Definition For $s > 0$, the *Dirichlet eta function* is defined by

$$\eta(s) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}. \tag{1}$$

Theorem η is infinitely differentiable on $(0, \infty)$ and its n-th derivatives are obtained by applying termwise differentiation on (1) n times.

To prove this, we need some preliminaries.

Lemma 1 (Abel's test) Suppose $A \subset \mathbb{R}^m$ and f_n, φ_n are real valued functions defined on A. If
(1) φ_n is monotonic,
(2) φ_n are uniformly bounded, and
(3) $\sum_{n=1}^{\infty} f_n$ converges uniformly,
then $\sum_{n=1}^{\infty} f_n \varphi_n$ converges uniformly.

This is a typical application of summation by parts.

Lemma 2 For $s > 0$, define $f_k(s) = \dfrac{(-1)^{k-1}}{k^s} \log^{n+1} k$. Then for any $a > b > 0$, $\sum_{k=1}^{\infty} f_k$ converges uniformly on (a, b).

proof. Put $h(\alpha) = \dfrac{\log^{n+1} \alpha}{\alpha^s}$. Then $h'(\alpha) = \dfrac{\log^n \alpha}{\alpha^{1+s}}(n + 1 - s \log \alpha)$. Then it suffices to show that

$$f_{m+1} + \cdots f_{m+2k} = (-1)^m (h(m+1) - h(m+2) + \cdots + h(m+2k-1) - h(m+2k))$$

converges uniformly for $m = 0, 1$. But since

$$\begin{aligned}
|f_{m+2k-1} + f_{m+2k}| &= |h(m+2k-1) - h(m+2k)| \\
&= |h'(c)| \qquad (\text{for some } m+2j-1 < c < m+2j) \\
&\le \frac{(n+1)\log^n c + s \log^{n+1} c}{c^{1+s}} \\
&\le \frac{(n+1)\log^n (2k+1) + b \log^{n+1}(2k+1)}{(2k-1)^{1+a}}
\end{aligned}$$

and

$$\sum_{j=1}^{\infty} \frac{(n+1)\log^n (2j+1) + b \log^{n+1}(2j+1)}{(2j-1)^{1+a}} < \infty,$$

Weierstrass M-test assures that $f_1 + \cdots + f_k$ converges uniformly on (a, b). ////

Now we are ready to prove the theorem.

proof. We use induction to prove that for nonnegative n and $s > 0$, we have

$$\eta^{(n)}(s) = (-1)^n \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s} \log^n k. \tag{2}$$

By definition, (2) holds for $n = 0$. Now assume (2) holds for some nonnegative integer n. Assume $b > a > 0$ and put $A = \{(s,t) \mid s,t \in (a,b) \text{ and } s \le t\}$. Put $h(x) = \dfrac{e^{-x} - 1 + x}{x}$. Then it is easy to show that h is monotone increasing and $h(x) \to 1$ as $x \to \infty$. Let f_k, φ_k be functions on A defined by

$$f_k(s,t) = \frac{(-1)^{k-1}}{k^s} \log^{n+1} k, \quad \varphi_k(s,t) = 1 - h((t-s)\log k).$$

Note that $0 \le \varphi_{k+1}(s,t) \le \varphi_k(s,t) \le \varphi_1(s,t) = 1$ and $f_1 + \cdots + f_k$ converges uniformly on A by Lemma 2. Then by Abel's test, $\sum_{k=1}^{\infty} f_k \varphi_k$ converges uniformly. But

$$\left| \frac{\eta^{(n)}(t) - \eta^{(n)}(s)}{t-s} - (-1)^{n+1} \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k^s} \log^{n+1} k \right| = \left| \sum_{k=1}^{\infty} f_k(s,t) - \sum_{k=1}^{\infty} f_k(s,t)\varphi_k(s,t) \right|. \tag{3}$$

By the uniform convergence and continuity of each $f_k\varphi_k$, we find that the right hand side of (3) converges to 0 as $t - s \to 0^+$. This completes the proof. ////