LECTURE NOTE : INTRODUCTION TO ERGODIC THEORY II

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1. Examples of ergodic transformations

A measure preserving transformation T on $(X, \mathcal{B}, \mu)$ is called ergodic if there is no invariant set (modulo measure zero) except for $\emptyset$ or X. Another definition of the ergodicity is there is no invariant function except for constant functions, i.e., $f \circ T(x) = f(x)$ (modulo measure zero set) implies that $f(x)$ is constant. We have examples of ergodic transformations:

(1) Irrational rotations on the unit circle: Let $T : [0,1) \to [0,1)$ by $x \mapsto x + \alpha \pmod 1$, where α is irrational. The Lebesgue measure m preserves T.

Suppose that $f \circ T(x) = f(x)$ and $f \in L^2(m)$. Expand $f(x)$ in Fourier series as $f(x) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}$. Then

$$f \circ T(x) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n (x+\alpha)} = \sum_{-\infty}^{\infty} a_n e^{2\pi i n \alpha} e^{2\pi i n x}.$$

From $f \circ T(x) = f(x)$, we have

$$a_n e^{2\pi i n \alpha} = a_n \text{ for all } n.$$

Since α is irrational, $e^{2\pi i n \alpha}$ cannot be 1 unless $n = 0$. Therefore $a_n = 0$ for all $n \neq 0$, which implies that $f(x)$ is constant.

(2) $2x$ map on the unit circle: Let $T : [0,1) \to [0,1)$ by $x \mapsto 2x \pmod 1$. The Lebesgue measure m preserves T.

Suppose that $f \circ T(x) = f(x)$ and $f \in L^2(m)$. Expand $f(x)$ in Fourier series as $f(x) = \sum_{-\infty}^{\infty} a_n e^{2\pi i n x}$. Then

$$f \circ T(x) = \sum_{-\infty}^{\infty} a_n e^{4\pi i n x}.$$

From $f \circ T(x) = f(x)$, we have $a_n = a_{2n}$ for all $n \in \mathbb{Z}$, that is

$$a_1 = a_2 = a_4 = a_8 = \dots$$

But since $\|f\|_2 = \sum_{-\infty}^{\infty} a_n^2 < \infty$, we have $a_1 = a_2 = a_4 = a_8 = \cdots = 0$ and $a_3 = a_6 = a_{12} = \cdots = 0$ and so on. Therefore $a_n = 0$ for all $n \neq 0$, which implies that $f(x)$ is constant.

Note that a measure preserving transformation T on $(X,\mathcal{B},\mu)$ is ergodic if for all $A,B\in\mathcal{B}$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}\mu(T^{-i}A\cap B)=\mu(A)\mu(B).$$

Definition 1. (1) *A measure preserving transformation T on $(X,\mathcal{B},\mu)$ is called weak-mixing, if for all $A,B\in\mathcal{B}$*

$$\lim_{n\to\infty}\frac{1}{n}\sum_{i=0}^{n-1}|\mu(T^{-i}A\cap B)-\mu(A)\mu(B)|=0.$$

(2) *A measure preserving transformation T on $(X,\mathcal{B},\mu)$ is called strong-mixing, if for all $A,B\in\mathcal{B}$*

$$\lim_{n\to\infty}\mu(T^{-n}A\cap B)=\mu(A)\mu(B).$$

Clearly, strong-mixing implies weak-mixing and weak-mixing implies ergodicity.

(1) Irrational rotations on the unit circle is not weak-mixing: Let T be the rotation by α and A, B be small intervals. Then there are many n's (more than half if A and B are small enough) such that $T^{-n}A\cap B=\emptyset$. For the such n's we have $|\mu(T^{-n}A\cap B)-\mu(A)\mu(B)|=\mu(A)\mu(B)$ and $\frac{1}{n}|\mu(T^{-n}A\cap B)-\mu(A)\mu(B)|$ cannot converge to 0.
(2) $2x$ map on the unit circle is strong-mixing: Let $T:x\mapsto 2x \pmod 1$ and B be an interval. Then for any A we have

$$\mu(T^{-n}A\cap B)\approx\frac{\text{number of }2^{-n}\text{ subintervals which intersect in }B}{2^n}\mu(A).$$

Precisely, we have

$$\frac{2^n\mu(B)-2}{2^n}\mu(A)\le\mu(T^{-n}A\cap B)\le\frac{2^n\mu(B)+2}{2^n}\mu(A).$$

Thus we have $\lim_{n\to\infty}\mu(T^{-n}A\cap B)=\mu(A)\mu(B)$.

Let X be a compact metric space and $T:X\to X$ be continuous. Let $M(X,T)$ be a set of T-invariant Borel probability measure.

Theorem 1. *(i) $M(X,T)$ is not empty.*
(ii) $M(X,T)$ is convex.
(iii) μ is extreme point if and only if μ is ergodic measure.
(iv) μ, $\nu\in M(X,T)$ are ergodic, then μ and ν are mutually singular.

Proof. (i) Pick an $x\in X$. Denote δ_x by the Dirac delta measure, i.e., $\delta_x(E)=1$ if $x\in E$ and 0 if $x\notin E$. Then the sequence measure $\frac{1}{n}(\delta_x+\delta_{Tx}+\cdots+\delta_{T^{n-1}x})$ has convergent subsequence to an invariant measure in weak *-topology. See Ahn's lecture note for the detail.

(ii) If μ and ν are invariant measures, then $p\mu + (1-p)\nu$ $(0 < p < 1)$ is also an invariant measure.

(iv) $\mu \perp \nu$ means that $X = E \cup F$, $E \cap F =$ and $\mu(E) = 1$, $\nu(F) = 1$. Consider the Birkhoff average $\frac{1}{n}\sum_{k=0}^{n-1} f(T(x))$. If we choose x is the "support" of μ then $\frac{1}{n}\sum_{k=0}^{n-1} f(T(x))$ goes to $\int f d\mu$. If x belongs to the "support" of ν then $\frac{1}{n}\sum_{k=0}^{n-1} f(T(x))$ converges to $\int f d\nu$. □

There are many invariant measure for the map $T : x \mapsto 2x \pmod 1$ besides the Lebesgue measure. One trivial invariant measure is δ_0 since 0 is a fixed point of T. Another invariant measure is $(p, 1-p)$-Bernoulli measure μ_p $(0 < p < 1)$, which is obtained by $\mu_p[0, \frac{1}{2}) = p$, $\mu_p[\frac{1}{2}, 1) = 1 - p$, $\mu_p[0, \frac{1}{4}) = p^2$, $\mu_p[\frac{1}{4}, \frac{1}{2}) = p(1-p)$ $\mu_p[\frac{1}{2}, \frac{3}{4}) = (1-p)p, \mu_p[\frac{3}{4}, 1) = (1-p)^2$, and so on. Choose x as a "typical" point of μ_p then in x's binary expansion 0 appears in probability p and 1 appears in probability $1-p$.

If there is only one measure in $M(X,T)$ then T is said to be uniquely ergodic. Let T be uniquely ergodic with the ergodic measure μ. Then the Birkhoff average $\frac{1}{n}\sum_{k=0}^{n-1} f(T(x))$ converges to $\int f d\mu$ for every $x \in X$.

T is said to be minimal, if every orbit of T is dense. If T is uniquely ergodic, then T is minimal.

An interval exchange map is a piecewise isometry bijection on the unit interval.

- If the length of subintervals are rationally independent and the permutation is irreducible, then the it is minimal (Keans's condition)
- Not every minimal interval exchange is uniquely ergodic.
- Almost every interval exchange is uniquely ergodic (Veech, Masur).
- Interval exchange is never strong-mixing.
- Almost every interval exchange is weak-mixing (Avila-Forni).

Let U_T be the unitary operator defined by $U_T(f) = f \circ T$. We call λ an eigenvalue of U_T if $U_T(f) = f \circ T = \lambda f$ for some f.

Equivalent conditions for the weak mixing property.

(1) $T \times T$ is ergodic.
(2) $T \times T$ is weak-mixing.
(3) 1 is the only eigenvalue and T is ergodic.

Note that if T is the irrational rotation by α then $T \times T : [0,1) \times [0,1) \to [0,1) \times [0,1)$ acts as $(x,y) \mapsto (x+\alpha, y+\alpha)$. Then the diagonal "strip" $\{(x,y) : |x-y| < \delta\}$ is invariant and $T \times T$ is not ergodic.

If we choose $f(x) = e^{2\pi i n x}$ then we have

$$f(T(x)) = e^{2\pi i n(x+\alpha)} = e^{2\pi i n\alpha} f(x)$$

so $e^{2\pi i n\alpha}$ is an eigenvalue which is different from 1.

Theorem 2 (Poincaré's Recurrence Theorem)**.** *Let T be a measure preserving transformation on $(X, \mathcal{B}, \mu)$. If $\mu(E) > 0$, then for almost every $x \in E$ is recurrent to E.*

Proof. Let F be the subset of E which is not recurrent to E. Then we have

$$\begin{aligned} F &= E \setminus \bigcup_{n=1}^{\infty} T^{-n}E \\ &= E \cap T^{-1}(X \setminus E) \cap T^{-2}(X \setminus E) \cap \dots \end{aligned}$$

Therefore, $F \cap T^{-n}F = \emptyset$ for all n. Since $\mu(X) \geq \mu(F) + \mu(T^{-1}F) + \cdots = \mu(F) + \mu(F) + \dots$, we have $\mu(F) = 0$. □

Theorem 3 (Furstenberg's Szemerédi theorem)**.** *Let T_1, T_2, ..., T_ℓ be commuting measure preserving transformations on $(X, \mathcal{B}, \mu)$. For any $E \in \mathcal{B}$ with $\mu(E) > 0$, we can choose $n \in \mathbb{N}$ such that*

$$\mu(E \cap T_1^{-n}E \cap \dots T_\ell^{-n}E) > 0.$$

Note that if $\ell = 1$, then the proof is directly obtained by the Poincaré recurrence theorem. For a transformation T, choose $T_1 = T$, $T_2 = T^2$, ..., and $T_\ell = T^\ell$. Then the set of points x such that $x \in E, T^n x \in E, T^{2n}x \in E, \dots, T^{\ell n}x \in E$ has positive measure.

By the Poincaré recurrence theorem we can define the recurrence time to a set E with $\mu(E) > 0$.

Definition 2. *Define the recurrence time E_E to E by*

$$R_E(x) = \min\{j \geq 1 : T^j(x) \in E\}.$$

Define the induced map T_E by

$$T_E(x) = T^{R_E(x)}(x).$$

Then is is not difficult to show that for $\mu(E) > 0$ (i) If T preserve μ, then T_E preserve μ_E (ii) If T is ergodic, then T_E is ergodic. Here μ_E is the induced measure defined by $\mu_E(A) = \mu(A)/\mu(E)$ for $A \subset E$.

Theorem 4 (Kac's Theorem)**.** *Let T be a measure preserving transformation on $(X, \mathcal{B}, \mu)$ and $\mu(E) > 0$. Then we have*

$$\int_E R_E(x) d\mu \leq 1.$$

If T is ergodic, the equality holds.

Proof. We will consider the case that T is ergodic and invertible. Let

$$E_n = \{x \in E : R_E(x) = n\}.$$

Then $E = \cup_{n=1}^{\infty} E_n$ and

$$X = E_1 \bigcup (E_2 \cup TE_2) \bigcup (E_3 \cup TE_3 \cup T^2 E_3) \bigcup \dots$$

Since $T^i E_n$'s $(0 \le i < n)$ are all disjoint,

$$1 = \mu(X) = \mu(E_1) + 2\mu(E_2) + 3\mu(E_3) + \dots$$

Therefore, we have

$$\int_E R_E(x) d\mu = \sum_{n=1}^{\infty} n\mu(E_n) = 1.$$

We have another proof for general ergodic transformation:
Let $N = \sum_{\ell=0}^{L-1} R_E({T_E}^{\ell} x)$. Then N is the time until the orbit of x under T visit E L times, so we have $\sum_{n=1}^{N} 1_E(T^n x) = L$. By the Birkhoff ergodic theorem

$$\int_E R_E d\mu = \lim_{L\to\infty} \frac{\sum_{\ell=0}^{L-1} R_E({T_E}^{\ell} x)}{L} = \lim_{N\to\infty} \frac{N}{\sum_{n=1}^{N} 1_E(T^n x)} = \frac{1}{\mu(E)}.$$

(The proof is from [1].) □

Group extension (Skew-product) Let (Y, T) be a dynamical systems and K be a compact group. If $\psi : Y \to K$ is continuous, then we can define a new dynamical system called a group extension on $X = Y \times K$, by

$$(y, k) \mapsto (Ty, \psi(y)k).$$

For an example, let $Y = \{0, 1\}^{\mathbb{N}}$ and T be the left shift map. Let $K = \mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ and $\psi(y) = (-1)^{y_1}$, $y = y_1 y_2 y_3 \dots$. Then the group extension $(y, k) \mapsto (Ty, k + \psi(y))$ can be interpreted as a random walker at k in $\mathbb{Z}_n$ jumps to $k + 1$ if $y_1 = 0$ and to $k - 1$ if $y_1 = 1$ and for the next turn the random walker jumps according to y_2's value.

References

[1] G.H. Choe, *Computational Ergodic Theory*, Springer-Verlag, 2005.