MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 1

1 (SHE Section 1.3)

40. $|x-2|<2 \implies -2<x-2<2 \implies 0<x<4 \implies x\in[0,4]$.

2 (SHE Section 1.5)

18. If $x>0$ we have $f(x)=\frac{x}{x}=1$, otherwise if $x<0$ we have $f(x)=\frac{x}{(-x)}=-1$. If $x=0$, then $f(x)$ is undefined. Therefore, f takes on the value 1 for all values $x>0$.

28. For $g(x)=\sqrt{x-1}-1$, we require all values x such that $x-1\geq 0$, so the domain is $x\in[1,\infty)$. Since $\sqrt{x-1}\geq 0$ for all values in the domain, it follows that the range must be $y\in[-1,\infty)$.

3 (SHE Section 1.6)

78. The lines $y=4x+2$ and $y=-19x$ intersect when $4x+2=-19x \implies x=-\frac{2}{23},\ y=-\frac{38}{23}$. By question 75, the angle between the lines is given by

$$\tan\alpha=\frac{4-19}{1-76}=\frac{1}{5}.$$

4 (SHE Section 1.7)

26. $f(x)=x^2+x,\ g(x)=\sqrt{x} \implies (f\circ g)(x)=f(g(x))=x+\sqrt{x}$. It follows that the domain of $f\circ g$ is $[0,\infty)$.

5 (i) Factoring, we have $x^4-x^3-3x^2+x+2=(x-1)(x-2)(x+1)^2=0$ whenever $x=1$, $x=2$, or $x=-1$.

(ii) Squaring both sides, we have

$$\begin{aligned}\cos x+1=\sin x &\implies (\cos x+1)^2=\sin^2 x \implies \cos^2 x+2\cos x+1=\sin^2 x\\ &\implies \cos^2 x+2\cos x+1=1-\cos^2 x \implies 2\cos^2 x+2\cos x=0\\ &\implies \cos x(\cos x+1)=0,\end{aligned}$$

so $\cos x=0$ or $\cos x=-1$. This yields the solutions $x=\frac{\pi}{2},\frac{3\pi}{2},\pi$. However, plugging in our solutions shows that $x=\frac{3\pi}{2}$ is not a solution, so the only solutions are $x=\frac{\pi}{2},\pi$.

6 (i) $3-|2x+4|\leq 1 \implies |2x+4|\geq 2 \implies |x+2|\geq 1 \implies x\in(-\infty,-3]\cup[-1,\infty)$.

(ii) First note that $\dfrac{3}{x-1}-\dfrac{4}{x}\geq 1 \iff \dfrac{3}{x-1}-\dfrac{4}{x}-1\geq 0$. Obtaining a common denominator gives us

$$\frac{3x-4(x-1)-x(x-1)}{x(x-1)}\geq 0 \implies \frac{4-x^2}{x(x-1)}\geq 0 \implies \frac{(2-x)(2+x)}{x(x-1)}\geq 0.$$

Again we create a chart of factors and signs. The expression is zero when $x=\pm 2$ and the expression is undefined at $x=0,1$.

	$x<-2$	$-2<x<0$	$0<x<1$	$1<x<2$	$x>2$
$2-x$	$+$	$+$	$+$	$+$	$-$
$2+x$	$-$	$+$	$+$	$+$	$+$
x	$-$	$-$	$+$	$+$	$+$
$x-1$	$-$	$-$	$-$	$+$	$+$
$\dfrac{(2-x)(2+x)}{x(x-1)}$	$-$	$+$	$-$	$+$	$-$

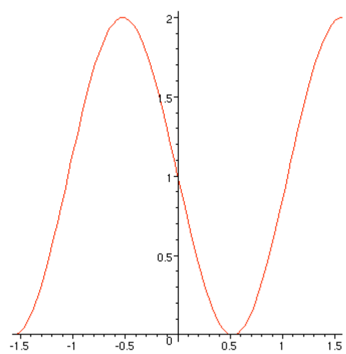
Therefore, the inequality holds when $x \in [-2,0) \cup (1,2]$.

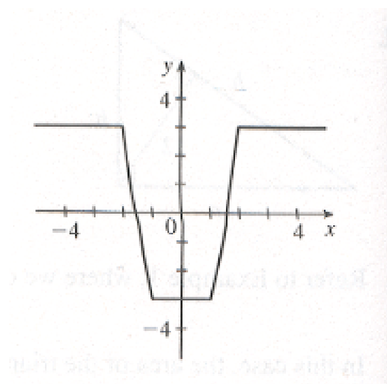
(iii) The simple way to solve the inequality is to square both sides (since both expressions are positive):

$$|x-3| < |2x+1| \implies (x-3)^2 < (2x+1)^2 \implies x^2-6x+9 < 4x^2+4x+1 \implies 3x^2+10x-8 > 0.$$

Factoring we get $(3x-2)(x+4) > 0$, which is true when $x \in (-\infty,-4) \cup (\frac{2}{3},\infty)$.

7 (i) Take the graph of $\cos x$. The graph of $\cos 3x$ decreases the period to $\frac{2\pi}{3}$. We shift the graph left to get the graph of $\cos(3x+\frac{\pi}{2})$. Finally, we shift the graph up by 1. This yields the graph of $f(x) = 1+\cos(3x+\frac{\pi}{2})$ which is provided below left.





(ii) By definition of absolute value,

$$|x^2-1| = \begin{cases} x^2-1, & |x| \geq 1 \\ 1-x^2, & |x| < 1, \end{cases} \qquad |x^2-4| = \begin{cases} x^2-4, & |x| \geq 2 \\ 4-x^2, & |x| < 2. \end{cases}$$

So for $0 \leq |x| < 1$, $g(x) = 1-x^2-(4-x^2) = -3$. For $1 \leq |x| < 2$, $g(x) = x^2-1-(4-x^2) = 2x^2-5$, and for $|x| \geq 2$, $g(x) = x^2-1-(x^2-4) = 3$. This yields the graph above right.

8 By the triangle inequality, $|a+b| \leq |a|+|b|$. Let $a = x+y$ and $b = -y$. Then $|x+y-y| \leq |x+y|+|-y|$ or $|x| \leq |x+y|+|y|$. The result immediately follows.

9 (i) $(f+g)\circ h = (f+g)(h(x)) = f(h(x))+g(h(x)) = f\circ h+g\circ h$, so the statement is always true.

(ii) Let $f(x) = x^2$, $g(x) = x$, and $h(x) = x^3$. Then

$$f\circ(g+h) = f(x+x^3) = x^2+2x^4+x^6, \qquad f\circ g+f\circ h = x^2+x^6,$$

so it does not follow that the statement is true for all functions f, g, h.

10 Suppose $f_E(x) = f(x) + f(-x)$ and $f_O(x) = f(x) - f(-x)$.

(a) Note that

$$f_E(-x) = f(-x) + f(-(-x)) = f(-x) + f(x) = f_E(x) \text{ and}$$
$$f_O(-x) = f(-x) - f(-(-x)) = f(-x) - f(x) = -f_O(x).$$

Hence f_E is even and f_O is odd.

(b) Since $f_E(x)/2$ and $f_O(x)/2$ are both odd functions, note that any function $f(x)$ can be written as

$$f(x) = \frac{f_E(x)}{2} + \frac{f_O(x)}{2},$$

since $\frac{1}{2}(f_E(x) + f_O(x)) = \frac{1}{2}(f(x) + f(-x) + f(x) - f(-x)) = f(x)$. Hence any function f can be written as the sum of an even function and an odd function.

11 (a) $\sin(2\pi - x) = \sin 2\pi \cos x - \cos 2\pi \sin x = -\sin x$.

(b) Using part (a), we notice that

$$\sin \tfrac{\pi}{100} = \sin(2\pi - \tfrac{199\pi}{100}) = -\sin \tfrac{199\pi}{100},$$
$$\sin \tfrac{2\pi}{100} = \sin(2\pi - \tfrac{198\pi}{100}) = -\sin \tfrac{198\pi}{100},$$

and so on. In other words, each outer two terms of the sum must cancel out. This leaves $\sin \frac{100\pi}{100}$, which is zero. Hence the sum is zero.