MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 3

1 (SHE Section 1.2)

74. If $a = b = \sqrt{2}$ then $a+b = 2\sqrt{2}$, which is irrational. If $a = \sqrt{2}$ and $b = -\sqrt{2}$, then $a+b = 0$, which is rational.
    If $a = \sqrt{2}$ and $b = 1+\sqrt{2}$, then $ab = \sqrt{2}+2$ is irrational (see 1.2 #71). However, if $a = b = \sqrt{2}$, then $ab = 2$, which is rational.

76. Suppose $\sqrt{3}$ is rational. Then $\sqrt{3} = a/b$, where a and b are integers and a and b have no common factors. Squaring both sides gives us $3 = a^2/b^2$ or $a^2 = 3b^2$. Since $3b^2$ is clearly divisible by 3, then a^2 is also divisible by 3. It also follows that a must be divisible by 3: suppose not, then $a = 3k+1$ or $a = 3k+2$ for some integer k, but

$$(3k+1)^2 = 9k^2+6k+1 = 3(k^2+2k)+1(3k+2)^2 \quad = 9k^2+12k+4 = 3(k^2+4k+1)+1$$

so a^2 can't be divisible by 3, a contradiction. Thus $a = 3m$ for some integer m, so $(3m)^2 = 3b^2$, which implies $b^2 = 3m^2$. This implies that b^2 is divisible by 3, which thereby implies that b is divisible by 3. But since a and b are both divisible by 3, this contradicts the assumption that a and b have no common factors, hence $\sqrt{3}$ must be irrational.

2 Since $14a+21b = 7(2a+3b)$ and $2a+3b$ is an integer, it follows that $7(2a+3b)$ is divisible by 7. But 100 is not divisible by 7, so there are no integers a and b such that $14a+21b = 100$.

3 We prove for all integers $n \geq 0$ that $n^3+(n+1)^3+(n+2)^3$ is divisible by 9 by induction. The statement is true for $n = 0$, since $n^3+(n+1)^3+(n+2)^3 = 9$ is clearly divisible by 9. Now suppose $k^3+(k+1)^3+(k+2)^3$ is divisible by 9. Then

$$(k+1)^3+(k+2)^3+(k+3)^3 = (k+1)^3+(k+2)^3+k^3+9k^2+27k+27$$

which is clearly divisible by 9 since $(k+1)^3+(k+2)^3+k^3$ is divisible by 9 by the induction hypothesis, and $9k^2+27k+27 = 9(k^2+3k+3)$ is also divisible by 9, hence the sum is divisible by 9. Hence the statement is true for all integers $n \geq 0$.

4 We prove that $\sum_{j=1}^{n} j^{-1/2} > 2(\sqrt{n+1}-1)$ for all positive integers n. The statement is true for $n = 1$ since $\sum_{j=1}^{1} j^{-1/2} = 1 > 2(\sqrt{2}-1)$ (since $8 < 9 \implies 2\sqrt{2} < 3 \implies 1 > 2\sqrt{2}-2$). Now suppose $\sum_{j=1}^{k} j^{-1/2} > 2(\sqrt{k+1}-1)$ for some integer k. Then

$$\begin{aligned}\sum_{j=1}^{k+1} j^{-1/2} &= \sum_{j=1}^{k} j^{-1/2} + (k+1)^{-1/2} > 2(\sqrt{k+1}-1) + \frac{1}{\sqrt{k+1}} = \frac{2(k+1)-2\sqrt{k+1}+1}{\sqrt{k+1}} \\ &= \frac{2k+3}{\sqrt{k+1}} - 2 > 2(\sqrt{k+2}-1)\end{aligned}$$

which is true for all positive integers k since

$$4k^2+12k+9 > 4k^2+12k+8 \implies (2k+3)^2 > 4(k+2)(k+1) \implies \frac{(2k+3)^2}{k+1} > 4(k+2)$$
$$\implies \frac{2k+3}{\sqrt{k+1}} > 2\sqrt{k+2}.$$

5 We prove this using strong induction on n. The statement is true for $n=2$ and $n=3$ since

$$F_1+F_3 = 1+2 = 3 = L_2, \qquad F_2+F_4 = 1+3 = 4 = L_3.$$

Now suppose the statement $L_n = F_{n-1}+F_{n+1}$ is true for $n = 1,2,3,4,\dots,k$. We wish to show the statement is true for $n = k+1$, that is, $L_{k+1} = F_{(k+1)-1}+F_{(k+1)+1}$. The result follows since

$$\begin{aligned} F_{(k+1)-1}+F_{(k+1)+1} &= F_k+F_{k+2} = (F_{k-1}+F_{k-2})+(F_{k+1}+F_k) = (F_{k-1}+F_{k+1})+(F_{k-2}+F_k) \\ &= L_k+L_{k-1} = L_{k+1}, \end{aligned}$$

so the statement is true for all n, thus completing the proof.

6 (SHE 2.2)

12. Since $x \to 0^-$, then $x<0$, so $\displaystyle\lim_{x\to 0^-}\frac{x}{|x|} = \lim_{x\to 0^-}\frac{x}{-x} = -1$.
22. Using the figure, only ε_1 works.
26. For $\varepsilon = \frac{1}{10}$, we choose $\delta = \frac{1}{2}$. If $0<|x-2|<\frac{1}{2}$, then $|\frac{1}{5}x-\frac{2}{5}| = \frac{1}{5}|x-2| < \frac{1}{5}\cdot\frac{1}{2} = \frac{1}{10} = \varepsilon$.

7 We choose $\delta = \frac{1}{10}$. If $|x-2| < \frac{1}{10}$, then $|5x-10| = 5|x-2| < 5\cdot\frac{1}{10} = \frac{1}{2}$.

8 We choose $\delta = \frac{1}{17500}$. If $0<|x-3|<\frac{1}{17500}$, then $|x^4-81| = |x^2+9|\cdot|x+3|\cdot|x-3|$. In particular, $\delta<1$, which means $|x-3|<1$, so $2<x<4$, which implies $|x^2+9|<25$ and $|x+3|<7$, so $|x^4-81| = |x^2+9|\cdot|x+3|\cdot|x-3| < 25\cdot 7\cdot\frac{1}{17500} = \frac{1}{100}$, which is what we require.

9 Here we choose $\delta = 1$. If $0<|x-3|<1$, then $|x^4-81| = |x^2+9|\cdot|x+3|\cdot|x-3|$. In particular, $\delta = 1$, which means $|x-3|<1$, so $2<x<4$, which implies $|x^2+9|<25$ and $|x+3|<7$, so $|x^4-81| = |x^2+9|\cdot|x+3|\cdot|x-3| < 25\cdot 7\cdot 1 = 175 < 1000$, which is what we require.

10 (i) We show that $\displaystyle\lim_{x\to 2}(3x-1) = 5$, in other words, for any $\varepsilon>0$ there exists $\delta>0$ such that $0<|x-2|<\delta$ implies $|(3x-1)-5|<\varepsilon$. Choose $\delta = \frac{\varepsilon}{3}$. Then if $0<|x-2|<\delta$, we have $|(3x-1)-5| = |3x-6| = 3|x-2| < 3\delta = 3\cdot\frac{\varepsilon}{3} = \varepsilon$, or $|(3x-1)-5|<\varepsilon$, as required.

(ii) We show that $\displaystyle\lim_{x\to 0}(2-5x) = 2$, in other words, for any $\varepsilon>0$ there exists $\delta>0$ such that $0<|x|<\delta$ implies $|(2-5x)-2|<\varepsilon$. Choose $\delta = \frac{\varepsilon}{5}$. Then if $0<|x|<\delta$, $|(2-5x)-2| = |-5x| = 5|x| < 5\delta = 5\cdot\frac{\varepsilon}{5} = \varepsilon$, or $|(2-5x)-2|<\varepsilon$, as required.

(iii) We show that $\displaystyle\lim_{x\to 4}\frac{x^2-1}{x+1} = 3$, so we need to show that for any $\varepsilon>0$ there exists $\delta>0$ such that $0<|x-4|<\delta$ implies $\left|\dfrac{x^2-1}{x+1}-3\right|<\varepsilon$. Choose $\delta = \min(1,\varepsilon)$. Then $0<|x-4|<\delta$ implies

$$\left|\frac{x^2-1}{x+1}-3\right| = \left|\frac{x^2-1-3(x+1)}{x+1}\right| = \left|\frac{x^2-3x-4}{x+1}\right| = \left|\frac{(x-4)(x+1)}{x+1}\right| = |x-4|$$

since $\delta < 1$ implies $|x-4| < 1$ or $3 < x < 5$, therefore $x \neq -1$. Since $|x-4| < \delta < \varepsilon$, thus $\left|\dfrac{x^2-1}{x+1} - 3\right| < \varepsilon$ as required.

(iv) We show that $\lim_{x\to 2} \sqrt{x^2+5} = 3$, so we need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x-2| < \delta$ implies $|\sqrt{x^2+5} - 3| < \varepsilon$. Choose $\delta = \min(1, \frac{3}{5}\varepsilon)$. Then $0 < |x-2| < \delta$ and $0 < |x-2| < 1$, so $1 < x < 3$. Hence

$$\begin{aligned} |\sqrt{x^2+5} - 3| = \left|\frac{(x^2+5)-9}{\sqrt{x^2+5}+3}\right| = \left|\frac{x^2-4}{\sqrt{x^2+5}+3}\right| &= \left|\frac{1}{\sqrt{x^2+5}+3}\right| \cdot |x+2| \cdot |x-2| \\ &< \frac{1}{3} \cdot 5 \cdot \delta < \frac{5}{3} \cdot \frac{3}{5}\varepsilon = \varepsilon \end{aligned}$$

since $1/(\sqrt{x^2+5}+3) < \frac{1}{3}$ regardless of the value of x.

(v) We show that $\lim_{x\to \frac{3}{2}} \dfrac{x^2}{x-2} = -\dfrac{9}{2}$, so we need to show that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $0 < |x-\frac{3}{2}| < \delta$ implies $\left|\frac{x^2}{x-2} + \frac{9}{2}\right| < \varepsilon$. Choose $\delta = \min(\frac{1}{4}, \frac{\varepsilon}{31})$. We have $\delta \leq \frac{1}{4}$ so $0 < |x-\frac{3}{2}| < \delta$ implies $|x-\frac{3}{2}| < \frac{1}{4} \implies \frac{5}{4} < x < \frac{7}{4}$, thus $|x+6| < \frac{31}{4}$ and $1/|x-2| < \frac{1}{\frac{1}{4}} = 4$. If $0 < |x-\frac{3}{2}| < \delta$, then

$$\left|\frac{x^2}{x-2} + \frac{9}{2}\right| = \left|\frac{2x^2+9x-18}{2(x-2)}\right| = \left|\frac{(2x-3)(x+6)}{2(x-2)}\right| = |x+6| \cdot \frac{1}{|x-2|} \cdot |x-\tfrac{3}{2}| < \frac{31}{4} \cdot 4 \cdot \delta < \varepsilon,$$

which is what is required.