MAT 137Y 2007-08 Winter Session, Solutions to Problem Set 12

1 (SHE 8.1)

18. Let $u = 3-2\cos\theta$. Then $\displaystyle\int \frac{\sin\theta}{3-2\cos\theta}\,d\theta = \frac{1}{2}\int\frac{du}{u} = \frac{1}{2}\ln|u|+C = \frac{1}{2}\ln|3-2\cos\theta|+C$.

28. Let $u = x^2$, then $du = 2x\,dx$, so
$$\int \frac{x}{9+(x^2)^2}\,dx = \frac{1}{2}\int\frac{du}{9+u^2} = \frac{1}{2}\cdot\frac{1}{3}\arctan\frac{u}{3}+C = \frac{1}{6}\arctan\frac{x^2}{3}+C.$$

38. $\displaystyle\int_0^{1/2}\frac{1+x}{\sqrt{1-x^2}}\,dx = \left[\sin^{-1}x - \sqrt{1-x^2}\right]_0^{1/2} = \frac{\pi}{6}+1-\frac{\sqrt{3}}{2}.$

50. For parts (a) and (b), the results are fairly obvious by applying the appropriate substitution. Despite the fact that the answers are different, it turns out that they differ by a constant, since
$$\frac{1}{2}\tan^2 x + C_1 = \frac{1}{2}(\sec^2 x - 1)+C_1 = \frac{1}{2}\sec^2 x + \left(C_1 - \frac{1}{2}\right),$$
so C_1 and C_2 differ by $\frac{1}{2}$, which is a constant.

2 (SHE 8.2)

6. We integrate by parts and let $u = x^2$ and $dv = xe^{-x^2}\,dx$. Then $du = 2x\,dx$ and $v = -\frac{1}{2}e^{-x^2}$. Therefore
$$\int x^3e^{-x^2}\,dx = -\frac{x^2}{2}e^{-x^2} + \int xe^{-x^2}\,dx = -\frac{1}{2}x^2e^{-x^2} - \frac{1}{2}e^{-x^2}+C.$$

18. We multiply the expression and integrate by parts, giving us
$$\begin{aligned}\int (2^x+x^2)^2\,dx = \int 2^{2x}+2x^22^x+x^4\,dx &= \frac{2^{2x}}{2\ln 2}+\frac{x^5}{5}+\int 2x^22^x\,dx\\ &= \frac{4^x}{\ln 4}+\frac{x^5}{5}+\frac{2x^22^x}{\ln 2}-\int\frac{4x2^x}{\ln 2}\,dx\\ &= \frac{4^x}{\ln 4}+\frac{x^5}{5}+\frac{2x^22^x}{\ln 2}-\frac{4x2^x}{(\ln 2)^2}+\int\frac{4\cdot 2^x}{(\ln 2)^2}dx\\ &= \frac{4^x}{\ln 4}+\frac{x^5}{5}+2^x\left[\frac{2x^2}{\ln 2}-\frac{4x}{(\ln 2)^2}+\frac{4}{(\ln 2)^3}\right]+C\end{aligned}$$

28. Let $u = e^{3x}$ and $dv = \cos 2x\,dx$. Then $du = 3e^{3x}\,dx$ and $v = \frac{1}{2}\sin 2x$. Hence
$$\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x - \int\frac{3}{2}e^{3x}\sin 2x\,dx.$$
Integrating by parts again, we let $u = \frac{3}{2}e^{3x}$ and $dv = \sin 2x\,dx$. Then $du = \frac{9}{2}e^{3x}\,dx$ and $v = -\frac{1}{2}\cos 2x$. Therefore we have
$$\begin{aligned}&\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x+\frac{3}{4}e^{3x}\cos 2x-\int\frac{9}{4}e^{3x}\cos 2x\,dx\\ \Longrightarrow\ &\frac{13}{4}\int e^{3x}\cos 2x\,dx = \frac{1}{2}e^{3x}\sin 2x+\frac{3}{4}e^{3x}\cos 2x\\ \Longrightarrow\ &\int e^{3x}\cos 2x\,dx = \frac{2}{13}e^{3x}\sin 2x+\frac{3}{13}e^{3x}\cos 2x+C.\end{aligned}$$

40. Let I be the integral we need to solve. Let $u = (\ln x)^2$ and $dv = x^2\ dx$. Then $du = \frac{2\ln x}{x}\ dx$ and $v = \frac{x^3}{3}$. Therefore

$$I = \int_1^{2e} x^2(\ln x)^2\ dx = \left[\frac{x^3}{3}(\ln x)^2\right]_1^{2e} - \int_1^{2e} \frac{2}{3}x^2 \ln x\ dx.$$

Integrating by parts again, we let $u = \ln x$, $dv = \frac{2x^2}{3}\ dx$; so $du = \frac{1}{x}\ dx$ and $v = \frac{2x^3}{9}$. Hence,

$$\begin{aligned} I &= \left[\frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x\right]_1^{2e} + \int_1^{2e} \frac{2x^2}{9}\ dx = \left[\frac{x^3}{3}(\ln x)^2 - \frac{2x^3}{9}\ln x + \frac{2x^3}{27}\right]_1^{2e} \\ &= \frac{8e^3}{3}\left[(\ln 2e)^2 - \frac{2}{3}\ln 2e + \frac{2}{9}\right] - \frac{2}{27}. \end{aligned}$$

3 (SHE 8.3)

6. $\displaystyle\int \sin^3 x \cos^2 x\ dx = \int \cos^2 x(1-\cos^2 x)\sin x\ dx = -\frac{1}{3}\cos^3 x + \frac{1}{5}\cos^5 x + C.$

8. Using trig identities, we get

$$\begin{aligned} \int \sin^2 x \cos^4 x\ dx &= \int (\sin x \cos x)^2 \cos^2 x\ dx = \int \frac{1}{4}\sin^2 2x \left(\frac{1}{2} + \frac{1}{2}\cos 2x\right)\ dx \\ &= \frac{1}{8}\int \sin^2 2x\ dx + \frac{1}{8}\int \sin^2 2x \cos 2x\ dx = \frac{1}{8}\left(\frac{1}{2}x - \frac{1}{8}\sin 4x\right) + \frac{1}{48}\sin^3 2x + C. \end{aligned}$$

20.

$$\begin{aligned} \int_0^{\pi/2} \cos^4 x\ dx &= \int_0^{\pi/2} \left(\frac{1+\cos 2x}{2}\right)^2\ dx = \int_0^{\pi/2} \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{\cos^2 2x}{4}\ dx \\ &= \int_0^{\pi/2} \frac{1}{4} + \frac{1}{2}\cos 2x + \frac{1+\cos 4x}{8}\ dx = \left[\frac{3}{8}x + \frac{1}{4}\sin 2x + \frac{1}{32}\sin 4x\right]_0^{\pi/2} \\ &= \frac{3\pi}{16}. \end{aligned}$$

30. Applying trig identities and making a substitution, we have

$$\begin{aligned} \int \frac{\sin^3 x}{\cos x}\ dx &= \int \frac{(1-\cos^2 x)\sin x}{\cos x}\ dx = \int \frac{u^2-1}{u}\ du = \int u - \frac{1}{u}\ du = \frac{1}{2}u^2 - \ln u \\ &= \frac{1}{2}\cos^2 x - \ln|\cos x| + C. \end{aligned}$$

4 (SHE 8.4)

4. If we let $u = x^2 \implies du = 2x\ dx$, then

$$\int \frac{x}{\sqrt{4-x^2}}\ dx = -\frac{-2x}{2\sqrt{4-x^2}}\ dx = -\int \frac{du}{2\sqrt{u}} = -\sqrt{u} + C = -\sqrt{4-x^2} + C.$$

12. Let $x = 4\sin u$. Then $dx = 4\cos u\, du$, and in particular, $\sqrt{16-x^2} = 4\cos u$. Then

$$\begin{aligned}\int \frac{x^2}{\sqrt{16-x^2}}\, dx &= \int \frac{16\sin^2 u \cdot 4\cos u}{4\cos u}\, du = 16\int \sin^2 u\, du \\ &= 8u - 4\sin 2u + C = 8u - 8\sin u \cos u \\ &= 8\arcsin\frac{x}{4} - \frac{1}{2}x\sqrt{16-x^2} + C.\end{aligned}$$

Hence, $\displaystyle\int_0^2 \frac{x^2}{\sqrt{16-x^2}}\, dx = 8\arcsin\frac{1}{2} - \sqrt{12} = \frac{4\pi}{3} - 2\sqrt{3}.$

18. Let $x = \sec u$, then $du = \sec u \tan u\, du$, so

$$\begin{aligned}\int \frac{\sqrt{x^2-1}}{x}\, dx &= \int \frac{\tan u}{\sec u} \sec u \tan u\, du = \int \tan^2 u\, du = \int \sec^2 u - 1\, du = \tan u - u + C \\ &= \sqrt{x^2-1} - \sec^{-1} x + C.\end{aligned}$$

24. We make a double substitution $e^x = 2\tan u$. Then $e^x\, dx = 2\sec^2 u\, du$. Hence

$$\begin{aligned}\int \frac{dx}{e^x\sqrt{4+e^{2x}}} &= \int \frac{e^x dx}{e^{2x}\sqrt{4+e^{2x}}} = \int \frac{2\sec^2 u}{4\tan^2 u \cdot 2\sec u}\, du = \frac{1}{4}\int \frac{\cos u}{\sin^2 u}\, du = -\frac{1}{4}\csc u + C \\ &= -\frac{1}{4}\frac{\sqrt{4+e^{2x}}}{e^x} + C.\end{aligned}$$

46. The area of the region is

$$A = 4\int_0^b \frac{a}{b}\sqrt{b^2+y^2}\, dy = \frac{4a}{b}\left[\frac{y}{2}\sqrt{b^2+y^2} + \frac{b^2}{2}\ln\left(y + \sqrt{b^2+y^2}\right)\right]_0^b = 2ab[\sqrt{2} + \ln(1+\sqrt{2})].$$

5 (SHE 8.5)

10. We apply partial fractions so that

$$\begin{aligned}\frac{x}{(x+1)(x+2)(x+3)} &= \frac{A}{x+1} + \frac{B}{x+2} + \frac{C}{x+3} \\ &\implies A(x+2)(x+3) + B(x+1)(x+3) + C(x+1)(x+2) = x \\ &\implies (A+B+C)x^2 + (5A+4B+3C)x + (6A+3B+2C) = x.\end{aligned}$$

Matching coefficients give us the system of equations

$$\begin{aligned} A+B+C &= 0 \\ 5A+4B+3C &= 1 \\ 6A+3B+2C &= 0 \end{aligned} \implies \begin{aligned} A &= -\tfrac{1}{2} \\ B &= 2 \\ C &= -\tfrac{3}{2}. \end{aligned}$$

Hence

$$\begin{aligned}\int \frac{x}{(x+1)(x+2)(x+3)}\, dx &= \int \frac{(-1/2)}{x+1} + \frac{2}{x+2} - \frac{3/2}{x+3}\, dx \\ &= -\frac{1}{2}\ln|x+1| + 2\ln|x+2| - \frac{3}{2}|x+3| + C.\end{aligned}$$

16. Applying long division first, we have $\int \frac{x^2+3}{x^2-3x+2}\,dx = \int \left(1+\frac{3x+1}{x^2-3x+2}\right)\,dx$. By partial fractions,

$$\frac{A}{x-1}+\frac{B}{x-2} = \frac{3x+1}{(x-1)(x-2)} \implies A(x-2)+B(x-1) = 3x+1.$$

Instead of matching coefficients, we can find the constants A and B by choosing values of x. If $x=2$, then $B=7$, and if $x=1$, then $-A=4$, or $A=-4$. Hence

$$\int \frac{x^2+3}{x^2-3x+2}\,dx = \int \left(1-\frac{4}{x-1}+\frac{7}{x-2}\right)\,dx = x-4\ln|x-1|+7\ln|x-2|+C.$$

30. Using partial fractions, we have

$$\frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)} = \frac{Ax+B}{x^2+1}+\frac{Cx+D}{x^2+3} \implies (Ax+B)(x^2+3)+(Cx+D)(x^2+1) = x^3+x^2+x+3.$$

Here we have no choice but to match coefficients, so

$$(A+C)x^3+(B+D)x^2+(3A+C)x+(3B+D) = x^3+x^2+x+3 \implies \begin{array}{c} A+C=1 \\ B+D=1 \\ 3A+C=1 \\ 3B+D=3. \end{array}$$

This gives our solutions $A=0$, $C=1$, $B=1$, $D=0$. Hence

$$\int \frac{x^3+x^2+x+3}{(x^2+1)(x^2+3)}\,dx = \int \frac{1}{x^2+1}+\frac{x}{x^2+3}\,dx = \tan^{-1}x+\frac{1}{2}\ln(x^2+3)+C$$

48. Note that $y = \frac{1}{x^2-1} = \frac{1}{2}\left[\frac{1}{x-1}-\frac{1}{x+1}\right]$ by partial fractions. We prove the statement is true for all $n \geq 1$. If $n=1$, we have

$$y' = \frac{1}{2}\left[-\frac{1}{(x-1)^2}+\frac{1}{(x+1)^2}\right] = \frac{(-1)^1 1!}{2}\left[\frac{1}{(x-1)^2}-\frac{1}{(x+1)^2}\right],$$

so the statement is true for $n=1$. Now suppose

$$y^{(k)} = \frac{(-1)^k k!}{2}\left[\frac{1}{(x-1)^k}-\frac{1}{(x+1)^k}\right],$$

then by differentiating,

$$y^{(k+1)} = \frac{(-1)^k k!}{2}\left[\frac{-(k+1)}{(x-1)^{k+1}}-\frac{-(k+1)}{(x+1)^{k+1}}\right] = \frac{(-1)^{k+1}(k+1)!}{2}\left[\frac{1}{(x-1)^{k+1}}-\frac{1}{(x+1)^{k+1}}\right],$$

which is obviously what we need to show. Hence the statement is true by induction.

6 (i) Let $u=\ln(\tan x)$. Then $du = \frac{\sec^2 x}{\tan x}\,dx = \frac{1}{\sin x\cos x}\,dx$. Hence

$$\int_{\pi/3}^{\pi/4} \frac{\ln(\tan x)}{\sin x\cos x}\,dx = \int_{\ln\sqrt{3}}^{0} u\,du = -\frac{1}{8}(\ln 3)^2.$$

(ii) Multiply top and bottom by $\cos x$. Then

$$\int \frac{1-\tan x}{1+\tan x}\,dx = \int \frac{\cos x - \sin x}{\cos x + \sin x}\,dx = \ln|\sin x + \cos x| + C.$$

(iii) Integrate by parts. Let $u = x$ and $dv = \cos^3 x \sin x\,dx$. Then $du = dx$ and $v = -\frac{1}{4}\cos^4 x$. Hence

$$\int x\cos^3 x \sin x\,dx = -\frac{1}{4}x\cos^4 x + \int \frac{1}{4}\cos^4 x\,dx = -\frac{1}{4}x\cos^4 x + \frac{3}{32}x + \frac{1}{16}\sin 2x + \frac{1}{128}\sin 4x + C,$$

where we re-use the result from Section 8.3 #20.

(iv) We integrate by parts. Let $u = \tan^{-1} x$ and $dv = x\,dx$. Then $du = 1/(1+x^2)\,dx$ and $v = \frac{1}{2}x^2$. This gives

$$\begin{aligned}\int x\tan^{-1} x\,dx &= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2}{1+x^2}\,dx = \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}\int \frac{x^2+1-1}{1+x^2}\,dx\\ &= \frac{1}{2}x^2\tan^{-1} x - \frac{1}{2}x + \frac{1}{2}\tan^{-1} x + C.\end{aligned}$$

(v) Multiply top and bottom by e^x. Then $\displaystyle\int \frac{dx}{e^x - e^{-x}} = \int \frac{e^x}{e^{2x}-1}\,dx$. Letting $u = e^x$ and $du = e^x\,dx$ gives

$$\int \frac{du}{u^2-1} = \frac{1}{2}\left(\int \frac{1}{u-1} - \frac{1}{u+1}\,du\right) = \frac{1}{2}\left(\ln(e^x-1) - \ln(e^x+1)\right) = \frac{1}{2}\ln\frac{e^x-1}{e^x+1} + C.$$

(vi) We integrate by parts: let $u = \ln(x+4)$ and $dv = \frac{1}{x^2}\,dx$. Then $du = \frac{1}{x+4}\,dx$ and $v = -\frac{1}{x}$. Hence

$$\begin{aligned}\int \frac{\ln(x+4)}{x^2}\,dx &= -\frac{\ln(x+4)}{x} + \int \frac{dx}{x(x+4)} = -\frac{\ln(x+4)}{x} + \int \frac{1/4}{x} - \frac{1/4}{x+4}\,dx\\ &= -\frac{\ln(x+4)}{x} + \frac{1}{4}\ln x - \frac{1}{4}\ln|x+4| + C,\end{aligned}$$

where the work involving partial fractions is omitted.