University of Toronto, MAT 137Y, 2007–2008
Problem Set #15
Not to be handed in.

This assignment will not count towards your course mark. The following exercises cover new material which you are responsible for the final exam.

1. SHE Section 12.5: 4, 6, 12, 16, 28, 32, 46.
2. SHE Section 12.8: 6, 8, 10, 14, 16, 26, 46.
3. SHE Section 12.9: 2, 6, 14, 28, 42.
4. Find the radius of convergence for the following power series.
   (i) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{k!} x^k.$
   (ii) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{(k!)^2} x^k.$
   (iii) $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{(k!)^{3/2}} x^k.$
   (iv) $\displaystyle\sum_{k=1}^{\infty} (k!)^{1/k} x^k.$
5. Starting with the formula $\dfrac{1}{1-x} = 1 + x + x^2 + \cdots$, show that
$$x\frac{d}{dx}\left(x\frac{d}{dx}\left(\frac{1}{1-x}\right)\right) = x + 2^2x^2 + 3^2x^2 + \cdots,$$
   and hence derive a formula for $\displaystyle\sum_{k=1}^{\infty} k^2 x^k$.
6. Let $\{a_n\}$ be the Fibonacci sequence defined by $a_0 = a_1 = 1$ and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 0$.
   (a) Prove that $a_n \leq 2^n$.
   (b) Let $f(x) = \displaystyle\sum_{n=0}^{\infty} a_n x^n$. Show that this power series has radius of convergence $R \geq \frac{1}{2}$.
   (c) By computing $f(x) - xf(x) - x^2 f(x)$, show that $f(x) = 1/(1-x-x^2)$.
   (d) Deduce that $a_n = \dfrac{f^{(n)}(0)}{n!}$.
7. Evaluate the following series by converting each one to a power series.
   (i) $\displaystyle\sum_{k=1}^{\infty} \frac{k}{2^k}.$
   (ii) $\displaystyle\sum_{k=0}^{\infty} \frac{k(k+1)}{3^k}.$
   (iii) $\displaystyle\sum_{k=0}^{\infty} \frac{k^2}{2^k}.$

(iv) $\sum_{k=1}^{\infty} \frac{2^k}{k(k+1)}$.

8. Find a power series expansion for $f(x) = \int_0^x e^{t^2}\,dt$ valid for all x.

9. Suppose $a_0 = 1$, $a_1 = 1$, and $a_{n+2} = a_{n+1} + 6a_n$ for $n \geq 0$.

   (a) Find a_2, a_3, a_4.

   (b) Prove that $a_n \leq 6^n$.

   (c) Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$. Show that this power series has radius of convergence $R \geq \frac{1}{6}$.

   (d) Show that $f(x) = 1/(1-x-6x^2)$. (Hint: This is similar to an exercise in Problem Set 21.)

   (e) Use partial fractions to write $f(x) = \frac{A}{1-3x} + \frac{B}{1+2x}$ for some constants A and B.

   (f) Use the power series expansions of $\frac{1}{1-3x}$ and $\frac{1}{1+2x}$ to show that $a_n = \frac{3}{5} \cdot 3^n + \frac{2}{5} \cdot (-2)^n$.