Lec Notes II

Let $f(x)$ be defined on the interval $-L \leq x \leq L$ and $f(x+2L)=f(x)$. We can write a fourier series expansion of $f(x)$ as follows.

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

where

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx.$$

Important Identity:

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\cos\frac{n\pi x}{L}\,dx = 0$$

for any pair of integers m & n.

$$\int_{-L}^{L} \sin\frac{m\pi x}{L}\sin\frac{n\pi x}{L}\,dx = \begin{cases} 0 & m\neq n \\ L & m=n \end{cases}$$

m, n integers.

$$\int_{-L}^{L} \cos\frac{m\pi x}{L}\cos\frac{m\pi x}{L}\,dx = \begin{cases} 0 & m\neq n \\ L & m=n\neq 0 \\ 2L & m=n=0 \end{cases}$$

m, n integers.

Let

$$f_N(x) = a_0 + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)$$

be an approximation of $f(x)$, we define an error function

$$E_N = \int_{-L}^{L}\left[f(x) - f_N(x)\right]^2 dx$$

E_N is computed as follows:

$$E_N = \int_{-L}^{L} f^2(x)\,dx$$

$$- \int_{-L}^{L} 2 f(x) f_N(x)\,dx$$

$$+ \int_{-L}^{L} f_N^2(x)\,dx$$

$$\int_{-L}^{L} f(x)\, f_N(x)\, dx$$

$$= \int_{-L}^{L} f(x) \left[a_0 + \sum_{n=1}^{N} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \right] dx$$

$$= a_0 \int_{-L}^{L} f\, dx + \sum_{n=1}^{N} \left(a_n \int_{-L}^{L} f \cos \frac{n\pi x}{L}\, dx + b_n \int_{-L}^{L} f \sin \frac{n\pi x}{L}\, dx \right)$$

$$= a_0\, 2L\, a_0 + \sum_{n=1}^{N} a_n L a_n + b_n L b_n \,.$$

$$= L \left[2 a_0^2 + \sum_{n=1}^{N} \left(a_n^2 + b_n^2 \right) \right]$$

$$\int_{-L}^{L} f_N^2(x)\,dx$$

$$= \int_{-L}^{L} \left[a_0 + \sum_{n=1}^{N}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)\right]^2 dx$$

$$= a_0^2\, 2L + \sum_{n=1}^{N}\left(a_n^2 L + b_n^2 L\right)$$

$$= L\left[2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right]$$

It follows that

$$\boxed{E_N = \int_{-L}^{L} f^2(x)\,dx - L\left[2a_0^2 + \sum_{n=1}^{N}\left(a_n^2 + b_n^2\right)\right]}$$

If $E^* = \underset{N\to\infty}{Lt} E_N$ we obtain.

$$E^* = \underset{N\to\infty}{Lt} E_N = \int_{-L}^{L} f^2(x)\,dx - L\left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$$

It turns out that for a large class of functions (★) $f(x)$, $E^*=0$ and we obtain

$$\frac{1}{L}\int_{-L}^{L} f^2(x)\,dx = \left[2a_0^2 + \sum_{n=1}^{\infty}(a_n^2+b_n^2)\right]$$

called Parseval's Equality.

(★): The class is defined precisely those functions defined $[-L, L]$:

$\int_{-L}^{L} f^2(x)\,dx$ is finite.

$\boxed{\text{Frequency and amplitude spectra}}$

Let $f(x)$ be as in page 2·1.

Denote

$$\omega_0 = \frac{\pi}{L}$$

we have

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left[a_n \cos n\omega_0 x + b_n \sin n\omega_0 x \right]$$

$$= a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left[\frac{a_n}{\sqrt{a_n^2+b_n^2}} \cos n\omega_0 x + \frac{b_n}{\sqrt{a_n^2+b_n^2}} \sin n\omega_0 x \right]$$

Define δ_n : $\cos \delta_n = \dfrac{a_n}{\sqrt{a_n^2+b_n^2}}$

$$\sin \delta_n = \frac{-b_n}{\sqrt{a_n^2+b_n^2}}$$

We have

$$f(x) = a_0 + \sum_{n=1}^{\infty} A_n \cos(n\omega_0 x + \delta_n)$$

where

$$A_n = \sqrt{a_n^2 + b_n^2} \quad n = 1, 2, \cdots$$

$$\omega_0 = \frac{\pi}{L}.$$

$$\delta_n = \tan^{-1}\left(-\frac{b_n}{a_n}\right)$$

$\omega_0, 2\omega_0, 3\omega_0 \cdots$ ← frequency spectrum of $f(x)$.

$n\omega_0$ is the n^{th} harmonic frequency of $f(x)$

δ_n is the n^{th} phase angle of $f(x)$.

$A_0, A_1, A_2, \cdots$ $A_0 = |a_0|$ is called the amplitude spectrum of $f(x)$.

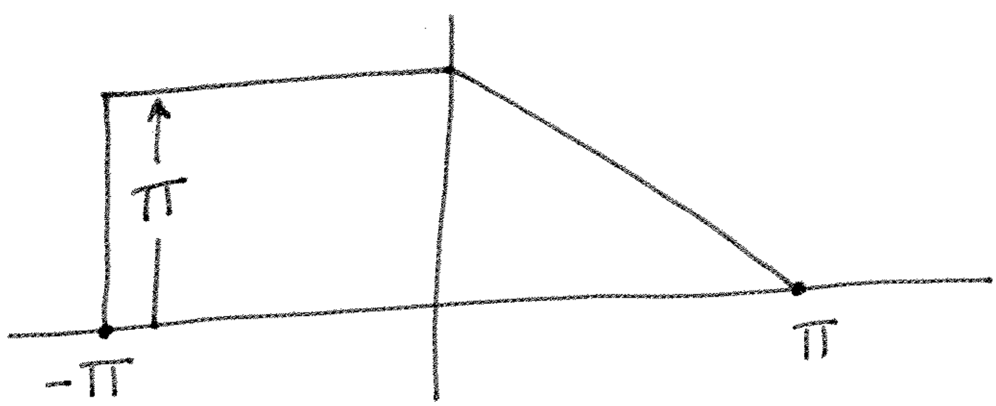
The function

$$\cos(n\omega_0 x + \delta_n)$$

is called the n^{th} harmonic of the function $f(x)$.

Example 2·1

$$f(x) = \begin{cases} \pi & -\pi < x < 0 \\ \pi - x & 0 \leq x \leq \pi \end{cases}$$



For this problem $L = \pi$, hence $\omega_0 = 1$.

The Fourier series expansion turns out to be

$$f(x) = \frac{3\pi}{4}$$
$$+ \frac{2}{\pi}\cos x - \sin x$$
$$+ \frac{1}{2}\sin 2x$$
$$+ \frac{2}{9\pi}\cos 3x - \frac{1}{3}\sin 3x$$
$$+ \frac{1}{4}\sin 4x$$
$$+ \frac{2}{25\pi}\cos 5x - \frac{1}{5}\sin 5x$$
$$+ \cdots$$
$$+ \cdots$$

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$$A_0 = |a_0| = \frac{3\pi}{4}.$$

$$A_1 = \sqrt{\left(\frac{2}{\pi}\right)^2 + (-1)^2} = \sqrt{1 + \frac{4}{\pi^2}}.$$

$$A_2 = \frac{1}{2} \qquad A_3 = \sqrt{\frac{4}{81\pi^2} + \frac{1}{9}}.$$

$$A_4 = \frac{1}{4}. \qquad A_5 = \sqrt{\left(\frac{2}{25\pi}\right)^2 + \frac{1}{25}}.$$

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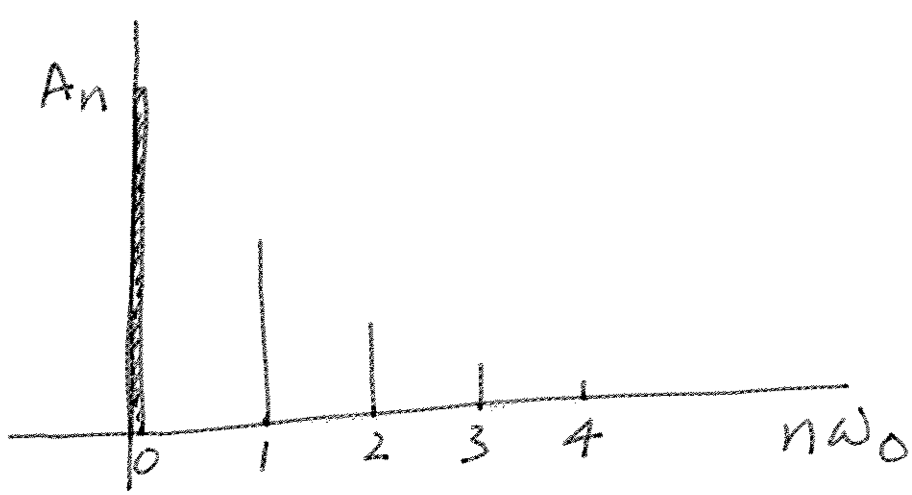
$$A_{2n} = \frac{1}{2n} \leftarrow n = 1, 2, \ldots$$

In general

$$A_{2n-1} = \frac{1}{2n-1}\sqrt{1 + \frac{4}{(2n-1)^2\pi^2}}.$$



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$$\delta_1 = \tan^{-1}\frac{\pi}{2}.$$

$$\delta_2 = \tan^{-1}(-\infty)$$

$$\delta_3 = \tan^{-1}\frac{3\pi}{2}.$$

$$\delta_4 = \tan^{-1}(-\infty)$$

$$\delta_5 = \tan^{-1}\left(\frac{5\pi}{2}\right)$$