the largest integer such that $a_s \ge s-1$. Then the sequence $\{a_1, a_2, \ldots, a_p\}$ is graphical if and only if $\sum_{k=1}^{p} a_k$ is even and for each integer n, $1 \le n \le s$,*

$$\sum_{k=1}^{n} a_k \le n(n-1) + \sum_{k=n+1}^{p} a_k.$$

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The lemma states that the number of inequalities to check in Theorem EG can be reduced. The purpose of this note is prove a refined form of Theorem EG mentioned in the introduction. We show that in case of multiple occurrences of numbers in the degree sequence, it suffices to check the inequality in Theorem EG only at the end of each segment of repeated values. Throughout this paper we shall employ the notation $(a)_m$ to denote m occurrences of the integer a. Thus, we may denote a typical degree sequence by

$$s := (a_1)_{m_1}, (a_2)_{m_2}, \ldots, (a_\ell)_{m_\ell}, \tag{1}$$

where $a_1 > a_2 > \cdots > a_\ell$ and each $m_k \geq 1$ with $m_1 + m_2 + \cdots + m_\ell = p$.

We shall write

$$\sigma_k \doteq \sum_{i=1}^{k} m_i, \quad \text{with} \quad \sigma_0 \doteq 0, \quad \text{and} \quad S_{r,t} \doteq \sum_{i=r}^{t} a_i m_i.$$

Our main result is the

Theorem: A sequence (1) is graphical if and only if $S_{1,\ell}$ is even and the inequality in Theorem EG holds for $n = \sigma_k$, $1 \leq k \leq \ell$.

Proof. By Theorem EG, we only need to prove that checking the inequality at each σ_k implies the inequality holds at each n. Suppose the inequality holds at each σ_k, but is not valid for some $n = N$. Let N_0 be the *least* such N, and write

$$N_0 = \sigma_k + n', \quad \text{where} \quad 1 \leq n' < m_{k+1} \quad \text{and} \quad 0 \leq k < \ell.$$

Thus,

$$\begin{aligned} S_{1,k} + a_{k+1} n' \quad &> \quad (\sigma_k + n')(\sigma_k + n' - 1) + (m_{k+1} - n') \min(a_{k+1}, \sigma_k + n') \\ &\quad + m_{k+2} \min(a_{k+2}, \sigma_k + n') + \cdots + m_\ell \min(a_\ell, \sigma_k + n') \end{aligned} \tag{2}$$

and

$$\begin{aligned} S_{1,k} + a_{k+1}(n' - 1) \quad &\leq \quad (\sigma_k + n' - 1)(\sigma_k + n' - 2) + (m_{k+1} - n' + 1) \min(a_{k+1}, \sigma_k + n' - 1) \\ &\quad + m_{k+2} \min(a_{k+2}, \sigma_k + n' - 1) + \cdots + m_\ell \min(a_\ell, \sigma_k + n' - 1) \end{aligned} \tag{3}$$

Suppose now that $a_{k+1} < \sigma_k + n'$. Then subtracting (3) from (2) gives the inequality

$$a_{k+1} > 2(\sigma_k + n' - 1) - a_{k+1},$$

which contradicts our assumption. Thus,

$$a_{k+1} \geq \sigma_k + n' \tag{4}$$

and (2) reduces to

$$\begin{aligned} S_{1,k} + a_{k+1} n' \quad &> \quad (\sigma_k + n')(\sigma_{k+1} - 1) + m_{k+2} \min(a_{k+2}, \sigma_k + n') \\ &\quad + \cdots + m_\ell \min(a_\ell, \sigma_k + n') \end{aligned} \tag{5}$$

Let r be such that $a_r < \sigma_k + n' \le a_{r-1}$. Such an r exists because $\sigma_k + n' < a_\ell$ together with (5) would imply

$$a_1(\sigma_k + n') \ge S_{1,k} + a_{k+1}n' > (\sigma_k + n')(\sigma_\ell - 1),$$

which is impossible since $a_1 \le p - 1 = \sigma_\ell - 1$. From (4), $r \ge k+2$, and (5) further reduces to

$$S_{1,k} + a_{k+1}n' > (\sigma_k + n')(\sigma_{r-1} - 1) + S_{r,\ell} \tag{6}$$

and (3) similarly to

$$S_{1,k} + a_{k+1}(n' - 1) \le (\sigma_k + n' - 1)(\sigma_{r-1} - 1) + S_{r,\ell} \tag{7}$$

Now subtracting (7) from (6) yields

$$a_{k+1} \ge \sigma_{r-1} \tag{8}$$

Let n'' be the *largest* integer $\le m_{k+1}$ for which the inequality in Theorem EG is not valid for $\sigma_k + n''$; from the definition, $n' \le n'' < m_{k+1}$. Furthermore, analogous to (2) and (3), we have

$$\begin{aligned} S_{1,k} + a_{k+1}n'' \;\; > \;\; & (\sigma_k + n'')(\sigma_k + n'' - 1) + (m_{k+1} - n'')\ \min(a_{k+1}, \sigma_k + n'') \\ & + m_{k+2}\ \min(a_{k+2}, \sigma_k + n'') + \cdots + m_\ell\ \min(a_\ell, \sigma_k + n'') \end{aligned} \tag{9}$$

and

$$\begin{aligned} S_{1,k} + a_{k+1}(n'' + 1) \;\; \le \;\; & (\sigma_k + n'')(\sigma_k + n'' + 1) + (m_{k+1} - n'' - 1)\ \min(a_{k+1}, \sigma_k + n'' + 1) \\ & + m_{k+2}\ \min(a_{k+2}, \sigma_k + n'' + 1) + \cdots + m_\ell\ \min(a_\ell, \sigma_k + n'' + 1) \end{aligned} \tag{10}$$

By (8), $\sigma_k + n'' < \sigma_{k+1} \le \sigma_{r-1} \le a_{k+1}$. Define s such that $a_s \le \sigma_k + n'' < a_{s-1}$, and note that $k + 1 < s \le r$. Now, the difference between the right hand sides of (9) and (10) equals

$$2(\sigma_k + n'') + (m_{k+1} - \sigma_k - 2n'' - 1) + m_{k+2} + \cdots + m_{s-1} = \sigma_{s-1} - 1.$$

Since a_{k+1} is the difference between the left hand sides of (9) and (10), the inequalities

$$\sigma_{s-1} - 1 < \sigma_{s-1} \le \sigma_{r-1} \le a_{k+1},$$

lead to a contradiction. This completes the proof of our result. □

References

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