MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 5

1 (i) Consider $S = \{x \ : \ x \in (0,1)\}$. The least upper bound is 1. To see this, note that 1 is an upper bound since $x < 1$ for all x in the set. Suppose a lower upper bound exists; then the lower bound is of the form $1-\varepsilon$, where $\varepsilon > 0$ is a sufficiently small number. But $1-\varepsilon < \max(\frac{1}{2}, 1-\frac{\varepsilon}{2}) < 1$, and $\max(\frac{1}{2}, 1-\frac{\varepsilon}{2}) \in S$, which contradicts the assumption that $1-\varepsilon$. Since this is true for any $\varepsilon > 0$, it must follow that 1 is the least upper bound.

Similarly, the greatest lower bound is zero. Since $x > 0$ for all $x \in S$, then zero is a lower bound. Suppose $\varepsilon > 0$ is a lower bound; then $\min(\frac{1}{2}, \frac{\varepsilon}{2}) \in S$, which contradicts that ε is a lower bound since $\varepsilon > \min(\frac{1}{2}, \frac{\varepsilon}{2})$.

The maximum value and minimum value does not exist. If a minimum value existed, then $0 < \varepsilon < 1$ would be the minimum value. But $\frac{\varepsilon}{2}$ is in the set, contradicting that ε is the minimum value. Similarly, if a maximum value existed then the maximum value would be $1-\varepsilon$, where $0 < \varepsilon < 1$. But $1-\varepsilon < 1-\frac{\varepsilon}{2} < 1$, and $1-\frac{\varepsilon}{2} \in S$, which contradicts the assumption that $1-\varepsilon$ is the maximum value.

(ii) The sequence is decreasing, so it follows that the maximum value occurs when $n = 0$; hence 1 is the maximum value. Since the maximum value is in the set, it also follows that 1 is the least upper bound.

We now claim that 0 is the least upper bound of the set. It follows for all $n > 0$ that $\frac{1}{3^n} > 0$, so 0 is an upper bound; hence the least upper bound exists. Suppose $\varepsilon > 0$ were the least upper bound. Then choose k such that $\frac{1}{3^k} < \varepsilon$. Then $\frac{1}{3^k}$ is in the set, but is smaller than the least upper bound, which is a contradiction.

There is no minimum value for the set. If a minimum value existed, then it would have to be of the form $\frac{1}{3^m}$ for some integer m. But $\frac{1}{3^{m+1}} < \frac{1}{3^m}$, which would contradict the assumption that $\frac{1}{3^m}$ is the least element.

(iii) The set $\{\frac{(-1)^n}{n} \ : \ n = 1,2,\ldots\} = \{-1, -\frac{1}{3}, -\frac{1}{5}, \ldots\} \cup \{\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \ldots\}$. From this we see that maximum value of the set is $\frac{1}{2}$ and the minimum value is -1. Since the maximum and minimum values both are in the set, it follows that they are also the least upper bound and greatest lower bounds of the set respectively.

(iv) Solving $|\sin x| < \frac{1}{2}$ on the interval $[0, 2\pi]$, we have $x \in [0, \frac{\pi}{6}) \cup (\frac{5\pi}{6} \cup \frac{7\pi}{6}) \cup (\frac{11\pi}{6}, 2\pi]$. Hence the maximum value and least upper bound is 2π and the minimum value and greatest lower bound is 0.

(v) For parts (v) and (vi), we assume that the sets are subsets of the real numbers. Therefore for the set $\{x \in \mathbb{Q} \ : \ -\sqrt{2} < x < 2\}$, the least upper bound is 2 and the greatest lower bound is $-\sqrt{2}$. To show the least upper bound is 2, it is clear that $x < \sqrt{2}$ for all x in the set, so an upper bound exists. Suppose $\sqrt{2}$ is not the least upper bound, then for some sufficiently small $\varepsilon > 0$, $\sqrt{2}-\varepsilon$ is the least upper bound (since the least upper bound can not be greater than $\sqrt{2}$). However, the rationals are dense; in other words, between any two distinct numbers there exists a rational number, so there must exist a rational number r such that $\sqrt{2}-\varepsilon < r < \sqrt{2}$. But r is in the set, so this contradicts the assumption that $\sqrt{2}-\varepsilon$ is the least upper bound. Hence $\sqrt{2}$ is the least upper bound.

By a similar argument, $-\sqrt{2}$ is the greatest lower bound. It also follows that no maximum and minimum value exists. For example, if a maximum value existed, the maximum value would have to be less than 2. Let $2-\varepsilon$ be the maximum value. But there exists $r \in \mathbb{Q}$ such that

$2-\varepsilon < r < 2$, which contradicts that $2-\varepsilon$ is the maximum value. Proving that there is no minimum value is similar.

(vi) We follow an identical argument to part (v), since the irrationals are dense as well. We omit the details of the proof; the least upper bound and greatest lower bound are 2 and $-\sqrt{2}$ respectively; and the minimum and maximum values do not exist.

2 (SHE Section 11.1)

22. We have $S = \{1,2,3,4\}$. We have glb $S = 1$. By Theorem 10.1.4, for $\varepsilon = \frac{1}{1000}$, there exists $s \in S$ such that $1 \le s < 1 + \frac{1}{1000}$. In this case, $s = 1$.
24. We have $S = \{\frac{1}{2}, \frac{1}{4}, \ldots, (\frac{1}{2})^n, \ldots\}$. In this case, glb $S = 0$. By Theorem 10.1.4, for $\varepsilon = (\frac{1}{4})^k$, there exists $s \in S$ such that $0 \le s < (\frac{1}{4})^k$. Let $n = 2k+1$. Then $s = (\frac{1}{2})^{2k+1} \in S$ and $0 \le (\frac{1}{2})^{2k+1} = \frac{1}{2} \cdot (\frac{1}{4})^k < (\frac{1}{4})^k$.
26. Let $S = \{a_1, a_2, \ldots, a_n\}$ be a non-empty set of real numbers.
    (a) Consider the number $M = |a_1| + |a_2| + \ldots + |a_n|$. Then $|a_k| \le M$ for all $k = 1,2,\ldots,n$. Hence S is bounded.
    (b) Since S is bounded above and below, then glb S and lub S both exist. Since S is a set of finitely many numbers, it follows that
    $$\text{glb } S = \max\{a_1, a_2, \ldots, a_n\} \in S; \qquad \text{lub } S = \min\{a_1, a_2, \ldots, a_n\} \in S.$$

3 (a) Since $f(x)$ is a decreasing function, it follows that for all $x < a$, then $f(x) < f(a)$. Hence, $f(a)$ is a lower bound for the set $\{f(x) \,:\, x < a\}$. Since a lower bound exists, the set must be bounded below.

(b) Since the lower bound exists, it follows that the greatest lower bound also exists. Let $m = \text{glb}\{f(x) \,:\, x < a\}$. We need to show that $\lim_{x \to a^-} f(x)$ exists, so we need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $a - \delta < x < a$ implies $|f(x) - m| < \varepsilon$. Given $\varepsilon > 0$, there is some $f(x)$ for $x < a$ such that $f(x) < m + \varepsilon$, since m is the greatest lower bound of the set. Choose $\delta = a - x_0$, where x_0 is arbitrary and less than a. If $a - \delta < x < a$, then $x_0 < x < a$. But f is decreasing, so $f(x_0) > f(x)$ if $x_0 < x < a$. So $m \le f(x) < m + \varepsilon$, or $0 \le f(x) - m < \varepsilon$, which implies $|f(x) - m| < \varepsilon$, which is what we needed to show.

4 (a) Suppose that A and B are bounded above, and α and β are the upper bounds of A and B respectively. Therefore, for all $x \in A$ and for all $y \in B$, $x \le \alpha$ and $y \le \beta$.

Now suppose $x \in A \cup B$. Then $x \in A$ or $x \in B$. Therefore, either $x \le \alpha$ or $x \le \beta$, which implies $x \le \max(\alpha, \beta)$, so $A \cup B$ must also be bounded above.

Similarly, if $x \in A \cap B$, then $x \in A$ and $x \in B$, so $x \le \alpha$ and $x \le \beta$. Therefore, $x \le \min(\alpha, \beta)$, so $A \cap B$ is also bounded above.

(b) We prove that lub $(A \cup B) = \max(\text{lub } A, \text{lub } B)$. Assume that $\alpha = \text{lub } A$ and $\beta = \text{lub } B$. Now suppose $x \in A \cup B$. Then from part (a), it follows that $x \le \max(\alpha, \beta)$. Therefore lub $(A \cup B) \le \max(\text{lub } A, \text{lub } B)$.

To prove that lub $(A \cup B) \ge \max(\text{lub } A, \text{lub } B)$, we first note that for all $\varepsilon > 0$, there exists $x \in A$ such that $x > \alpha - \varepsilon$ and there exists $x \in B$ such that $x > \beta - \varepsilon$ (Theorem 10.1.2). Then it follows that for any $\varepsilon > 0$, there exists $x \in A$ or $x \in B$ (that is, $x \in A \cup B$) such that $x > \max(\alpha, \beta) - \varepsilon$. Therefore,
$$\text{lub } (A \cup B) > \max(\alpha, \beta) - \varepsilon$$

for all $\varepsilon > 0$. Since this is true for all $\varepsilon > 0$, then it follows that

$$\text{lub } (A \cup B) \geq \max(\alpha, \beta).$$

Since lub $(A \cup B) \leq \max(\alpha, \beta)$ and lub $(A \cup B) \geq \max(\alpha, \beta)$, it must follow that lub $(A \cup B) = \max(\alpha, \beta)$, which completes the proof.

(c) Let α and β be defined in the same way as in part (b). If $x \in A \cap B$, then $x \in A$ and $x \in B$, so by part (a), $x \leq \min(\alpha, \beta)$. Since this is true for all $x \in A \cap B$, then lub $(A \cap B) \leq \min(\alpha, \beta)$.

(d) It is not true that lub $(A \cap B) = \min(\alpha, \beta)$. Consider the sets $A = \{0, 1\}$ and $B = \{0, 2\}$. Then $\alpha = 1$ and $\beta = 2$, so $\min(\alpha, \beta) = 1$. However, lub $(A \cap B) = 0 \neq 1$.