MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 4

1 (SHE Section 2.2)

58. Suppose $\lim_{x\to c} f(x) = L$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-c| < L \implies |f(x) - L| = |(f(x) - L) - 0| = \varepsilon$, so it immediately follows that $\lim_{x\to c} f(x) = L$ if and only if $\lim_{x\to c}(f(x) - L) = 0$ by choosing the same δ.

60. The statement is clearly false. Let $f(x) = \begin{cases} 1, & x \neq 0 \\ 0, & x = 0. \end{cases}$ Then $\lim_{x\to 0} f(x) = 0$, but there is no such interval of the form $(0-\gamma, 0+\gamma)$ such that $f(x) > 0$ for all x in the interval.

62. Suppose $\lim_{x\to c} f(x) = L$. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-c| < \delta \implies |f(x) - L < \varepsilon \implies L - \varepsilon < f(x) < L + \varepsilon$. Let $B = \max(|L-\varepsilon|, |L+\varepsilon|)$, then it follows that $|f(x)| < B$, as required.

2 (a) $\lim_{x\to 2} \frac{x^2-4}{x^3-8} = \lim_{x\to 2} \frac{(x+2)(x-2)}{(x-2)(x^2+2x+4)} = \lim_{x\to 2} \frac{x+2}{x^2+2x+4} = \frac{4}{12} = \frac{1}{3}$.

(b) By the definition of absolute value,

$$|5x+1| = \begin{cases} 5x+1, & x \geq -\frac{1}{5}, \\ -5x-1, & x < -\frac{1}{5}, \end{cases} \qquad |5x-1| = \begin{cases} 5x-1, & x \geq \frac{1}{5}, \\ -5x+1, & x < \frac{1}{5}. \end{cases}$$

Therefore, for values close to zero, $|5x+1| = 5x+1$ and $|5x-1| = -5x+1$.
Hence $\lim_{x\to 0} \frac{|5x+1| - |5x-1|}{x} = \lim_{x\to 0} \frac{(5x+1) - (-5x+1)}{x} = \lim_{x\to 0} \frac{10x}{x} = 10$.

(c) Multiplying top and bottom by the conjugate, we have

$$\lim_{x\to 3} \frac{\sqrt{x+6}-x}{x^3-3x^2} = \lim_{x\to 3} \frac{x+6-x^2}{(x^3-3x^2)(\sqrt{x+6}+x)} = \lim_{x\to 3} \frac{-(x-3)(x+2)}{x^2(x-3)(\sqrt{x+6}+x)} = \lim_{x\to 3} \frac{-(x+2)}{x^2(\sqrt{x+6}+x)}$$
$$= -\frac{5}{54}.$$

3 (SHE 2.3 54)

(a) Suppose $\lim_{x\to c} f(x) = L$. Then given $\varepsilon > 0$ there exists $\delta_f > 0$ such that $0 < |x-c| < \delta_f$ implies $|f(x) - L| < \varepsilon$. Since g differs from f at finitely many points $x_1, x_2, \ldots, x_n$, find x_k which is closest to $x = c$. Now to show that $\lim_{x\to c} g(x) = L$, choose $\delta_g = \min(\delta_f, |c - x_k|)$. Then for all $x \in (c - \delta_g, c + \delta_g)$, it follows that $f(x) = g(x)$, so if $|f(x) - L| < \varepsilon$, then it is also true that $|g(x) - L| < \varepsilon$, which completes the proof.

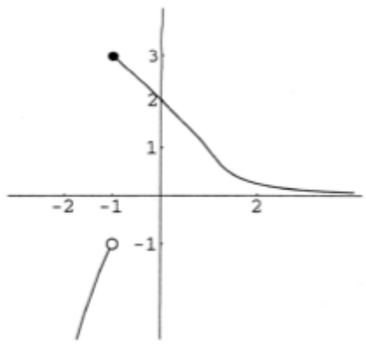
(b) Now we need to show that if $\lim_{x\to c} f(x)$ does not exist, then $\lim_{x\to c} g(x)$ also does not exist. The proof is by contradiction: suppose $\lim_{x\to c} g(x)$ exists. But f and g only differ at finitely many points, so by part (a), it follows that $\lim_{x\to c} f(x)$ must exist, which contradicts the assumption that $\lim_{x\to c} f(x)$ does not exist. Hence $\lim_{x\to c} g(x)$ also does not exist.

4 (SHE 2.4)

26. The sketch is given below. Since

$$\lim_{x\to -1^-} g(x) = \lim_{x\to -1^-} -x^2 = -1, \quad \lim_{x\to -1^+} g(x) = \lim_{x\to -1^+}(2-x) = 3,$$
$$\lim_{x\to 1^-} g(x) = \lim_{x\to 1^-}(2-x) = 1 = \lim_{x\to 1^+} \frac{1}{x^2} = \lim_{x\to 1^+} g(x),$$

so we only have a jump discontinuity at $x = -1$.



50(a). Suppose f is continuous at c. Then for any $\varepsilon > 0$ there exists $\delta > 0$ such that $0 < |x-c| < \delta$ implies $|f(x) - f(c)| < \varepsilon$. In particular, choose $\varepsilon = f(c)$, then there exists $\delta > 0$ such that $0 < |x-c| < \delta \implies |f(x) - f(c)| < f(c) \implies 0 < f(x) < 2f(c)$, which implies $f(x) > 0$ when $0 < |x-c| < \delta$, or $x \in (c-\delta, c+\delta)$.

50(c). Define the function $h(x) = g(x) - f(x)$. Then $h(c) = g(c) - f(c) > 0$, so by part (a), there exists $\delta > 0$ such that $h(x) = g(x) - f(x) > 0$, or $f(x) < g(x)$, for all $x \in (c-\delta, c+\delta)$.

56. Let $f_e(x) = \frac{1}{2}[f(x) + f(-x)]$ and $f_o(x) = \frac{1}{2}[f(x) - f(-x)]$. Then f_e is even and f_o is odd, and each continuous on $(-\infty, \infty)$, and $f = f_e + f_o$.

5 (SHE 2.5)

10. $\displaystyle \lim_{x\to 0} \frac{\sin^2 x^2}{x^2} = \lim_{x\to 0}(\sin x^2)\left(\frac{\sin x^2}{x^2}\right) = 0(1) = 0.$

18. $\displaystyle \lim_{x\to 0} \frac{x^2 - 2x}{\sin 3x} = \lim_{x\to 0}\left(\frac{x-2}{3}\right)\left(\frac{3x}{\sin 3x}\right) = -\frac{2}{3}\cdot 1 = -\frac{2}{3}.$

44. Since $0 \le \cos^2[1/(x-\pi)] \le 1$ for all $x \ne \pi$, we have $|[(x-\pi)\cos^2[1/(x-\pi)]| \le |x-\pi|$. Thus,

$$-|x-\pi| \le |(x-\pi)\cos[1/(x-\pi)]| \le |x-\pi|.$$

Since $\lim_{x\to\pi}(-|x-\pi|) = \lim_{x\to\pi}(|x-\pi|) = 0$, the result follows by the pinching theorem.

6 (SHE 2.6)

8. Set $f(x) = \sqrt{x^2 - 3x} - 2$. Then f is continuous on $[3,5]$ and $f(3) = -2 < 0$, $f(5) = \sqrt{10} + 2 > 0$. By the intermediate value theorem there is a c in $[1,2]$ such that $f(c) = 0$.

12. Set $f(x) = x^2$. Then $f(x)$ is continuous on $[1,2]$. $f(1) = 1 < 2$ and $f(2) = 4 > 2$. By IVT there is a $c \in (1,2)$ such that $f(c) = 2$.

26. Set $h(x) = f(x) - g(x)$. Then h is continuous on $[a,b]$, and $h(a) = f(a) - g(a) < 0$ and $h(b) = f(b) - g(b) > 0$. By IVT, there exists a number $c \in (a,b)$ such that $h(c) = 0$. Thus, $f(c) = g(c)$.

7 Consider the function $f(x) = x^3 + \cos x$. Then $f(0) = 1 > 0$ and $f(-1) = -1 + \cos(-1) < 0$ since $0 < \cos(-1) < 1$. Since f is continuous on $(-1,0)$, then by IVT there exists $c \in (-1,0)$ such that $f(c) = c^3 + \cos c$, so a solution $x = c$ exists.