MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 2

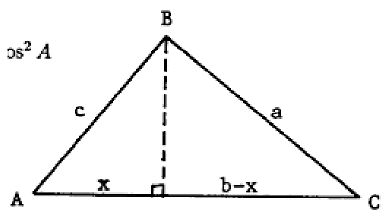
1 (SHE Section 1.6)

62. It is known that $\sin x$ has period 2π, so $\sin ax$ has period $2\pi/a$. Therefore $\sin\frac{1}{2}x$ has period 4π.

72. $g(-x) = \sec(-x) = \frac{1}{\cos(-x)} = \frac{1}{\cos x} = \sec x = g(x)$ (since $\cos x$ is even). Therefore g is also even.

84. We use the figure below. $h = c\sin A$ and $x = c\cos A$. Therefore,

$$a^2 = h^2 + (b-x)^2 = c^2\sin^2 A + b^2 - 2bc\cos A + c^2\cos^2 A = b^2 + c^2 - 2bc\cos A.$$



2 (a) Notice that $k\sin(x+\phi) = k\sin x\cos\phi + k\cos x\sin\phi$. Therefore

$$k\sin x\cos\phi + k\cos x\sin\phi = A\sin x + B\cos x \implies A = k\cos\phi,\ B = k\sin\phi.$$

Observe that $A^2 + B^2 = k^2\cos^2\phi + k^2\sin^2\phi = k^2$. Hence,

$$k = \sqrt{A^2+B^2}, \quad \cos\phi = \frac{A}{\sqrt{A^2+B^2}}, \quad \sin\phi = \frac{B}{\sqrt{A^2+B^2}}.$$

(b) For (i), we have $A = \frac{1}{\sqrt{3}}$ and $B = 1$. Then by part (a) we have $k = \sqrt{\frac{1}{3}+1} = \frac{2}{\sqrt{3}}$, $\cos\phi = \frac{1/\sqrt{3}}{2/\sqrt{3}} = \frac{1}{2}$ and $\sin\phi = \frac{\sqrt{3}}{2}$. It follows that $\phi = \frac{\pi}{3}$. Hence $\frac{1}{\sqrt{3}}\sin x + \cos x = \frac{2}{\sqrt{3}}\sin(x+\frac{\pi}{3})$.

For (ii), we have $A = -5$ and $B = 5$. Then by part (a) we have $k = \sqrt{25+25} = 5\sqrt{2}$, $\cos\phi = \frac{-5}{5\sqrt{2}} = -\frac{1}{\sqrt{2}}$ and $\sin\phi = \frac{1}{\sqrt{2}}$. It follows that $\phi = \frac{3\pi}{4}$.
Hence $-5\sin 2x + 5\cos 2x = 5\sqrt{2}\sin(2x+\frac{3\pi}{4})$.

3 (a) Using the addition formulas for $\cos x$,

$$\frac{1}{2}[\cos(u+v)+\cos(u-v)] = \frac{1}{2}[\cos u\cos v - \sin u\sin v + \cos u\cos v + \sin u\sin v] = \cos u\cos v,$$
$$\frac{1}{2}[\cos(u-v)-\cos(u+v)] = \frac{1}{2}[\cos u\cos v + \sin u\sin v - \cos u\cos v + \sin u\sin v] = \sin u\sin v.$$

(b) Using part (a),

$$\tan\frac{7\pi}{24}\tan\frac{11\pi}{24} = \frac{\sin\frac{7\pi}{24}\sin\frac{11\pi}{24}}{\cos\frac{7\pi}{24}\cos\frac{11\pi}{24}} = \frac{\frac{1}{2}[\cos(\frac{11\pi}{24}-\frac{7\pi}{24})-\cos(\frac{11\pi}{24}+\frac{7\pi}{24})]}{\frac{1}{2}[\cos(\frac{11\pi}{24}+\frac{7\pi}{24})+\cos(\frac{11\pi}{24}-\frac{7\pi}{24})]} = \frac{\cos\frac{\pi}{6}-\cos\frac{3\pi}{4}}{\cos\frac{3\pi}{4}+\cos\frac{\pi}{6}}$$
$$= \frac{\frac{\sqrt{3}}{2}+\frac{\sqrt{2}}{2}}{-\frac{\sqrt{2}}{2}+\frac{\sqrt{3}}{2}} = \frac{\sqrt{3}+\sqrt{2}}{\sqrt{3}-\sqrt{2}} = (\sqrt{3}+\sqrt{2})^2$$

4 We only consider the solutions within the interval $[0, 2\pi]$. Factoring the expression on the left side gives us

$$g(x) = \cos x(2\sin^2 x + 3\sin x - 2) = \cos x(2\sin x - 1)(\sin x + 2) > 0.$$

The expression is zero when $\cos x = 0$ or $\sin x = \frac{1}{2}$, which is $x = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{\pi}{6}, \frac{5\pi}{6}$. We can draw a chart of signs to find where the expression is positive or negative.

	$0 < x < \frac{\pi}{6}$	$\frac{\pi}{6} < x < \frac{\pi}{2}$	$\frac{\pi}{2} < x < \frac{5\pi}{6}$	$\frac{5\pi}{6} < x < \frac{3\pi}{2}$	$\frac{3\pi}{2} < x < 2\pi$
$\cos x$	$+$	$+$	$-$	$-$	$+$
$2\sin x - 1$	$-$	$+$	$+$	$-$	$-$
$\sin x + 2$	$+$	$+$	$+$	$+$	$+$
$g(x)$	$-$	$+$	$-$	$+$	$-$

so $g(x) > 0$ when $\frac{\pi}{6} < x < \frac{\pi}{2}$ or $\frac{5\pi}{6} < x < \frac{3\pi}{2}$. Since we want all solutions on the entire real line, the inequality is satisfied for $x \in (\frac{\pi}{6} + 2\pi n, \frac{\pi}{2} + 2\pi n) \cup (\frac{5\pi}{6} + 2\pi n, \frac{3\pi}{2} + 2\pi n)$ for all integers n.

5 (i) $\dfrac{1}{\cot x - \csc x} = \dfrac{1}{\frac{\cos x}{\sin x} - \frac{1}{\sin x}} = \dfrac{\sin x}{\cos x - 1}$.

(ii) $\csc^2\beta - \csc\beta\cot\beta = \csc\beta(\csc\beta - \cot\beta) = \dfrac{\frac{1}{\sin\beta} - \frac{\cos\beta}{\sin\beta}}{\sin\beta} = \dfrac{1 - \cos\beta}{\sin^2\beta} = \dfrac{1 - \cos\beta}{1 - \cos^2\beta}$

$= \dfrac{1 - \cos\beta}{(1 - \cos\beta)(1 + \cos\beta)} = \dfrac{1}{1 + \cos\beta}$.

(iii) $\csc^4\theta - \cot^4\theta = (\csc^2\theta - \cot^2\theta)(\csc^2\theta + \cot^2\theta) = \csc^2\theta + \cot^2\theta$ (since $\csc^2\theta - \cot^2\theta = 1$).

(iv) $\dfrac{\sin 2x}{1 - \cos 2x} = \dfrac{2\sin x\cos x}{1 - \cos^2 x + \sin^2 x} = \dfrac{2\sin x\cos x}{2\sin^2 x} = \dfrac{\cos x}{\sin x} = \cot x$.