MAT 137Y, 2007–2008, Solutions to Problem Set 15

1. (SHE 12.5)

4. Given $a_k = (-1)^k \frac{1}{k \ln k}$, we know that $\sum |a_k| = \sum \frac{1}{k \ln k}$ diverges by the integral test (see 11.2 #21), but $\sum a_k$ converges by the alternating series test, so the series converges conditionally and not absolutely.

6. As $\lim\limits_{k \to \infty} \dfrac{k}{\ln k} \overset{\mathrm{H}}{=} \infty$, the alternating series test does not apply and the series diverges.

12. The values of $\sin \frac{k\pi}{4}$ alternate between $0, \pm 1/\sqrt{2}$, and ± 1. The limit as $k \to \infty$ clearly does not exist, so again the series diverges by the basic divergence test.

16. Since $\lim\limits_{k \to \infty} \dfrac{1}{\sqrt{k(k+1)}} = 0$ and the series alternate in sign, the series converges by the alternating series test.

    Using the limit comparison test with $b_k = \frac{1}{k}$, the series $\sum \left| \frac{(-1)^k}{\sqrt{k(k+1)}} \right| = \sum \frac{1}{\sqrt{k(k+1)}}$ diverges, so the series in question conditionally converges (and not absolutely).

28. Since every absolutely convergent series converges, it is sufficient to show that the series absolutely converges. Note that $|a_k| = \left| \frac{\sin(k\pi/2)}{k\sqrt{k}} \right| < \frac{1}{k^{3/2}}$, so by the comparison test, the series converges absolutely.

32. With any alternating series, the error estimate of the actual series and the partial sum s_n is approximately a_{n+1} (Equation 11.4.5). Therefore the error estimate between $\sum (-1)^{k+1} \frac{1}{k}$ and s_{20} is $a_{21} = \frac{1}{21}$.

46. If the sequence of terms $\{a_n\}$ is nonincreasing instead of decreasing, the alternating series still converges. To see this, we can make slight changes of the proof of Theorem 11.4.4. The even partial sums s_{2m} are now nonnegative. Since $s_{2m+2} \le s_{2m}$, the sequence of even terms converges, so $s_{2m} \to L$. Since $s_{2m+1} = s_{2m} - a_{2m+1}$ and $a_{2m+1} \to 0$, we have $s_{2m+1} \to L$. Hence $s_n \to L$.

2. (SHE 12.8)

6. Applying the ratio test, $\lim\limits_{k \to \infty} \left| \dfrac{2^{k+1} x^{k+1}}{k^2} \dfrac{k^2}{2^k x^k} \right| = \lim\limits_{k \to \infty} \left| 2x \left(\dfrac{k}{k+1} \right)^2 \right| = 2|x|$.

    Hence the series converges when $2|x| < 1$, or $|x| < \frac{1}{2}$ and diverges when $|x| > \frac{1}{2}$. (Therefore the radius of convergence is $\frac{1}{2}$.) To find the interval of convergence, we check the points $|x| = \frac{1}{2}$. If $x = \frac{1}{2}$, then we have $\sum \frac{2^k}{k^2} \cdot \frac{1}{2^k} = \sum \frac{1}{k^2}$, which is a convergent p-series. if $x = -\frac{1}{2}$, then we have $\sum \frac{(-1)^k}{k^2}$ which converges by the alternating series test. Therefore the interval of convergence is $[-\frac{1}{2}, \frac{1}{2}]$.

8. Applying the ratio test, $\lim\limits_{k \to \infty} \left| \dfrac{(-1)^{k+1} x^{k+1}}{\sqrt{k+1}} \cdot \dfrac{\sqrt{k}}{(-1)^k x^k} \right| = \lim\limits_{k \to \infty} \sqrt{\dfrac{k}{k+1}} |x| = |x|$.

    Hence the series converges when $|x| < 1$ and diverges when $|x| > 1$. If $x = 1$, then we have $\sum \frac{(-1)^k}{\sqrt{k}}$ which converges by the alternating series test. If $x = -1$, then we have $\sum \frac{(-1)^k \cdot (-1)^k}{\sqrt{k}} = \sum \frac{1}{\sqrt{k}}$, which diverges by the p-series test. Therefore the interval of convergence is $(-1, 1]$.

10. Applying the ratio test, $\lim\limits_{k \to \infty} \left| \dfrac{x^{k+1}}{(k+1)^2 2^{k+1}} \dfrac{k^2 2^k}{x^k} \right| = \lim\limits_{k \to \infty} \dfrac{1}{2} |x| \left(\dfrac{k}{k+1} \right)^2 = \dfrac{1}{2} |x|$,

    so the series converges when $|x| < 2$ and diverges when $|x| > 2$. If $x = 2$, then we have $\sum \frac{1}{k^2}$ which converges. If $x = -2$ then we have $\sum \frac{(-1)^k}{k^2}$ which also converges by the alternating series test. Therefore the interval of convergence is $[-2, 2]$.

14. Applying the ratio test, $\lim\limits_{k \to \infty} \left| \dfrac{x^{k+1} \ln k}{\ln(k+1) \cdot x^k} \right| = \lim\limits_{k \to \infty} |x| \dfrac{\ln k}{\ln(k+1)} = |x|$.

    The series converges when $|x| < 1$ and diverges when $|x| > 1$. If $x = 1$, we have $\sum \frac{1}{\ln k}$ which diverges by comparison with the harmonic series. If $x = -1$, then we have $\sum \frac{(-1)^k}{\ln k}$ which converges by the alternating series test. Hence the interval of convergence is $[-1, 1)$.

16. Applying the ratio test, $\lim_{k\to\infty}\left|\frac{(k+1)a^{k+1}x^{k+1}}{ka^kx^k}\right| = \lim_{k\to\infty}\frac{k+1}{k}|ax| = |ax|$.

 The series converges when $|x| < \frac{1}{|a|}$ and diverges if $|x| > \frac{1}{|a|}$. If $x = \frac{1}{|a|}$, then we have $\sum k\frac{a^k}{|a|^k}$, which diverges by the basic divergence test. The case for $x = -\frac{1}{|a|}$ is identical, so the interval of convergence is $(-\frac{1}{|a|}, \frac{1}{|a|})$.

26. Applying the ratio test, $\lim_{k\to\infty}\left|\frac{(-e)^{k+1}x^{k+1}}{(k+1)^2}\frac{k^2}{(-e)^kx^k}\right| = \lim_{k\to\infty}|x|e\left(\frac{k}{k+1}\right)^2 = e|x|$.

 The series converges when $|x| < \frac{1}{e}$ and diverges when $|x| > \frac{1}{e}$. If $x = \frac{1}{e}$, then the series becomes $\sum\frac{(-1)^k}{k^2}$, which converges by the alternating series test. If $x = -\frac{1}{e}$, then the series becomes $\sum\frac{1}{k^2}$, which converges. The interval of convergence is $[-\frac{1}{e}, \frac{1}{e}]$.

46. It is easy to see (by induction) that $s_k = \sum_{n=1}^{k}\frac{1}{n} < k$ for all positive integers k. Furthermore $\sum kx^k$ converges for $|x| < 1$ by the ratio test. Therefore $\sum s_kx^k$ converges for at least $|x| < 1$ by the comparison test. But if $|x| = 1$, then we have either $\sum s_k$ or $\sum(-1)^ks_k$. But neither of these sums converge; by the basic divergence test

$$\lim_{k\to\infty}s_k = \sum_{n=1}^{\infty}\frac{1}{n}$$

 diverges, so the series in question must diverge for all $|x| > 1$ (Theorem 12.8.2, statement 2). The interval of convergence is therefore $(-1, 1)$.

3. (SHE 12.9)

 2. Using the geometric series, we have

$$\frac{1}{(1-x)^3} = \frac{1}{2}\frac{d^2}{dx^2}\left(\frac{1}{1-x}\right) = \frac{1}{2}\frac{d^2}{dx^2}\sum_{n=0}^{\infty}x^n = \frac{1}{2}\sum_{n=2}^{\infty}n(n-1)x^{n-2} = \sum_{n=2}^{\infty}\frac{n(n-1)}{2}x^{n-2}.$$

 6. $\ln(2-3x) = \ln 2 + \ln(1-\frac{3}{2}x) = \ln 2 - \frac{3}{2}x - \frac{1}{2}(\frac{3}{2})^2x^2 - \frac{1}{3}(\frac{3}{2})^3x^3 - \cdots - \frac{1}{n+1}(\frac{3}{2})^{n+1}x^{n+1} - \cdots$.

 14. $\frac{1-x}{1+x} = \frac{1}{1+x} - \frac{x}{1+x} = \sum_{k=0}^{\infty}(-1)^kx^k - x\sum_{k=0}^{\infty}(-1)^kx^k = 1 + 2\sum_{k=0}^{\infty}(-1)^{k+1}x^{k+1}$.

 28. Substituting the power series of $\cos x$ and integrating term-by-term,

$$\begin{aligned}\int_0^x\frac{1-\cos t}{t^2}\,dt &= \int_0^x\frac{1}{t^2}\left[1-\sum_{k=0}^{\infty}(-1)^k\frac{t^{2k}}{(2k)!}\right]dt\\ &= \int_0^x\frac{1}{t^2}\left[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{t^{2k}}{(2k)!}\right]dt = \int_0^x\left[\sum_{k=1}^{\infty}(-1)^{k+1}\frac{t^{2k-2}}{(2k)!}\right]dt\\ &= \sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{(2k)!}\frac{x^{2k-1}}{2k-1}.\end{aligned}$$

 42. First note that the power series for $e^x = \sum_{k=0}^{\infty}\frac{x^k}{k!}$. To obtain the power series required, note that

$$e^{x^3} = \sum_{k=0}^{\infty}\frac{x^{3k}}{k!} \implies xe^{x^3} = \sum_{k=0}^{\infty}\frac{x^{3k+1}}{k!}.$$

4. (i) $\lim_{k\to\infty}\left|\frac{(k+1)^{k+1}x^{k+1}}{(k+1)!}\frac{k!}{k^kx^k}\right| = \lim_{k\to\infty}\frac{(k+1)^k}{k^k}|x| = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k|x| = e|x|$.

 Hence by the ratio test, the series converges for $|x| < \frac{1}{e}$, so the radius of convergence is $R = \frac{1}{e}$.

(ii) $\lim_{k\to\infty}\left|\frac{(k+1)^{k+1}x^{k+1}}{[(k+1)!]^2}\frac{(k!)^2}{k^k x^k}\right| = \lim_{k\to\infty}\frac{(k+1)^{k-2}}{k^k}|x| = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k\cdot\frac{1}{(k+1)^2}|x| = 0.$
The limit is always less than 1, so by the ratio test, the series converges for all x. Therefore, the radius of convergence is $R=\infty$.

(iii) $\lim_{k\to\infty}\left|\frac{(k+1)^{k+1}x^{k+1}}{[(k+1)!]^{3/2}}\frac{(k!)^{3/2}}{k^k x^k}\right| = \lim_{k\to\infty}\frac{(k+1)^{k-\frac{1}{2}}}{k^k}|x| = \lim_{k\to\infty}\left(1+\frac{1}{k}\right)^k\cdot\frac{1}{(k+1)^{1/2}}|x| = 0.$
Again, the limit is always less than 1, so the series converges for all x, and the radius of convergence is $R=\infty$.

(iv) For this series, neither the ratio test nor the root test helps. However, note that $\sqrt[k]{k!} < \sqrt[k]{k^k} = k$, and $\sum kx^k$ converges for all $|x|<1$ by the ratio test, so by the comparison test, the series $\sum\sqrt[k]{k!}/x^k$ converges for $|x|<1$. What about $|x|\geq 1$? At $x=1$, the series is merely $\sum\sqrt[k]{k!}$, but $\sqrt[k]{k!} > \sqrt[k]{k}$ and $\lim_{k\to\infty}\sqrt[k]{k}=1$, so $\lim_{k\to\infty}\sqrt[k]{k!}\neq 0$, so the series must diverge by the basic divergence test. So by statement 2 of Theorem 12.8.2, the series in question must diverge for all $|x|>1$. Hence, the radius of convergence is $R=1$.

5. Starting with $\frac{1}{1-x} = \sum_{k=0}^{\infty} x^k$, we have

$$\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{d}{dx}\sum_{k=0}^{\infty}x^k = \sum_{k=1}^{\infty}kx^{k-1} \implies x\frac{d}{dx}\left(\frac{1}{1-x}\right) = x\sum_{k=1}^{\infty}kx^{k-1} = \sum_{k=1}^{\infty}kx^k$$
$$\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) = \sum_{k=1}^{\infty}k^2x^{k-1} \implies x\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) = \sum_{k=1}^{\infty}k^2x^k = \sum_{k=0}k^2x^k.$$

But $x\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) = \frac{x}{(1-x)^2}+\frac{2x^2}{(1-x)^3}$, which is the equation for $\sum_{k=0}^{\infty}k^2x^k$.

6. (a) To prove that $a_n \leq 2^n$ for all n, we use complete induction. It is obvious that the statement is true for $n=0$ and $n=1$. Now suppose $a_k\leq 2^k$ and $a_{k+1}\leq 2^{k+1}$ for some integer k. Then

$$a_{k+2} = a_{k+1}+a_k \leq 2^{k+1}+2^k = 2\cdot 2^k + 2^k = 3\cdot 2^k < 4\cdot 2^k = 2^{k+2},$$

which is what we needed to show.

(b) Since $\sum 2^n x^n = \sum(2x)^n$ converes for $|x|<\frac{1}{2}$ (which can be easily verified by the ratio test), then by part (a) and the comparison test, the series $\sum a_n x^n$ converges for $|x|<\frac{1}{2}$, so the radius of convergence must be at least $\frac{1}{2}$.

(c) Since $f(x) = \sum_{n=0}^{\infty}a_nx^n$, then

$$\begin{aligned} f(x)-xf(x)-x^2f(x) &= \sum_{n=0}^{\infty}a_nx^n - x\sum_{n=0}^{\infty}a_nx^n - x^2\sum_{n=0}^{\infty}a_nx^n \\ &= \sum_{n=0}^{\infty}a_nx^n - \sum_{n=1}^{\infty}a_{n-1}x^n - \sum_{n=2}^{\infty}a_{n-2}x^n \\ &= a_0+(a_1-a_0)x+\sum_{n=2}^{\infty}(a_n-a_{n-1}-a_{n-2})x^n. \end{aligned}$$

But $a_0=a_1=1$ and $a_n-a_{n-1}-a_{n-2}=0$ (by definition of the Fibonacci sequence), so $f(x)-xf(x)-x^2f(x)=1$. Solving for f gives us $f(x)=1/(1-x-x^2)$.

(d) Since $\sum_{n=0}^{\infty}a_nx^n$ is the power series for f, then it is also the Taylor series of f, so the result is obvious from Equation 12.6.4.

7. (i) From question 5, we have

$$x\frac{d}{dx}\left(\frac{1}{1-x}\right) = \frac{x}{(1-x)^2} = \sum_{k=1}^{\infty} kx^k.$$

In particular if $x = \frac{1}{2}$, we have $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k} = \frac{\frac{1}{2}}{(1-\frac{1}{2})^2} = 2.$

(ii) Using the geometric series $\displaystyle\sum_{k=0}^{\infty} x^k = \frac{1}{1-x}$, we multiply both sides by x and differentiate twice, giving us

$$\sum_{k=0}^{\infty} x^{k+1} = \frac{x}{1-x} \implies \sum_{k=1}^{\infty} (k+1)kx^{k-1} = \frac{d^2}{dx^2}\left(\frac{x}{1-x}\right) = \frac{2}{(1-x)^2} + \frac{2x}{(1-x)^3}$$
$$\implies \sum_{k=1}^{\infty} (k+1)kx^k = \frac{2x}{(1-x)^2} + \frac{2x^2}{(1-x)^3}.$$

(iii) We need to find the power series for $\sum k^2 x^k$. This is done in question 5:

$$\sum_{k=0}^{\infty} k^2 x^k = \frac{x}{(1-x)^2} + \frac{2x^2}{(1-x)^3} \implies \sum_{k=1}^{\infty} \frac{k^2}{2^k} = \frac{\frac{1}{2}}{\frac{1}{4}} + \frac{\frac{1}{2}}{\frac{1}{8}} = 6,$$

where we let $x = \frac{1}{2}$. Hence the sum is 6.

(iv) It can be easily shown either by the basic divergence test or the ratio test that the series must diverge.

8. The power series for e^{t^2} is obtained by using the power series for e^t:

$$e^t = \sum_{k=0}^{\infty} \frac{t^k}{k!} \implies e^{t^2} = \sum_{k=0}^{\infty} \frac{t^{2k}}{k!}, \quad t \in \mathbb{R}.$$

Now integrating both sides,

$$\int_0^x e^{t^2}\,dt = \int_0^x \sum_{k=0}^{\infty} \frac{t^{2k}}{k!}\,dt = \sum_{k=0}^{\infty} \left[\frac{t^{2k+1}}{(2k+1)k!}\right]_0^x = \sum_{k=0}^{\infty} \frac{x^{2k+1}}{(2k+1)k!}.$$

The original power series converges for all x. By Theorem 12.9.4, integrating term-by-term preserves the radius of convergence, so the power series converges to the integral for all x.

9. Let $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + 6a_n$ for $n \geq 0$.

(a) $a_2 = 7$, $a_3 = 13$, $a_4 = 55$.

(b) To prove that $a_n \leq 6^n$ for all n, we use complete induction. Since $a_0 = 1 \leq 6^0$ and $a_1 = 1 \leq 6^1$, the statement is true for $n = 0$ and $n = 1$. Now suppose $a_k \leq 6^k$ and $a_{k+1} \leq 6^{k+1}$ for some integer k. Then

$$a_{k+2} = a_{k+1} + a_k \leq 6^{k+1} + 6^k = 6 \cdot 6^k + 6^k = 7 \cdot 6^k < 36 \cdot 6^k = 6^{k+2},$$

which is what we needed to show.

(c) Since $\sum 6^n x^n = \sum (6x)^n$ converes for $|x| < \frac{1}{6}$ (which can be easily verified by the ratio test), then by part (b) and the comparison test, the series $\sum a_n x^n$ converges for $|x| < \frac{1}{6}$, so the radius of convergence must be at least $\frac{1}{6}$.

(d) Since $f(x)=\sum_{n=0}^{\infty} a_n x^n$, then

$$\begin{aligned} f(x)-xf(x)-6x^2 f(x) &= \sum_{n=0}^{\infty} a_n x^n - x\sum_{n=0}^{\infty} a_n x^n - 6x^2 \sum_{n=0}^{\infty} a_n x^n \\ &= \sum_{n=0}^{\infty} a_n x^n - \sum_{n=1}^{\infty} a_{n-1} x^n - 6\sum_{n=2}^{\infty} a_{n-2} x^n \\ &= a_0 + (a_1 - a_0)x + \sum_{n=2}^{\infty} (a_n - a_{n-1} - 6a_{n-2})x^n. \end{aligned}$$

But $a_0 = a_1 = 1$ and $a_n - a_{n-1} - 6a_{n-2} = 0$ (by definition), so $f(x)-xf(x)-6x^2 f(x) = 1$. Solving for f gives us $f(x) = 1/(1-x-6x^2)$.

(e) By partial fractions $f(x) = \frac{\frac{2}{5}}{2x+1} - \frac{\frac{3}{5}}{3x-1} = \frac{2}{5}\frac{1}{1+2x} + \frac{3}{5}\frac{1}{1-3x}$.

(f) Note that $1/(1+2x)$ and $1/(1-3x)$ can both be written as power series, so

$$f(x) = \frac{2}{5}\sum_{n=0}^{\infty}(-2)^n x^n + \frac{3}{5}\sum_{n=0}^{\infty} 3^n x^n = \sum_{n=0}^{\infty}\left[(-2)^n \frac{2}{5} + 3^n \cdot \frac{3}{5}\right] x^n.$$

But $f(x)=\sum_{n=0}^{\infty} a_n x^n$, so equating coefficients we get $a_n = (-2)^n \cdot \frac{2}{5} + 3^n \cdot \frac{3}{5}$, which is exactly what we need to prove. This gives a non-recursive formula for a_n!