MAT 137Y, 2003-2004 Test 3 Solutions

1. Compute the following integrals.

(10%) **(i)** $\int 5^{2+x}\,dx.$

$$\int 5^{2+x}\,dx = 25\int 5^x\,dx = 25\frac{5^x}{\ln 5}+C.$$

(10%) **(ii)** $\int x^3(\ln x)^2\,dx.$

Let I be the integral and integrate by parts. We let $u=(\ln x)^2$, $dv = x^3\,dx$. Then $du = \frac{2\ln x}{x}\,dx$ and $v=\frac{1}{4}x^4$. Hence,

$$\begin{aligned} I &= \frac{1}{4}x^4(\ln x)^2 - \int \frac{1}{2}x^3\ln x\,dx \\ &= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{2}\left[\frac{1}{4}x^4\ln x - \frac{1}{4}\int x^3\,dx\right] \qquad (u=\ln x,\ dv = x^3\,dx,\ du=\tfrac{1}{x}\,dx,\ v=\tfrac{1}{4}x^4) \\ &= \frac{1}{4}x^4(\ln x)^2 - \frac{1}{8}x^4\ln x + \frac{1}{32}x^4 + C. \end{aligned}$$

(10%) **(iii)** $\int \frac{dx}{x^2\sqrt{x^2+16}}.$

Let $x = 4\tan u$. Then $dx = 4\sec^2 u\,du$ and this gives

$$\begin{aligned} \int \frac{dx}{x^2\sqrt{x^2+16}} &= \int \frac{4\sec^2 u\,du}{16\tan^2 u\cdot 4\sec u} = \frac{1}{16}\int \frac{\sec u}{\tan^2 u}\,du = \frac{1}{16}\int \frac{1}{\cos u}\cdot\frac{\cos^2 u}{\sin^2 u}\,du \\ &= \frac{1}{16}\int \csc u\cot u\,du = -\frac{1}{16}\csc u = -\frac{\sqrt{x^2+16}}{16x}+C. \end{aligned}$$

(10%) **(iv)** $\int \frac{4x}{(x-1)^2(x+1)}\,dx.$

We separate the integral using partial fractions. For some constants A, B, and C:

$$\frac{A}{x-1}+\frac{B}{(x-1)^2}+\frac{C}{x+1} = \frac{4x}{(x-1)^2(x+1)} \implies A(x-1)(x+1)+B(x+1)+C(x-1)^2 = 4x.$$

If $x=1$, this gives $2B=4$, or $B=2$. If $x=-1$ we get $4C=-4$, or $C=-1$. If $x=0$, we get $-A+B+C=0$, or $A=1$. Hence,

$$\int \frac{4x}{(x-1)^2(x+1)}\,dx = \int \frac{1}{x-1}+\frac{2}{(x-1)^2}-\frac{1}{x+1}\,dx = \ln|x-1| - 2(x-1)^{-1} - \ln|x+1| + C.$$

2. Compute the following limits.

(8%) **(i)** $\lim_{x\to 1^+}(\ln x)\tan\left(\frac{\pi x}{2}\right).$

The limit is of the form $0\cdot(-\infty)$. Rewriting the limit,

$$\lim_{x\to 1^+}(\ln x)\tan\left(\frac{\pi x}{2}\right) = \lim_{x\to 1^+}\frac{\ln x}{\cot\frac{\pi x}{2}} \overset{\text{H}}{=} \lim_{x\to 1^+}\frac{\frac{1}{x}}{-\csc^2\left(\frac{\pi x}{2}\right)\cdot\frac{\pi}{2}} = -\frac{2}{\pi},$$

since $\csc\frac{\pi}{2}=1$.

(8%) **(ii)** $\lim_{x\to\infty}(e^x+x)^{1/x}$.

Let $L=\lim_{x\to\infty}(e^x+x)^{1/x}$. Then

$$\ln L=\lim_{x\to\infty}\frac{1}{x}\ln(e^x+x)\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x+1}{e^x+x}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{e^x+1}\overset{\text{H}}{=}\lim_{x\to\infty}\frac{e^x}{e^x}=1.$$

Since $\ln L=1$, it follows that $L=e$.

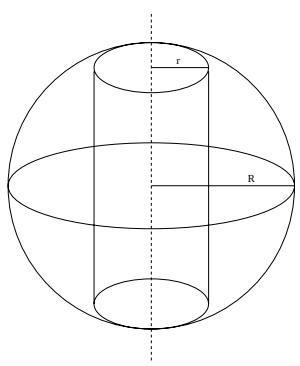
(12%) **3.** A hole of radius r is drilled through the center of a sphere of radius R (where $R>r$). Find the volume of the remaining portion of the sphere.

Consider the hole drilled from the top as illustrated in the diagram. Then it is easy to see that the remaining volume can be computed by using shells, so

$$\begin{aligned}V&=2\int_r^R 2\pi x\sqrt{R^2-x^2}\,dx=2\pi\int_{R^2-r^2}^0-\sqrt{u}\,du \qquad (u=R^2-x^2,\ du=-2x\,dx)\\&=2\pi\int_0^{R^2-r^2}\sqrt{u}\,du=2\pi\left[\frac{2}{3}u^{3/2}\right]_0^{R^2-r^2}=\frac{4}{3}\pi(R^2-r^2)^{3/2}.\end{aligned}$$

Alternatively, suppose the hole is drilled from the side through the center. The line $y=r$ intersects the semicircle $y=\sqrt{R^2-x^2}$ when $r=\sqrt{R^2-x^2}$ which implies $x^2=R^2-r^2$ or $x=\pm\sqrt{R^2-r^2}$. Rotating the region about the x-axis gives us

$$\begin{aligned}V&=\int_{-\sqrt{R^2-r^2}}^{\sqrt{R^2-r^2}}\pi\left[\left(\sqrt{R^2-x^2}\right)^2-r^2\right]dx=2\pi\int_0^{\sqrt{R^2-r^2}}(R^2-x^2-r^2)\,dx \qquad \text{(by symmetry)}\\&=2\pi\left[(R^2-r^2)x-\frac{1}{3}x^3\right]_0^{\sqrt{R^2-r^2}}=2\pi\left[(R^2-r^2)^{3/2}-\frac{1}{3}(R^2-r^2)^{3/2}\right]\\&=2\pi\cdot\frac{2}{3}(R^2-r^2)^{3/2}=\frac{4}{3}\pi(R^2-r^2)^{3/2}.\end{aligned}$$



(8%) **4.** Find $\dfrac{d^2}{dx^2}\left[\int_0^x\left(\int_1^{\sin t}e^{u^2}\,du\right)dt\right]$.

By Theorem 5.3.5,

$$\frac{d^2}{dx^2}\left[\int_0^x\left(\int_1^{\sin t}e^{u^2}\,du\right)dt\right]=\frac{d}{dx}\left[\int_1^{\sin x}e^{u^2}\,du\right]=e^{\sin^2 x}\cdot\cos x.$$

5. Suppose $f(x) = 2x + \cos x$.

(6%) **(a)** Show that $f(x)$ is one-to-one.

The function $f(x)$ is one-to-one if the function is strictly increasing. Note that $f'(x) = 2 - \sin x > 1 > 0$ for all x, so the function is strictly increasing and we are done.

(6%) **(b)** Find $(f^{-1})'(1)$.

We have $(f^{-1})'(1) = \dfrac{1}{f'(f^{-1}(1))} = \dfrac{1}{f'(0)} = \dfrac{1}{2}$.

6. Suppose f is continuous on $[a,b]$.

(8%) **(a)** Prove there exists $c \in [a,b]$ such that $\displaystyle\int_a^c f(t)\,dt = \int_c^b f(t)\,dt$.

Consider the function $g(x) = \displaystyle\int_a^x f(t)\,dt - \int_x^b f(t)\,dt$. If $\displaystyle\int_a^b f(t)\,dt = 0$ then the result is obvious (by choosing $c = a$ or $c = b$). Otherwise, $g(x)$ is continuous (by Theorem 5.3.5) and

$$g(a) = -\int_a^b f(t)\,dt, \qquad g(b) = \int_a^b f(t)\,dt,$$

so $g(a)$ and $g(b)$ are of opposite sign. Hence by the Intermediate Value Theorem there exists $c \in (a,b)$ such that $g(c) = \displaystyle\int_a^c f(t)\,dt - \int_c^b f(t)\,dt = 0$, thereby ending the proof.

(4%) **(b)** Must there exist $c \in (a,b)$ such that $\displaystyle\int_a^c f(t)\,dt = \int_c^b f(t)\,dt$? Justify your answer with an appropriate proof or counterexample.

No, consider the example $f(x) = \sin x$ and let $a = 0$ and $b = 2\pi$. It should be clear by a diagram that the statement is true for $c = 0$ and $c = 2\pi$, but nowhere else.