Lec 6

Most about Matrices:

Eigenvalues and Eigenvectors.

Def 6·1:

Eigenvalues of a square matrix are roots of the characteristic polynomial. They are usually denoted using the symbol λ.

Example 6·1

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

characteristic polynomial $p(\lambda)$

$$= \det(\lambda I - A)$$

$$= \begin{vmatrix} \lambda & -1 \\ 2 & \lambda - 3 \end{vmatrix} = \lambda^2 - 3\lambda + 2$$

$$= (\lambda - 1)(\lambda - 2)$$

Eigenvalues do not have to be real.

Example 6.2 :

$$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

characteristic polynomial $p(\lambda) =$

$$\begin{vmatrix} \lambda - \sigma & -\omega \\ \omega & \lambda - \sigma \end{vmatrix}$$

$$= (\lambda - \sigma)^2 + \omega^2$$

$$= \lambda^2 - 2\sigma\lambda + (\sigma^2 + \omega^2)$$

Roots are at

$$\sigma + i\omega, \ \sigma - i\omega$$

Eigenvector:

An n vector v is called an eigenvector of a matrix A w.r.t. the eigenvalue λ if

① $v \neq 0$ and ② $Av = \lambda v$

Example 6.1 (continued)

Find eigenvectors of A

$\lambda = 1$: $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$Av = \lambda v \Rightarrow Av = v$$

$$Av = \begin{pmatrix} v_2 \\ -2v_1 + 3v_2 \end{pmatrix}, \quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$$

$$Av = v \Rightarrow v_2 = v_1, \quad -2v_1 + 3v_2 = v_2$$

$$\Rightarrow \boxed{v_1 = v_2}$$

$$v = \begin{pmatrix} v_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} v_1$$

Any scalar multiple of $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ is an eigenvector of A w.r.t. eigenvalue $\lambda = 1$.

$\lambda = 2$: $\quad v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

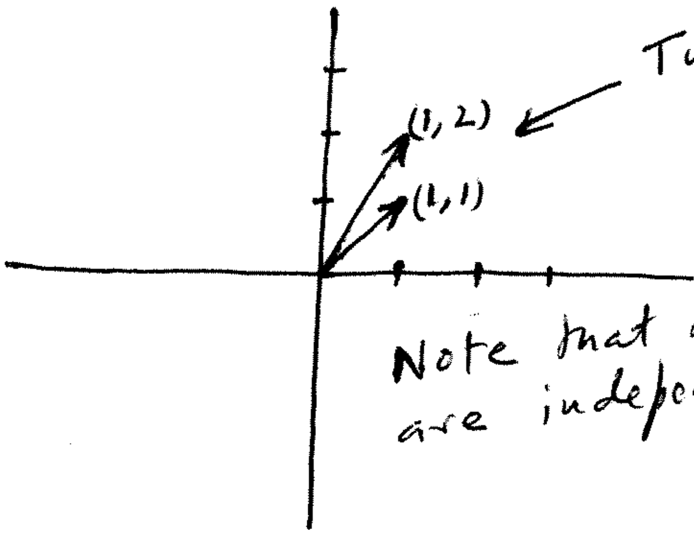
$$Av = \lambda v \Rightarrow Av = 2v$$

$$Av = \begin{pmatrix} v_2 \\ -2v_1 + 3v_2 \end{pmatrix}, \quad 2v = \begin{pmatrix} 2v_1 \\ 2v_2 \end{pmatrix}$$

$$Av = 2v \Rightarrow \left. \begin{array}{l} v_2 = 2v_1 \\ -2v_1 + 3v_2 = 2v_2 \end{array} \right\} \Rightarrow \boxed{v_2 = 2v_1}$$

$$v = \begin{pmatrix} v_1 \\ 2v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 2 \end{pmatrix} v_1$$

Any scalar multiple of $\begin{pmatrix} 1 \\ 2 \end{pmatrix}$ is an eigenvector of A w.r.t. eigenvalue $\lambda = 2$.



Two eigenvectors of A.

Note that the eigenvectors are independent.

<u>**Example 6.2 (continued)**</u>

$\lambda = \sigma + i\omega$: $\quad v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + i \begin{pmatrix} v_1 \\ u_2 \end{pmatrix}$

$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$,

$$A v = \begin{pmatrix} (\sigma u_1 + \omega u_2) + i(\sigma v_1 + \omega v_2) \\ (\sigma u_2 - \omega u_1) + i(\sigma v_2 - \omega v_1) \end{pmatrix}$$

$$\lambda v = \begin{pmatrix} (\sigma u_1 - \omega v_1) + i(\omega u_1 + \sigma v_1) \\ (\sigma u_2 - \omega v_2) + i(\omega u_2 + \sigma v_2) \end{pmatrix}$$

Equating the real and imaginary parts we get

$$u_2 = -v_1 \,, \quad v_2 = u_1$$

$$v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} + i \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$

$\lambda = \sigma - i\omega$: check that

$$v = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} - i \begin{pmatrix} -u_2 \\ u_1 \end{pmatrix}$$

$\lambda_1 = \sigma + i\omega$

$v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}$

$\lambda_2 = \sigma - i\omega$

$v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$

Fact:

Complex conjugate eigenvalues produce complex conjugate eigenvectors.

Fact: Eigenvectors v_1, v_2 corresponding to two distinct eigenvalues λ_1, λ_2 are l.i.

Why? Suppose $v_2 = k v_1$ for some scalar k it follows that

$A v_2 = \lambda_2 v_2$ because v_2 is an eigenvector corresponding to eigenvalue λ_2

Moreover

$A v_1 = \lambda_1 v_1$ because v_1 is an eigenvector corresponding to eigenvalue λ_1

$\Rightarrow k A v_1 = k \lambda_1 v_1$

$\Rightarrow A(k v_1) = \lambda_1 k v_1$

$\Rightarrow A v_2 = \lambda_1 v_2$

Thus

$$A v_2 = \lambda_2 v_2 = \lambda_1 v_2$$

$\Rightarrow$ Either $v_2 = 0$ or $\lambda_1 = \lambda_2$

But $v_2 \neq 0$ because v_2 is an eigenvector also $\lambda_1 \neq \lambda_2$ because they are distinct.

It follows that the original supposition is wrong and that v_1 and v_2 are l.i.

Fact: Eigenvectors $v_1, v_2, \ldots, v_n$ corresponding to distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ respectively are l.i. ie

$\{v_1, \ldots, v_n\}$ is a l.i. set of vectors.

Proof: Suppose not, assume without any loss of generality that $\{v_1, \ldots, v_{n-1}\}$ is a l.i. set and that

$$v_n = \alpha_1 v_1 + \alpha_2 v_2 + \cdots + \alpha_{n-1} v_{n-1}$$

for some choice of real numbers $\alpha_1, \ldots, \alpha_{n-1}$.

It follows that

$$A v_n = \alpha_1 A v_1 + \alpha_2 A v_2 + \cdots + \alpha_{n-1} A v_{n-1}$$

(premultiplying by A)

Hence we conclude that

$$\lambda_n v_n = \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \cdots + \alpha_{n-1} \lambda_{n-1} v_{n-1}$$

$$\Rightarrow \lambda_n (\alpha_1 v_1 + \cdots + \alpha_{n-1} v_{n-1})$$
$$= \alpha_1 \lambda_1 v_1 + \alpha_2 \lambda_2 v_2 + \cdots + \alpha_{n-1} \lambda_{n-1} v_{n-1}$$

$$\Rightarrow \alpha_1 (\lambda_n - \lambda_1) v_1 + \alpha_2 (\lambda_n - \lambda_2) v_2 + \cdots$$
$$\cdots + \alpha_{n-1} (\lambda_n - \lambda_{n-1}) v_{n-1} = 0$$

$$\Rightarrow \alpha_1 (\lambda_n - \lambda_1) = 0, \ \alpha_2 (\lambda_n - \lambda_2) = 0, \cdots$$
$$\cdots \alpha_{n-1} (\lambda_n - \lambda_{n-1}) = 0$$

Since $v_1, \ldots, v_{n-1}$ are l.i.

$$\Rightarrow \alpha_1 = \alpha_2 = \cdots = \alpha_{n-1} = 0$$

since the eigenvalues are all distinct.

6.10

$\Rightarrow v_n = 0$ which violates the assumption that v_n is an eigenvector.

Thus v_n is not a non-trivial l.c. of vectors $v_1, \ldots, v_{n-1}$. Hence

$$\{v_1, \ldots, v_n\}$$

is a l.i. set of vectors. $\blacksquare$

Upshot:

n distinct eigenvalues produce

n l.i. eigenvectors

$$\lambda_1 \quad \lambda_2 \quad \cdots \quad \lambda_n$$

$$v_1 \quad v_2 \quad \cdots \quad v_n$$

$\{v_1, \ldots, v_n\}$ is a l.i. set

Hence it forms a basis of $\mathbb{R}^n$.

Distinct eigenvalues are good because they give rise to n l.i. eigenvectors $v_1, \dots, v_n$ that form a basis of $\mathbb{R}^n$.

Actually we can say more :—

If $\lambda_1 \dots \lambda_n$ are n distinct eigenvalues
$v_1 \dots v_n$ are corresponding eigenvectors.

Define

$$T = (v_1 \dots v_n)$$

T is a $n \times n$ matrix, invertible because $v_1 \dots v_n$ are l.i.

Note that

$$A v_1 = \lambda_1 v_1$$
$$A v_2 = \lambda_2 v_2$$
$$\vdots$$
$$A v_n = \lambda_n v_n$$

$$\Rightarrow A(v_1 \cdots v_n) = (v_1 \cdots v_n)\begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

$$\Rightarrow AT = T\Lambda$$

where

$$\Lambda = \begin{pmatrix} \lambda_1 & & 0 \\ & \lambda_2 & \\ 0 & & \ddots \\ & & & \lambda_n \end{pmatrix}$$

a diagonal matrix

Similarity transformation.

Hence $T^{-1} A T = \Lambda$

If a square $n \times n$ matrix A has distinct eigenvalues, then A is similar to a diagonal matrix of all eigenvalues of A. The matrix T is constructed by stacking up the eigenvectors columnwise.

Remark: For whatever reason, similarity operation and diagonalization of a square matrix is important. The point is that a matrix is diagonalizable (by similarity operation) if it has distinct eigenvalues.

Example 6·3

Going back to the matrix

$$A = \begin{pmatrix} 0 & 1 \\ -2 & 3 \end{pmatrix}$$

A is similar to the diagonal matrix

$$\begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix} = \Lambda$$

where $T = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$

Check that

$$T^{-1} A T = \Lambda$$

For the matrix

$$A = \begin{pmatrix} \sigma & \omega \\ -\omega & \sigma \end{pmatrix}$$

A is similar to the diagonal matrix

$$\begin{pmatrix} \sigma + i\omega & 0 \\ 0 & \sigma - i\omega \end{pmatrix} = \Lambda$$

for

$$T = \begin{pmatrix} 1 & 1 \\ i & -i \end{pmatrix}$$

Example 6.4

Let B be the following matrix

$$B = \begin{pmatrix} 1.5000 & -0.8333 & -0.3333 \\ -5.5000 & -7.1667 & -3.6667 \\ 14.5000 & 24.1667 & 11.6667 \end{pmatrix}$$

The characteristic polynomial of B is $(\lambda-2)^3$ so that B has eigenvalues only at 2 repeated 3 times.

B has only 2 l. i. eigenvectors They are

$$v_1 = \begin{pmatrix} -.5 \\ -5.5 \\ 14.5 \end{pmatrix} \quad \text{and} \quad v_3 = \begin{pmatrix} -2/3 \\ 0 \\ 1 \end{pmatrix}$$

In fact

$$B v_1 = \begin{pmatrix} -1 \\ -11 \\ 29 \end{pmatrix} \quad \& \quad B v_3 = \begin{pmatrix} -4/3 \\ 0 \\ 2 \end{pmatrix}$$

indicating that

$$B v_1 = 2 v_1 \quad , \quad B v_3 = 2 v_3$$

A little algebra shows that

$\exists$ a vector $v_2 = \begin{pmatrix} 1/3 \\ 0 \\ 1 \end{pmatrix}$ such that

$$\boxed{B v_2 = 2 v_2 + v_1}$$

$$B v_2 = \begin{pmatrix} .1667 \\ -5.5 \\ 16.5 \end{pmatrix} , \quad B v_2 - v_1 = \begin{pmatrix} .6667 \\ 0 \\ 2 \end{pmatrix}$$

The vector v_2 is not an eigenvector but is called a generalized eigenvector.

Define

$$T = \begin{pmatrix} -0.5000 & 0.3333 & -0.6667 \\ -5.5000 & 0 & 0 \\ 14.5000 & 1.0000 & 1.0000 \end{pmatrix}$$

eigenvector

generalized eigenvector.

eigenvector

$$T^{-1} B T = \left(\begin{array}{cc|c} 2 & 1 & 0 \\ 0 & 2 & 0 \\ \hline 0 & 0 & 2 \end{array}\right)$$

Jordan canonical form

Example 6.5

Let 'B' be the following matrix.

B =

$$\begin{pmatrix} 1.1667 & -0.5101 & -0.2222 \\ -5.5000 & -7.1667 & -3.6667 \\ 13.5000 & 25.1364 & 12.0000 \end{pmatrix}$$

As in example 6.4, the characteristic polynomial of this matrix B is also $(\lambda-2)^3$ so that B has eigenvalues at 2 repeated 3 times. This time B has only one eigenvector

$$v_1 = \begin{pmatrix} .5 \\ 5.5 \\ -14.5 \end{pmatrix} \qquad \boxed{B v_1 = 2 v_1}$$

There is no other eigenvector which is independent w.r.t. v_1. However there are two additional vectors

$$v_2 = \begin{pmatrix} -.8333 \\ -5.5 \\ 13.5 \end{pmatrix} \qquad v_3 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$

such that

$$B v_2 = 2 v_2 + v_1$$

$$B v_3 = 2 v_3 + v_2$$

The vectors v_2 and v_3 are called generalized eigenvectors.

Define

$$T = \begin{pmatrix} v_1 & v_2 & v_3 \end{pmatrix}$$

eigenvector

generalized eigenvector

$$= \begin{pmatrix} 0.5000 & -0.8333 & 1.0000 \\ 5.5000 & -5.5000 & 0 \\ -14.5000 & 13.5000 & 0 \end{pmatrix}$$

We can see that

$$T^{-1} B T = \begin{pmatrix} 2 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 2 \end{pmatrix}$$

Jordan canonical form.

Examples 6.4 and 6.5 illustrate what might happen when the eigenvalues are not distinct.

There may not be enough number (3 in the case of examples 6.4 & 6.5) of l.i. eigenvectors.

The slack has to be filled up with generalized eigenvectors.

<u>Good News</u>: For every $n \times n$ matrix there $\boxed{\text{always exists}}$ n l.i. eigenvectors and generalized eigenvectors.

6.23

Not all matrices are diagonalizable but "all matrices can be reduced to a jordan canonical form"

Example 6.6

```
>> A=[2 0 0 0 0 0 1 0;0 3 0 0 0 0 0 1;0 0 2 0 0 0 0 0;0 0 0 3 1 0 0 0;0 0 0 0 3 0 0 0;0 1 ↙
 0 0 0 3 0 0 ;0 0 0 0 0 0 2 0 ; 0 0 0 0 0 0 0 3]

A =
```

$$
\begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 3 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 3 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}
$$

```
>> [v j]=jordan(A)

v =
```

generalized Eigenvectors

$$
\begin{array}{c}
\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & -5 \\
0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & -3 & -6 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & -3 \\
0 & 0 & 1 & 0 & 0 & 0 & -5 & -3 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0
\end{pmatrix} \\
\begin{matrix} V1 & V2 & V3 & V4 & V5 & V6 & V7 & V8 \end{matrix}
\end{array}
$$

Eigenvectors

```
j =
```

$$
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}
$$

2x2 jordan block $\lambda = 2$
3x3 " " $\lambda = 3$
1x1 " " $\lambda = 2$
2x2 " " $\lambda = 3$

```
>> inv(v) * A * v
```

$V^{-1} A V$))

```
ans =
```

$$
\begin{pmatrix}
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 3 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 3 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 3
\end{pmatrix}
$$

```
>>
```

$$AV_1 = 2 \cdot V_1$$
$$AV_2 = 2 \cdot V_2 + V_1$$
$$AV_3 = 3V_3$$
$$AV_4 = 3V_4 + V_3$$
$$AV_5 = 3V_5 + V_4$$
$$AV_6 = 2V_6$$
$$AV_7 = 3V_7$$
$$AV_8 = 3V_8 + V_7$$

Example 6·7:

A =

2.0000	0	0	0	0	0	1.0000	0
0	3.0000	0	0	0	0	0	1.0000
0	0	2.0000	0	0	0	0	0
0	0	0	3.0000	1.0000	0	0	0
0	0	0	0	3.0000	0	0	0
0	1.0000	0	0	0	3.0000	0	0
0	0	0	0	0	0	2.0000	0
0.1000	0	0	0	0	0	0	3.0000

perturbation to the A matrix

```
>> [v j]=jordan(A)
```

v =

1.0000	0	0	0	0	0	0	0
0.1000	0.2000	0	0.1000	-0.2000	0	0	0
0	1.0000	0	0	0	1.0000	0	0
0	0	0	1.0000	0	0	1.0000	0
0	0	0	0	1.0000	0	0	1.0000
-0.1000	-0.3000	0.1000	-0.2000	0.3000	0	0	0
0	1.0000	0	0	0	0	0	0
-0.1000	-0.1000	0	0	0.1000	0	0	0

Eigenvectors and gen. eigenvectors have changed

j =

2	1	0	0	0	0	0	0
0	2	0	0	0	0	0	0
0	0	3	1	0	0	0	0
0	0	0	3	1	0	0	0
0	0	0	0	3	0	0	0
0	0	0	0	0	2	0	0
0	0	0	0	0	0	3	1
0	0	0	0	0	0	0	3

Jordan canonical form does not change.

Example 6·8 :–

```
A =

    2.0000         0         0         0         0         0    1.0000         0
         0    3.0000         0         0         0         0         0    1.0000
         0         0    2.0000         0         0         0         0         0
         0         0         0    3.0000    1.0000         0         0         0
         0         0         0         0    3.0000         0         0         0
         0    1.0000         0         0         0    3.0000         0         0
         0         0    1.0000         0         0         0    2.0000         0
    0.2000         0         0         0         0         0         0    3.0000

>> [v j]=jordan(A)
```

perturbation to the A matrix.

```
v =

    1.0000         0         0         0         0         0         0         0
    0.2000    0.4000    0.6000         0    0.2000   -0.6000         0         0
         0         0    1.0000         0         0         0         0         0
         0         0         0         0    1.0000         0    1.0000         0
         0         0         0         0         0    1.0000         0    1.0000
   -0.2000   -0.6000   -1.2000    0.2000   -0.6000    1.2000         0         0
         0    1.0000         0         0         0         0         0         0
   -0.2000   -0.2000   -0.2000         0         0    0.2000         0         0


j =

     2     1     0     0     0     0     0     0
     0     2     1     0     0     0     0     0
     0     0     2     0     0     0     0     0
     0     0     0     3     1     0     0     0
     0     0     0     0     3     1     0     0
     0     0     0     0     0     3     0     0
     0     0     0     0     0     0     3     1
     0     0     0     0     0     0     0     3
```

Jordan canonical form has changed.

Eigenvalues and their multiplicity have not changed.

Example 6.9:


```
A =

    2.0000         0         0         0         0         0    1.0000         0
         0    3.0000         0         0         0         0         0    1.0000
         0         0    2.0000         0         0         0         0         0
         0         0         0    3.0000    1.0000         0         0         0
         0         0         0         0    3.0000    1.0000         0         0
         0    1.0000         0    0.1000         0    3.0000         0         0
         0         0    0.1000         0         0         0    2.0000         0
    0.2000         0         0         0         0         0         0    3.0000
```

perturbation to the A matrix.

```
>> [v j]=jordan(A)
```

This was matrix T in previous examples
The columns are eigenvectors and generalized eigenvectors.

$$V = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & .1 & 0 & 0 \\ 0 & 0 & 0 & .02 & -.06 & .02 & .04 & .06 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ -.2032+i.0858 & -.2032-i.0858 & .0458 & -.2 & .6 & -.0182 & -.086 & -.2394 \\ .0127-i.1016 & .0127+i.1016 & .0212 & 0 & -.2 & .0182 & .0678 & .1534 \\ .0379+i.0287 & .0379-i.0287 & .0099 & 0 & 0 & -.0182 & -.0496 & -.0856 \\ 0 & 0 & 0 & 0 & 0 & 0 & .1 & 0 \\ 0 & 0 & 0 & 0 & .02 & -.02 & -.02 & -.02 \end{pmatrix}$$

$$J = \left(\begin{array}{cc|c|cc|ccc}
2.7679+i\,.4020 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 2.7679-i\,.4020 & 0 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 3.4642 & 0 & 0 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\
\hline
0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2
\end{array}\right)$$

Block Diagonalization

Eigenvalues at $2.7679 \pm i\,.4020$, 3.4642, 3 & 3, 2 & 2 & 2

There are 4 Jordan Blocks.

The block diagonal matrix in the previous page is complex valued and obtained by a similarity transformation with a complex valued matrix 'v'. A different similarity transformation with a real valued matrix can be constructed such that the matrix 'A' in page 6.27 is similar to the following matrix:

$$J = \begin{pmatrix} 2.7679 & 0.4020 & 0 & 0 & 0 & 0 & 0 & 0 \\ -0.4020 & 2.7679 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 3.4642 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 3 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 3 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

Example 6.10:

A =

$$\begin{pmatrix} 2 & 3 & 1 & 0 \\ -3 & 2 & 0 & 1 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

>> [v j] = jordan (A)

gen eigenvector

v =

$$\begin{pmatrix} 0.5 & 0 & 0.5 & 0 \\ -0.5i & 0 & 0.5i & 0 \\ 0 & 0.5 & 0 & 0.5 \\ 0 & -0.5i & 0 & 0.5i \end{pmatrix}$$

Eigenvector

j =

$$\left(\begin{array}{cc|cc} 2-3i & 1 & 0 & 0 \\ 0 & 2-3i & 0 & 0 \\ \hline 0 & 0 & 2+3i & 1 \\ 0 & 0 & 0 & 2+3i \end{array}\right)$$

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Example 6.11:

A =

$$\begin{pmatrix} 2 & 3 & 0 & 0 \\ -3 & 2 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & -3 & 2 \end{pmatrix}$$

```
>> [v j] = jordan (A)
```

v =

$$\begin{pmatrix} 0.5 & 0.5 & 0 & 0 \\ -0.5i & 0.5i & 0 & 0 \\ 0.5 & 0.5 & 0.5 & 0.5 \\ -0.5i & 0.5i & -0.5i & 0.5i \end{pmatrix}$$

4 Eigenvectors

j =

$$\begin{pmatrix} 2-3i & 0 & 0 & 0 \\ 0 & 2+3i & 0 & 0 \\ 0 & 0 & 2-3i & 0 \\ 0 & 0 & 0 & 2+3i \end{pmatrix}$$